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AND COMPLEMENTS THE MATERIAL IN ANY CLASS TEXT

GRADE 12

KEVIN SMITH

SUITABLE FOR IEB AND NATIONAL CURRICULUM STUDENTS

KEY TO COLOUR CODING

🟦	Headings and Sub-headings	Text	Formulas
🟩	Rules and Steps	💡	Edison - ideas, reminders and suggestions
🟥	Explanations	🧢	Recap of previous work
Bold	Questions and Emphasis	💡	Homework exercises to complete
Text	Definitions and Explanations		

Exercises in this textbook:

💡 Complete Exercise 1 on page...

The textbook is laid out so that the exercises are at the end of each chapter after the content and worked through examples. This prevents exercises from interrupting the flow of the textbook when studying theory for tests and exams. It also allows exercises for an entire section to be completed at once when revising.

After each section you will find a **Homework Edison** in a **purple block as shown abov** The Homework Edison provides a clear reference to the exercise relevant to the section which has just been completed. It is essential that you perform these exercises before moving onto the next section. The exercises will ensure consolidation and reinforce the principles which have been demonstrated in the relevant section.

💡 Please note that within examples, colour coding has been used to link concepts. There may be inconsistencies, but the clarity of the message is the primary concern. In addition, correct English punctuation may have been compromised for the sake of clarity. Whilst every care has been taken to avoid errors within the book, should you feel that there is an error, please email kevin@berlutbooks.co.za

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1. **Read slowly**. A maths book does not flow like a novel - even missing a single word could cause a misunderstanding in a fundamental concept.
2. Take note of the **colours** used. If something is in colour it is there for a reason.
3. The **environment** in which a student studies is extremely important. **A quiet room with no distractions** is conducive to learning. A student will get very little out of a maths book if he/she cannot concentrate.
4. **Ensure that each sentence or line** of an equation **is understood** before proceeding.
5. **Do not leave out anything**. Each example illustrates a specific concept and these are rarely repeated. There is very little chance of understanding missed information by reading further in the textbook.
6. All **symbols** in mathematics have a very specific meaning. If there is any uncertainty about the meaning of a term or symbol, do not set it aside, look it up or ask a teacher to explain it to you. The **glossary** in the back of this book will explain most of the important terms.
7. Always study with a **pencil and paper**. Work out proofs and derivations yourself, using the textbook as a guide. **After reading an example, cover it up and attempt the problem yourself, taking note of any errors you made.**
8. Mathematics is a hands-on subject. A sportsman doesn't become a professional by watching on the side-lines. He/she **practises** for hours every day. In the same way, you cannot become a mathematician by watching others do maths. **Practice, practice** and more **practice** is the only way to truly master maths.
9. Your brain also needs to **rest**. Do not study for hours at a time without a break. Effective studying requires concentration. If you are tired take a **nap**.
10. When **preparing** for an exam or test, make sure that you do as many **past papers** as you can. If you are unsure of how to answer a question, try to find the method in this book before looking at the answer at the back. This will prove very satisfying and build your confidence. Before your exam, practise at least two papers under exam conditions and according to the allocated time.

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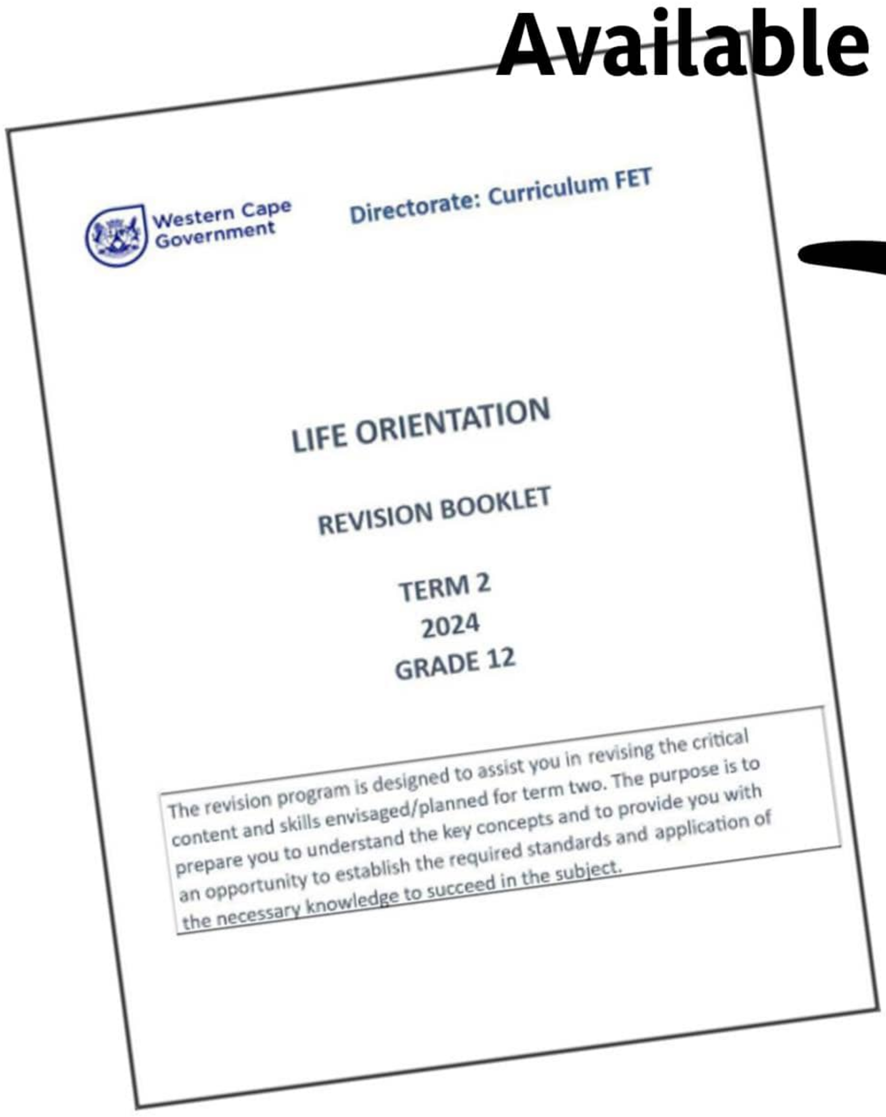

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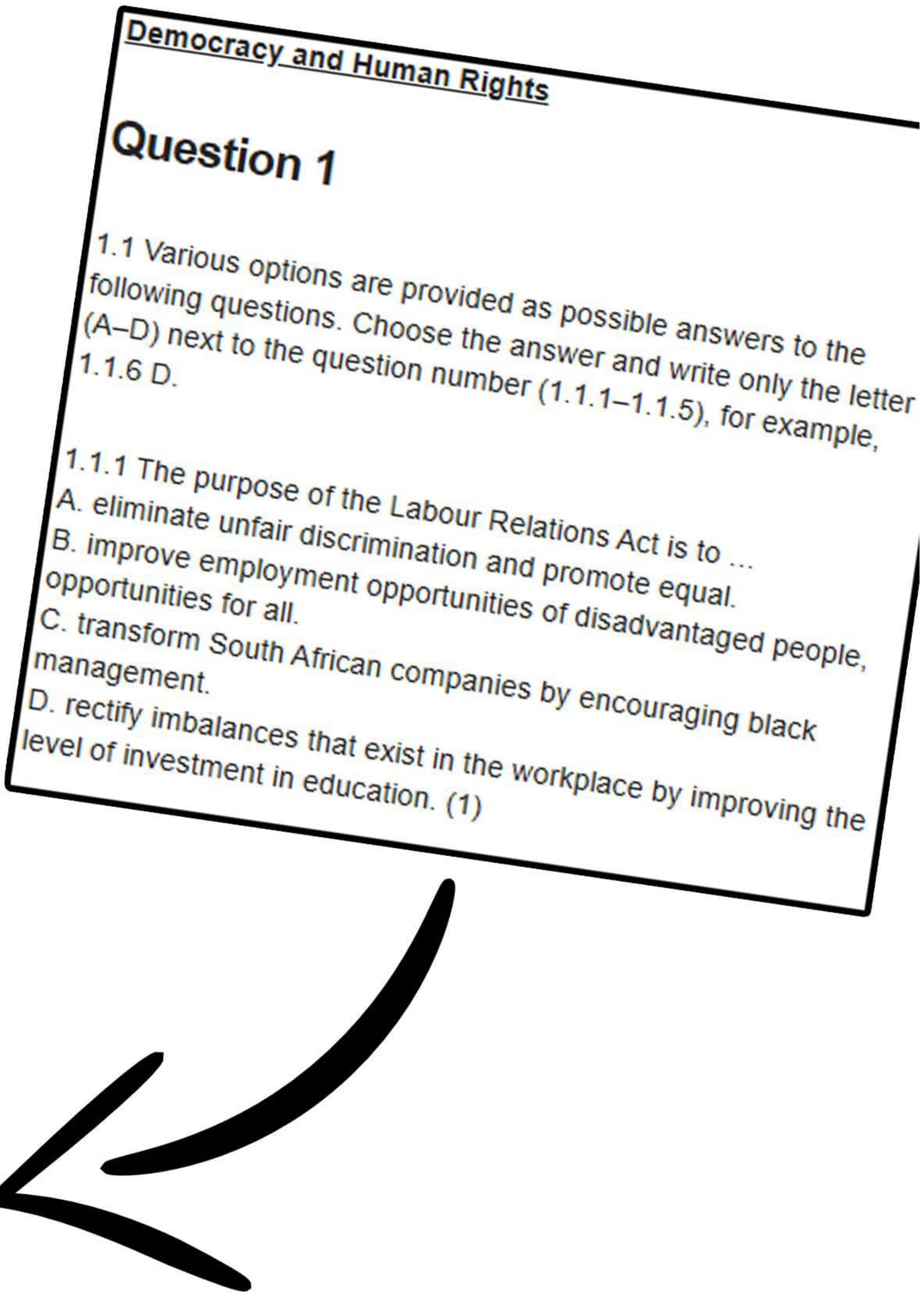
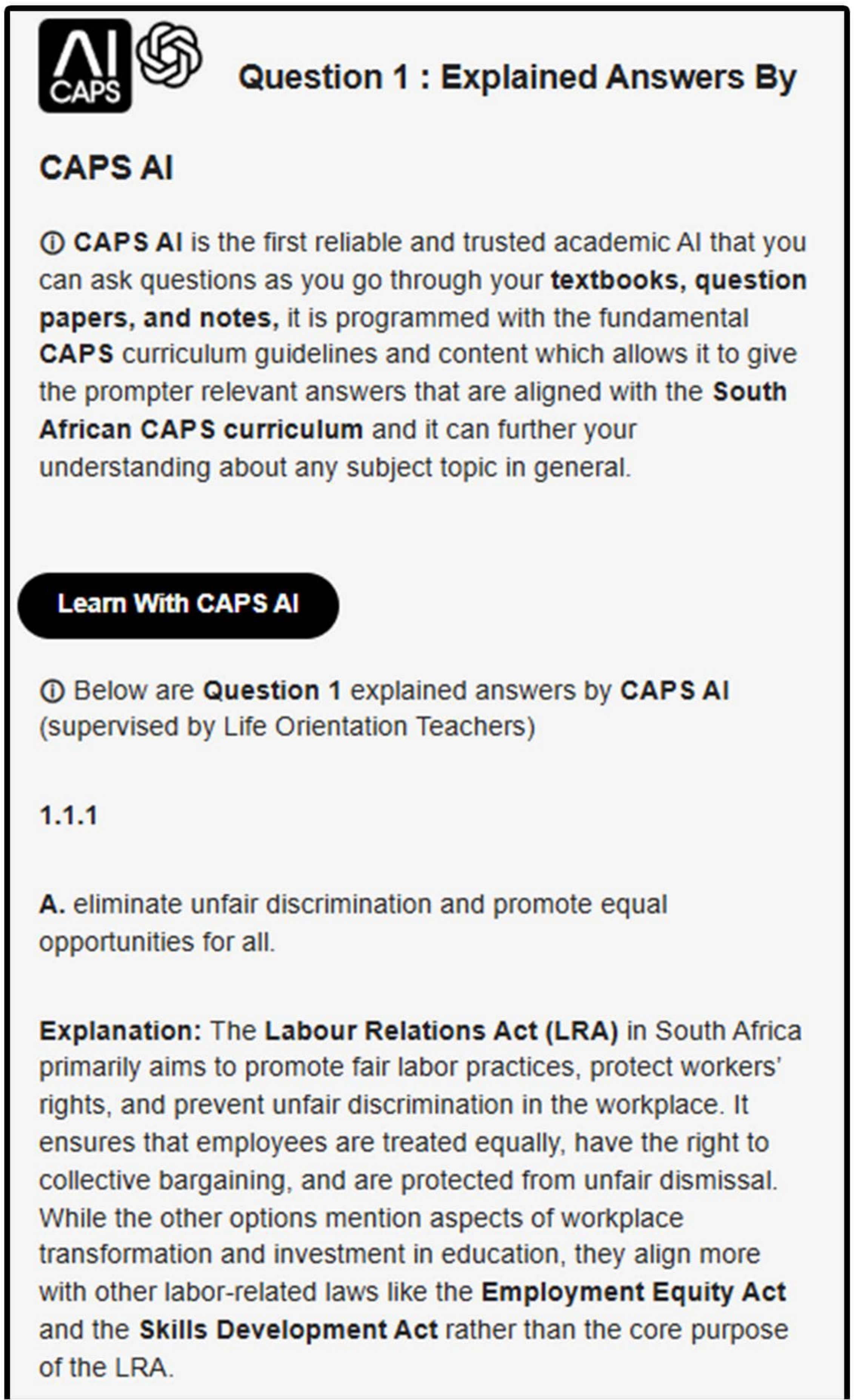


Democracy and Human Rights

Question 1

1.1 Various options are provided as possible answers to the following questions. Choose the answer and write only the letter (A–D) next to the question number (1.1.1–1.1.5), for example, 1.1.6 D.

1.1.1 The purpose of the Labour Relations Act is to …
A. eliminate unfair discrimination and promote equal opportunities for all.
B. improve employment opportunities of disadvantaged people.
C. transform South African companies by encouraging black management.
D. rectify imbalances that exist in the workplace by improving the level of investment in education. (1)



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ⓘ Below are **Question 1** explained answers by **CAPS AI** (supervised by Life Orientation Teachers)

1.1.1

A. eliminate unfair discrimination and promote equal opportunities for all.

Explanation: The **Labour Relations Act (LRA)** in South Africa primarily aims to promote fair labor practices, protect workers' rights, and prevent unfair discrimination in the workplace. It ensures that employees are treated equally, have the right to collective bargaining, and are protected from unfair dismissal. While the other options mention aspects of workplace transformation and investment in education, they align more with other labor-related laws like the **Employment Equity Act** and the **Skills Development Act** rather than the core purpose of the LRA.

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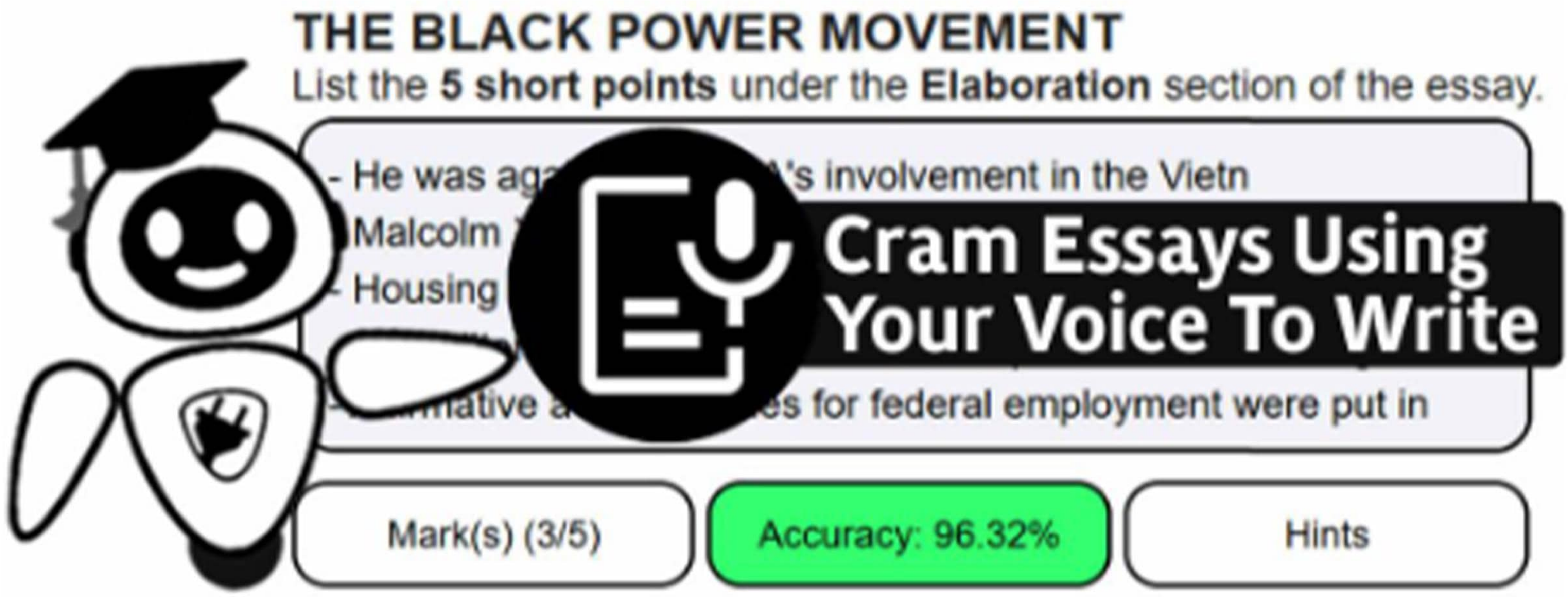

THE BLACK POWER MOVEMENT

List the **5 short points** under the **Elaboration** section of the essay.

- He was ag A's involvement in the Vietn
- Malcolm
- Housing
- rmative a es for federal employment were put in

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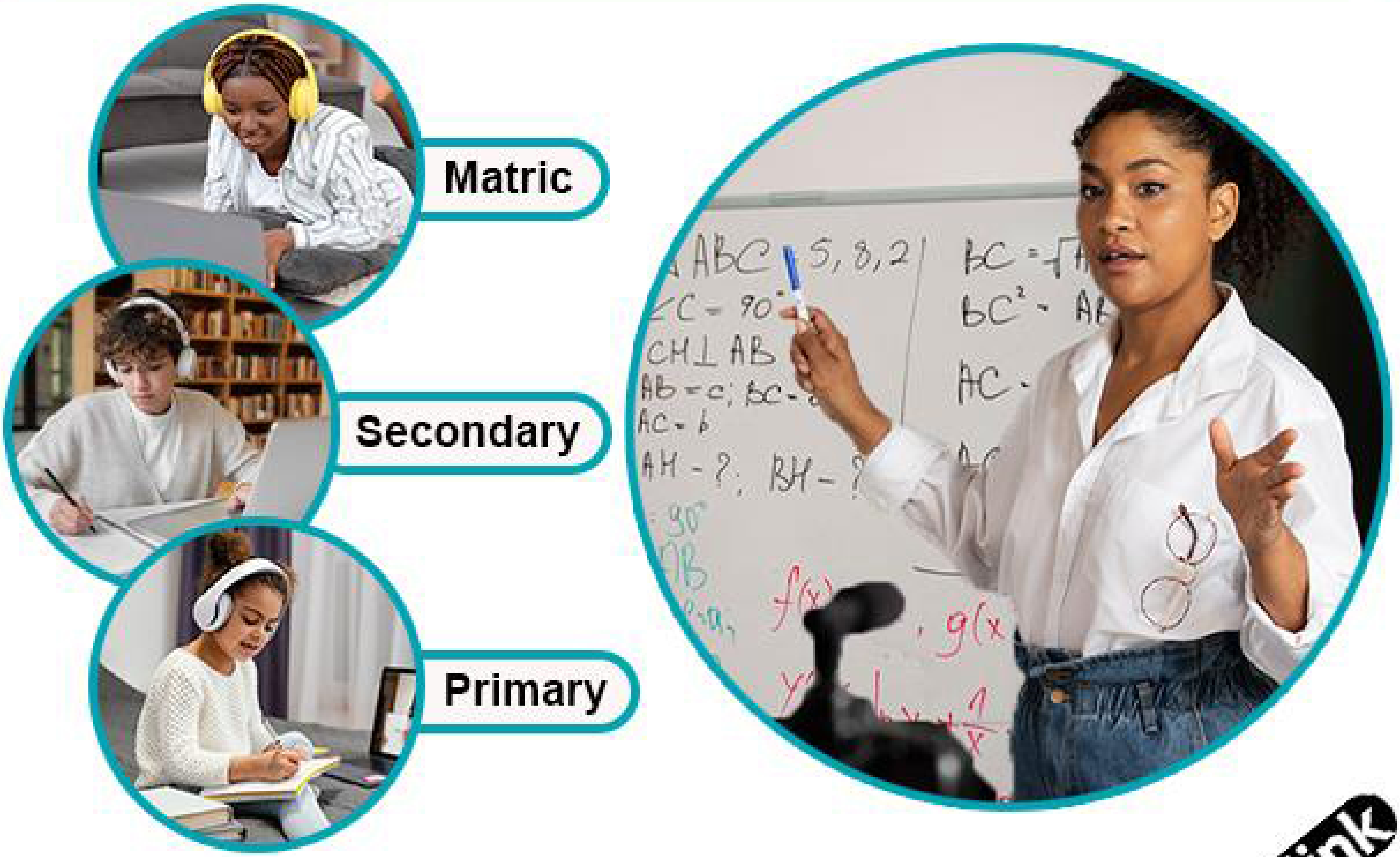

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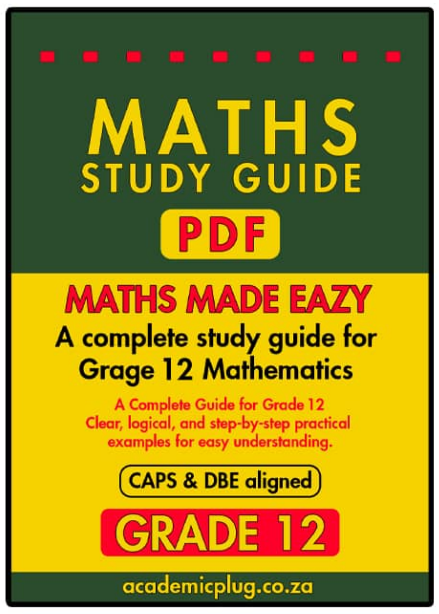
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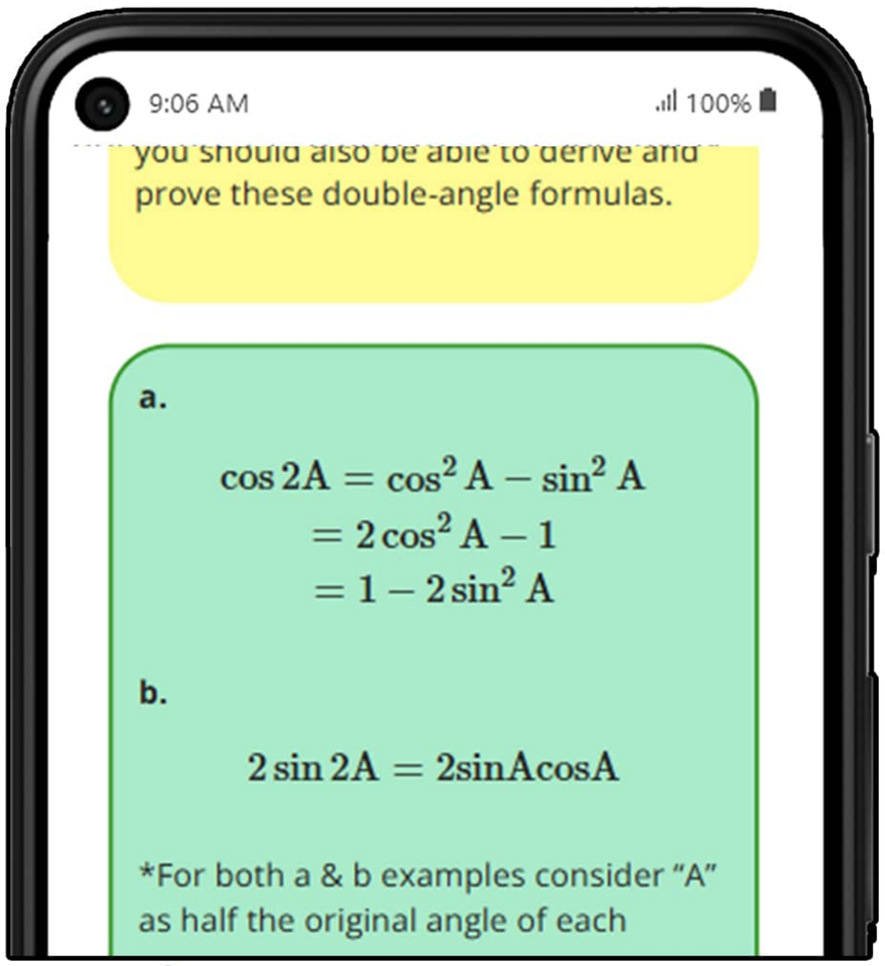

you should also be able to derive and prove these double-angle formulas.

a.

$$\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A \\ &= 2\cos^2 A - 1 \\ &= 1 - 2\sin^2 A\end{aligned}$$

b.

$$2\sin 2A = 2\sin A\cos A$$

*For both a & b examples consider "A" as half the original angle of each

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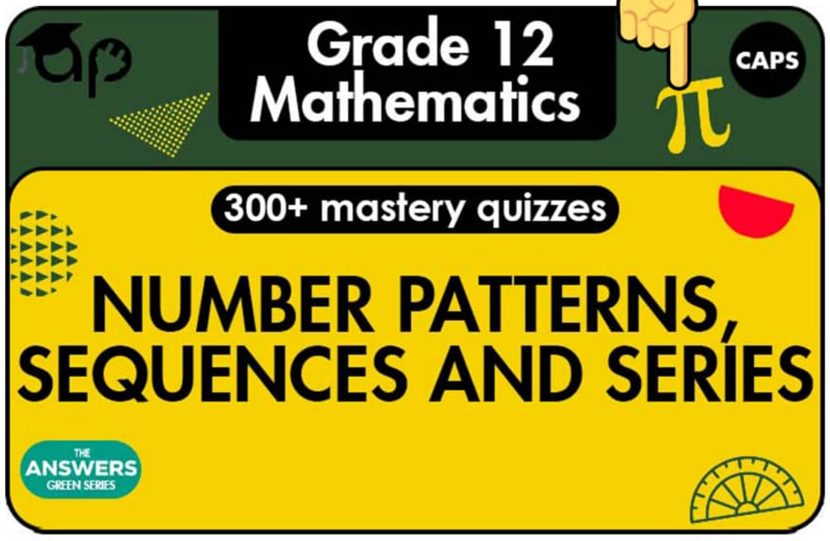
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CHAPTER 1

NUMBER PATTERNS, SEQUENCES AND SERIES

WHAT YOU WILL LEARN IN THIS CHAPTER

PAGE	CONTENT	CAPS	ASSESSMENTS YOU WILL BE ABLE TO:
3	**A. Grade 11 Revision of Quadratic Sequences**	2	• Identify and solve problems involving number patterns that lead to arithmetic and geometric sequences and series, including infinite geometric series
4	**B. Sequences (Progressions)**		
4	• Arithmetic and Geometric Sequences		
8	**C. Series**		
8	• Arithmetic Series		
8	• Proof		
9	• Geometric Series		
9	• Proof		
9	• Summary of sequences and Series Formulas		
10	• Arithmetic and Geometric Series Examples		
13	**D. Sigma Notation**		
14	**E. Infinite Geometric Sequences**		
15	• Sum to Infinity		
16	**F. Applications of Sequences and Series**		
18	• Percentages in Sequences and Series		

A. GRADE 11 REVISION OF QUADRATIC SEQUENCES

A quadratic sequence is a sequence in which the second difference is constant.
In a quadratic sequence the general equation for the n^{th} term in a sequence is:

$$T_n = an^2 + bn + c$$

Where: $n \rightarrow$ the position of the term in the sequence $(ie:$ term 1, term2, ...$)$

$a \rightarrow$ constant term that is equal to half of the second difference

$b;c \rightarrow$ constant terms

Example: **Determine an algebraic expression that represents the n^{th} term of the sequence 4, 7, 14, 25, ...**

Ensure that the sequence is quadratic:

4 ↘↙ 7 ↘↙ 14 ↘↙ 25
3 ↘↙ 7 ↘↙ 11 → first difference
4 4 → constant second difference

$\therefore$ Sequence is quadratic

$\therefore T_n = an^2 + bn + c$

Now calculate the value of a:

$\therefore 2a =$ second difference

$\therefore 2a = 4$

$\therefore a = 2$

$\therefore T_n = 2n^2 + bn + c$

Solve for b and c by substituting in two terms and using simultaneous equations:

Substitute $T_1 = 4$:

$4 = 2(1)^2 + b(1) + c$

$\therefore\ 2 = b + c \rightarrow \boxed{1}$

Substitute $T_2 = 7$:

$7 = 2(2)^2 + b(2) + c$

$-1 = 2b + c \rightarrow \boxed{2}$

$\boxed{2} - \boxed{1}$:

$\therefore -1 - 2 = 2b + c - (b + c)$

$\therefore b = -3 \rightarrow \boxed{3}$

Subsitute $\boxed{3}$ into $\boxed{1}$

$\therefore 2 = -3 + c$

$\therefore c = 5$

$\therefore T_n = 2n^2 - 3n + 5$

B. SEQUENCES (PROGRESSIONS)

A sequence is a set of ordered numbers.
There are two types of sequences that will be covered in Grade 12:

1. Arithmetic
2. Geometric

1. **An arithmetic** sequence is a sequence in which the **difference** between any two consecutive terms is **common**. This constant will be denoted as ***d***.
2. **A geometric** series is a sequence where the **ratio** of any two consecutive terms is **common**. This constant will be denoted as ***r***.

Notation

The first term of a sequence is denoted as T_1, the second term as T_2, the third term as T_3 and so on. In the arithmetic and geometric formula for the nth term, ***a*** is the first term in the series.

The **difference** between two terms will be called ***d***. (Arithmetic Sequence)
The **ratio** between two terms will be called ***r***. (Geometric Sequence)

ARITHMETIC AND GEOMETRIC SEQUENCES

FORMULA:

Arithmetic	**Geometric**
$d = T_n - T_{n-1} \rightarrow$ **common difference**	$r = \dfrac{T_n}{T_{n-1}} \rightarrow$ **common ratio**
$T_n = a + (n-1)d$	$T_n = ar^{(n-1)}$

Arithmetic	**Geometric**
Example 1:	**Example 1:**
Find the 10th term of the sequence 5 ; 12 ; 19 ; …	**Find the 6th term of the sequence 2 ; 6 ; 18 ; …**
$T_n = a + (n-1)d$	$T_n = ar^{(n-1)}$
$a = 5$ and $d = 12 - 5 = 7$	$a = 2$ and $r = \dfrac{6}{2} = 3$
$\therefore T_{10} = a + (10-1)d$	$\therefore T_6 = ar^{(6-1)}$
$= 5 + 9(7)$	$= 2(3)^5$
$= 5 + 63$	$= 486$
$= 68$	

Arithmetic

Example 2:

Which term of the sequence $\frac{1}{4};-1;-2\frac{1}{4};\ldots$ is equal to $-13\frac{1}{2}$?

$$T_n = a+(n-1)d$$

$$a = \frac{1}{4} \quad \text{and} \quad d = -1-\frac{1}{4} = \frac{-5}{4}$$

$$T_n = a+(n-1)d$$

$$\therefore -13\frac{1}{2} = \frac{1}{4}+(n-1)\left(\frac{-5}{4}\right)$$

$$\therefore -\frac{27}{2}-\frac{1}{4} = (n-1)\left(\frac{-5}{4}\right)$$

$$\therefore -\frac{55}{4} = -\frac{5}{4}(n-1) \rightarrow \text{mulitply both sides by 4}$$

$$\therefore 55 = 5(n-1)$$

$$\therefore \frac{55}{5} = n-1$$

$$\therefore n = 12$$

$$\therefore T_{12} = -13\frac{1}{2}$$

Geometric

Example 2:

Which term of the sequence $20;-4;\frac{4}{5};\ldots$ is equal to $\frac{4}{125}$?

$$T_n = ar^{(n-1)}$$

$$a = 20 \quad \text{and} \quad r = \frac{-4}{20} = \frac{-1}{5}$$

$$\therefore \frac{4}{125} = 20\left(-\frac{1}{5}\right)^{n-1}$$

$$\therefore \frac{1}{625} = \left(-\frac{1}{5}\right)^{n-1}$$

$$\therefore \frac{1}{5^4} = \left(-\frac{1}{5}\right)^{n-1}$$

$$\therefore \left(-\frac{1}{5}\right)^4 = \left(-\frac{1}{5}\right)^{n-1}$$

You can multiply by negative 1 as there is an even exponent.

$$\therefore 4 = n-1$$

$$\therefore n = 5$$

$$\therefore T_5 = \frac{4}{125}$$

Arithmetic

Example 3:

Determine the arithmetic sequence if the fourth term is –6 and the eleventh term is –34.

Solve for a and d:

$T_4 = -6$ and $T_{11} = -34$

$T_n = a + (n-1)d$

$\therefore T_4 = a + 3d$ and $T_{11} = a + 10d$

$\therefore -6 = a + 3d \rightarrow \boxed{1}$

and $-34 = a + 10d \rightarrow \boxed{2}$

Equation $\boxed{1} - \boxed{2}$:

$\therefore -6 + 34 = a + 3d - a - 10d$

$\therefore 28 = -7d$

$\therefore d = -4 \rightarrow \boxed{3}$

Substitute $\boxed{3}$ into $\boxed{1}$:

$\therefore -6 = a + 3(-4)$

$\therefore a = 6$

$T_n = a + (n-1)d$

$\therefore T_n = 6 + (n-1)(-4)$

$\therefore T_1 = 6\,;\, T_2 = 2\,;\, T_3 = -2$

$\therefore$ The sequence is : $6\,;\,2\,;\,-2, \ldots$

Geometric

Example 3:

Determine the geometric sequence if the second term is $\frac{1}{9}$ and the sixth term is $\frac{1}{729}$.

Solve for a and r:

$T_2 = \frac{1}{9}$ and $T_6 = \frac{1}{729}$

$T_n = ar^{n-1}$

$\therefore \frac{1}{9} = ar \rightarrow \boxed{1}$ and $\frac{1}{729} = ar^5 \rightarrow \boxed{2}$

Equation $\boxed{2} \div \boxed{1}$:

$$\therefore \frac{\frac{1}{729}}{\frac{1}{9}} = \frac{ar^5}{ar}$$

$$\therefore \frac{1}{81} = r^4$$

$$\therefore r^4 = \frac{1}{81}$$

$$\therefore r = \pm\frac{1}{3} \rightarrow \boxed{3}$$

Substitute $\boxed{3}$ into $\boxed{1}$:

$\therefore a = \frac{1}{3}$ or $a = -\frac{1}{3}$

$T_n = ar^{(n-1)} \rightarrow$ if a is '+ ve' then r is '+ ve'

$\therefore T_1 = \frac{1}{3}\,;\; T_2 = \frac{1}{3}\left(\frac{1}{3}\right)^1\,;\; T_3 = \frac{1}{3}\left(\frac{1}{3}\right)^2$

$\therefore$ The sequence is : $\frac{1}{3}; \frac{1}{9}; \frac{1}{27}, \ldots$

OR

$T_n = ar^{(n-1)} \rightarrow$ if a is '– ve' then r is '– ve'

$\therefore T_1 = -\frac{1}{3};\; T_2 = -\frac{1}{3}\left(-\frac{1}{3}\right)^1;\; T_3 = -\frac{1}{3}\left(-\frac{1}{3}\right)^2$

$\therefore$ The sequence is : $-\frac{1}{3}; +\frac{1}{9}; -\frac{1}{27}, \ldots$

Arithmetic

Example 4:

Given that T_5 is $x-2$, T_6 is $2x-6$ and T_7 is $4x-8$, determine:

a) x

b) **the sequence**

a) $T_7 - T_6 = T_6 - T_5$

$$\therefore 4x-8-2x+6 = 2x-6-x+2$$

$$\therefore 2x-2 = x-4$$

$$\therefore x = -2$$

b) Substitute $x=-2$ into $T_5,\ T_6,\ T_7$

$$\therefore T_5 = -4, \quad T_6 = -10, \quad T_7 = -16$$

$$\therefore d = T_6 - T_5 = -10-(-4) = -6$$

You now need to solve for a:

$$T_n = a+(n-1)d$$

$$\therefore T_5 = a+4d$$

$$\therefore -4 = a+4(-6)$$

$$\therefore -4 = a-24$$

$$\therefore a = 20$$

$\therefore$ The sequence is 20 ; 14 ; 8;...

Geometric

Example 4:

Given that T_4 is $x-4$, T_5 is $x+2$ and T_6 is $3x+1$ determine:

a) x

b) **the sequence**

a) $\dfrac{T_5}{T_4} = \dfrac{T_6}{T_5}$

$$\therefore \frac{x+2}{x-4} = \frac{3x+1}{x+2}$$

$$\therefore (x+2)(x+2) = (3x+1)(x-4)$$

$$\therefore x^2+4x+4 = 3x^2-11x-4$$

$$\therefore -2x^2+15x+8 = 0$$

$$\therefore 2x^2-15x-8 = 0$$

$$\therefore (2x+1)(x-8) = 0$$

$$\therefore x = -\frac{1}{2} \quad \text{or} \quad x = 8$$

b) $x = -\dfrac{1}{2}$ or $x = 8$

$$\therefore T_4 = -\frac{9}{2} \quad \text{or} \quad T_4 = 4$$

$$\therefore T_5 = \frac{3}{2} \quad \text{or} \quad T_5 = 10$$

$$\therefore T_6 = -\frac{1}{2} \quad \text{or} \quad T_6 = 25$$

$$\therefore r = \frac{-\frac{1}{2}}{\frac{3}{2}} = -\frac{1}{3} \quad \text{or} \quad r = \frac{25}{10} = \frac{5}{2}$$

$$T_6 = ar^5 \qquad\qquad T_6 = ar^5$$

$$\therefore -\frac{1}{2} = a\left(-\frac{1}{3}\right)^5 \quad \text{or} \quad \therefore 25 = a\left(\frac{5}{2}\right)^5$$

$$\therefore a = \frac{243}{2} \quad \text{or} \quad \therefore a = \frac{32}{125}$$

$\therefore$ The sequence is $\qquad$ $\therefore$ The sequence is

$$\frac{243}{2};-\frac{81}{2};\frac{27}{2};\ldots \quad \text{or} \quad \frac{32}{125};\frac{16}{25};\frac{8}{5};\ldots$$

Complete Exercise 1 on page 19.

C. SERIES

A sequence and a series are very similar. However, in a series, all the terms are added together.

Sequence $\rightarrow 1;3;5;7;9;...$

Series $\rightarrow 1+3+5+7+9+...$

Notation

The notation used in a series is an extension of the notation used in sequences.

S_n refers to the sum to n terms. Thus, S_1 refers to the sum of the 1st term and is equal to T_1. S_2 refers to the sum of the first two terms and is equal to the sum of T_1 and T_2.

ARITHMETIC SERIES

In general: $S_n = T_1 + T_2 + T_3 + T_4 + T_5 + ... + T_n$

From this we can obtain the following formulas for arithmetic series:

Arithmetic Series Formulas:

1. $S_n = \frac{n}{2}(a+L)$ **and** 2. $S_n = \frac{n}{2}\left[2a+(n-1)d\right]$

PROOF

$S_n = T_1 + T_2 + T_3 + ... + T_n$

$\therefore S_n = a+(a+d)+(a+2d)+...+\left[a+(n-1)d\right]$

Let $L = T_n = a+(n-1)d$

$\therefore S_n = a+(a+d)+...+L-d+L \rightarrow \boxed{1}$

Reverse the order of equation $\boxed{1}$ to obtain equation $\boxed{2}$:

$\therefore S_n = L+(L-d)+...+(a+d)+a \rightarrow \boxed{2}$

Equation $\boxed{1}$ + Equation $\boxed{2}$:

$\therefore 2S_n = (a+L)+(a+L)+...+(a+L)+(a+L)$

$\therefore 2S_n = n(a+L)$

$\therefore S_n = \frac{n(a+L)}{2} \rightarrow$ use if the last term is known

Substituting in for L:

$\therefore S_n = \frac{n\left[a+a+(n-1)d\right]}{2}$

$\therefore S_n = \frac{n\left[2a+(n-1)d\right]}{2} \rightarrow$ use if the last term is not known

GEOMETRIC SERIES

Geometric Series Formulas:

1. $S_n = \dfrac{a\left(r^n - 1\right)}{r-1} \rightarrow$ **use if** $r > 1$ **and** **2.** $S_n = \dfrac{a\left(1-r^n\right)}{1-r} \rightarrow$ **use if** $r < 1$

PROOF

$S_n = T_1 + T_2 + \ldots + T_{n-1} + T_n$

$\therefore S_n = a + ar + \ldots + ar^{n-2} + ar^{n-1} \rightarrow \boxed{1}$

In order to obtain equation $\boxed{2}$, multiply equation $\boxed{1}$ by r

$\therefore rS_n = ar + ar^2 + \ldots + ar^{n-1} + ar^n \rightarrow \boxed{2}$

Equation $\boxed{2} - \boxed{1}$:

$\therefore rS_n - S_n = -a + ar^n \rightarrow$ take out a common factor of a and S_n

$\therefore S_n\left(r-1\right) = a\left(r^n - 1\right)$

$\therefore S_n = \dfrac{a\left(r^n - 1\right)}{\left(r-1\right)}$

OR

Equation $\boxed{1} - \boxed{2}$:

$\therefore S_n - rS_n = a - ar^n \rightarrow$ take out a common factor of a and S_n

$\therefore S_n\left(1-r\right) = a\left(1-r^n\right)$

$\therefore S_n = \dfrac{a\left(1-r^n\right)}{\left(1-r\right)}$

SUMMARY OF SEQUENCES AND SERIES FORMULAS

Sequences Formulas:

Arithmetic	**Geometric**
$T_n = a + \left(n-1\right)d$	$T_n = ar^{(n-1)}$

Series Formulas:

Arithmetic	**Geometric**
$S_n = \dfrac{n\left[a+L\right]}{2} \rightarrow$ **last term known**	$S_n = \dfrac{a\left(r^n - 1\right)}{r-1} \rightarrow \; r > 1$
$S_n = \dfrac{n}{2}\left[2a + \left(n-1\right)d\right] \rightarrow$ **last term not known**	$S_n = \dfrac{a\left(1-r^n\right)}{1-r} \rightarrow \; r < 1$

ARITHMETIC AND GEOMETRIC SERIES EXAMPLES

Example 1: Evaluate the sum of each of the following:

a) $20+17+14+\ldots$ to 500 terms
b) $-14-10-6-\ldots$ to 80 terms
c) $1+3+9+\ldots$ to 8 terms

a) $T_2-T_1=-3$ and $T_3-T_2=-3$

$\therefore$ Sequence is arithmetic and $d=-3$

$$S_n=\frac{n}{2}\left[2a+(n-1)d\right]$$

$$\therefore S_{500}=\frac{500}{2}\left[2(20)+499(-3)\right]$$
$$=250[40-1497]$$
$$=-364\,250$$

b) $T_2-T_1=4$ and $T_3-T_2=4$

$\therefore$ Sequence is arithmetic and $d=4$

$$S_n=\frac{n}{2}\left[2a+(n-1)d\right]$$

$$\therefore S_{80}=\frac{80}{2}\left[2(-14)+79(4)\right]$$
$$=40[-28+316]$$
$$=11520$$

c) $\frac{T_2}{T_1}=\frac{3}{1}$ and $\frac{T_3}{T_2}=\frac{9}{3}=\frac{3}{1}$

$\therefore$ Sequence is geometric and $r=3$

$$S_n=\frac{a\left(r^n-1\right)}{r-1}\rightarrow\text{geometric}$$

$$\therefore S_8=\frac{1\left[(3)^8-1\right]}{3-1}$$
$$=3\,280$$

Example 2: Determine the sum of the first 1000 natural numbers?

series $\rightarrow 1+2+3+\ldots+1000 \rightarrow$ arithmetic series

It can easily be seen that this series contains 1 000 terms. However, the number of terms can also be calculated using the formula for arithmetic sequences as shown below.

$a=1$, $d=1$ and $T_n=1000$

$1000=1+(n-1)(1)\rightarrow T_n=a+(n-1)d$

$\therefore 999=n-1$

$\therefore n=1000$

$$\therefore S_n=\frac{n(a+L)}{2}\rightarrow\text{last term is known}$$

$$S_{1000}=\frac{1000\,(1+1000)}{2}$$
$$=500\,500$$

Example 3: Use your answer in example 2 to determine the sum of the natural numbers up to 1 000, which are not divisible by 5.

In order to calculate the sum of the numbers up to 1000 which are not divisible by 5, you have to calculate the sum of the numbers which **are** divisible by 5, and subtract them from the sum of the first 1000 natural numbers.

series $\rightarrow 5+10+15+\ldots+1000$

It can easily be seen that this series contains 200 terms. $\rightarrow \frac{1000}{5}$

However, the number of terms can also be calculated using the formula for arithmetic sequences as shown below:

$a=5,\ d=5$ and $T_n=1000$

$1000=5+(n-1)(5) \rightarrow T_n=a+(n-1)d$

$\therefore 995=(n-1)(5) \rightarrow$ divide both sides by 5

$\therefore 199=n-1$

$\therefore n=200$

$\therefore S_n=\frac{n}{2}(a+L) \rightarrow$ last term is known

$$\therefore S_{200}=\frac{200}{2}(5+1000)$$

$=100\,500 \rightarrow$ sum of the natural numbers up to 1000 which are divisible by 5

$\therefore$ The sum of the first 1000 natural numbers that are not divisible by 5 is:

$\therefore 500\,500-100\,500=400\,000$

Example 4: Find the sum of all multiples of 3 between 1 and 400.

series $\rightarrow 3+6+9+\ldots+399 \rightarrow$ note that 399 is the last mulitple of 3 less than 400

It can easily be seen that this series contains 133 terms. $\rightarrow \frac{399}{3}$

However, the number of terms can also be calculated using the formula for arithmetic sequences as shown below:

$a=3,\ d=3$ and $T_n=399$

$399=3+(n-1)(3) \rightarrow T_n=a+(n-1)d$

$\therefore 396=(n-1)(3) \rightarrow$ divide both sides by 3

$\therefore 132=n-1$

$\therefore n=133$

$S_n=\frac{n}{2}(a+L) \rightarrow$ last term is known

$$\therefore S_{133}=\frac{133}{2}(3+399)$$

$=26\,733$

Example 5: **Evaluate the following series: $1{,}25+2{,}5+3{,}75+\ldots+12{,}5$**

You first have to determine the number of terms in the series:

$a=1{,}25$, $d=T_2-T_1=1{,}25$ and $T_n=12{,}5$

$\therefore 12{,}5=1{,}25+(n-1)(1{,}25)\rightarrow T_n=a+(n-1)d$

$\therefore 11{,}25=(n-1)(1{,}25)\rightarrow$ divide both sides by 1,25

$\therefore 9=n-1$

$\therefore n=10$

$S_n=\frac{n}{2}(a+L)\rightarrow$ last term is known

$\therefore S_{10}=\frac{10}{2}[1{,}25+12{,}5]=68{,}75$

Example 6: **The sum of the first n terms of a sequence is $S_n=3n^2+2n+1$**

Determine:

- **a) the sum of the first 8 terms**
- **b) the sum of the first 7 terms**
- **c) the 8th term**

a) $S_8=3(8)^2+2(8)+1$

$=209$

b) $S_7=3(7)^2+2(7)+1$

$=162$

c) $T_n=S_n-S_{n-1}$

$\therefore T_8=S_8-S_7$

$\therefore T_8=209-162$

$\therefore T_8=47$

It is important to note that:

$T_n=S_n-S_{n-1}$

Complete Exercise 2 on page 19.

D. SIGMA NOTATION

$\sum$ (sigma) notation corresponds to S and denotes summation (addition).

Illustrative Example: $\sum_{k=0}^{n} k = 0+1+2+3+\ldots+n-1+n$

The illustrative example above shows the expansion for sigma notation. The value of k below sigma is the first integer to substitute for k. Each subsequent term substituted into k is then increased in ones until you reach n, which will be the last term in the series.

The number of terms (n) is equal to the top number – the bottom number + 1

For example: $\sum_{k=0}^{5} k = 0+1+2+3+4+5=15$

$\therefore$ The number of terms in the series $= 5-0+1=6$

Example 1: Evaluate $\sum_{r=1}^{13} 3+(r-1)\left(\frac{3}{2}\right)$.

$\sum_{r=1}^{13} 3+(r-1)\frac{3}{2} = 3+\frac{9}{2}+6+\frac{15}{2}+\ldots+21 \rightarrow$ always expand when given a question in sigma notation

$n = 13-1+1=13 \rightarrow$ determine the number of terms in the series

$$S_n = \frac{n(a+L)}{2}$$

$$\therefore S_{13} = \frac{13(3+21)}{2}$$

$$=156$$

Example 2: Determine the value of n if $\sum_{k=1}^{n} 3(2)^{k-1} = 381$.

$\sum_{k=1}^{n} 3(2)^{k-1} = 3+6+12+\ldots+3(2)^{n-1} = 381 \rightarrow$ always expand when given a question in sigma notation

$\frac{T_2}{T_1} = \frac{T_3}{T_2} = 2 \rightarrow$ geometric series

$\therefore r=2$ and $a=3$

$$S_n = \frac{a\left(r^n-1\right)}{r-1}$$

$\therefore 381 = \frac{3\left(2^n-1\right)}{2-1} \rightarrow$ divide both sides by 3

$\therefore 127 = 2^n - 1$

$\therefore 128 = 2^n$

$\therefore 2^7 = 2^n$

$\therefore n = 7$

Example 3: **What is the greatest integer value of m for which $\sum_{i=1}^{m}(2)^{i-1} < 511$?**

$$\sum_{i=1}^{m}(2)^{i-1} = 1+2+4+...+(2)^{m-1} \rightarrow \text{always expand when given a question in sigma notation}$$

$$S_m < 511$$

$$\frac{T_2}{T_1} = \frac{T_3}{T_2} = 2 \rightarrow \text{geometric series}$$

$$\therefore r = 2 \quad \text{and} \quad a = 1$$

$$S_m = \frac{a\left(r^m - 1\right)}{r-1}$$

$$\therefore \frac{1\left(2^m - 1\right)}{2-1} < 511$$

$$\therefore 2^m < 512$$

$$\therefore 2^m < 2^9$$

$$\therefore m < 9$$

$$\therefore m = 8 \rightarrow m \text{ has to be a natural number } (\mathbb{N})$$

Variables other than k, such as i, are often used in sigma notation as shown in the above example.

Complete Exercise 3 on page 20.

E. INFINITE GEOMETRIC SEQUENCES

An infinite geometric series has no end.
Although an infinite geometric series never ends, it is possible to calculate a value for certain infinite series. We say a series of this type **converges**.

If a series keeps on growing infinitely bigger or smaller, it is **not** possible to calculate a value for it and we say this series **diverges**.

Examples of diverging infinite geometric series are:
$\mathbf{2+4+8+16+32+...}$ **or** $\mathbf{2-4+8-16+32-...}$

An example of a converging infinite geometric series is:

$$\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\frac{1}{243}+...$$

In this case, the more terms that are added, the closer each term gets to zero. Therefore, this series will converge and a final answer to this series can be calculated.

A geometric series converges if $-1 < r < 1$.

SUM TO INFINITY

In order to determine the sum to infinity of a geometric sequence use the formula shown in the green block below:

Sum to infinity of a geometric series:

$$S_\infty = \frac{a}{1-r}$$

Where:

$a \to 1^{st}$ **term**

$r \to$ **common ratio and** $-1 < r < 1$

Example 1: Evaluate the series $\frac{5}{10} + \frac{5}{100} + \frac{5}{1000} + \ldots$

$a = \frac{5}{10}$ and $r = \frac{1}{10} \to$ series converges as $-1 < r < 1$

$$S_\infty = \frac{a}{1-r} = \frac{\frac{5}{10}}{1-\frac{1}{10}} = \frac{5}{9}$$

Example 2: Convert $4,\dot{8}\dot{1}$ **to a common fraction.**

$$4,\dot{8}\dot{1} = 4 + \frac{81}{100} + \frac{81}{10000} + \frac{81}{10^6} + \ldots \to 4,\dot{8}\dot{1} = 4,8181818181\ldots$$

$a = \frac{81}{100}$ and $r = \frac{1}{100} \to$ 4 is not part of the series

$$S_\infty = \frac{a}{1-r} = \frac{\frac{81}{100}}{1-\frac{1}{100}} = \frac{9}{11}$$

$$\therefore 4,\dot{8}\dot{1} = 4 + \frac{9}{11} = \frac{53}{11}$$

Example 3: Show that $0,\dot{9} = 1$.

$$0,\dot{9} = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \ldots \to 0,\dot{9} = 0,999999\ldots$$

$a = \frac{9}{10}$ and $r = \frac{1}{10}$

$$S_\infty = \frac{a}{1-r} = \frac{\frac{9}{10}}{1-\frac{1}{10}} = 1$$

Complete Exercise 4 on page 20.

F. APPLICATIONS OF SEQUENCES AND SERIES

Example 1: Tom increases the distance he cycles by 12km each week. If he cycles 20km in the first week, what distance does he cycle in the 8^{th} week and how far will he have cycled in total at the end of the 8^{th} week?

$$\text{series} \to 20+32+44+\ldots$$
$$T_n = a+(n-1)d$$
$$T_8 = a+7d$$
$$\therefore T_8 = 20+7(12)$$
$$=104$$

$\therefore$ He cycles 104 km in the 8^{th} week. → always put in units

$$S_8 = \frac{8}{2}(a+L) \to S_n = \frac{n}{2}\left[2a+(n-1)d\right] \text{ could also have been used}$$
$$\therefore S_8 = 4(20+104)$$
$$=496$$

$\therefore$ He would have cycled a total of 496 km after 8 weeks.

Example 2: On the first day of drilling, a driller reaches a depth of 120m. Each day after that he drills two thirds of the depth of the previous day.

a) If he earns R40/m how much will he earn by the end of the 5^{th} day?
b) What is the maximum amount of money he can earn?

$$\text{series} \to 120+120\left(\frac{2}{3}\right)+120\left(\frac{2}{3}\right)^2+\ldots \to \text{infinite series with } r=\frac{2}{3}$$

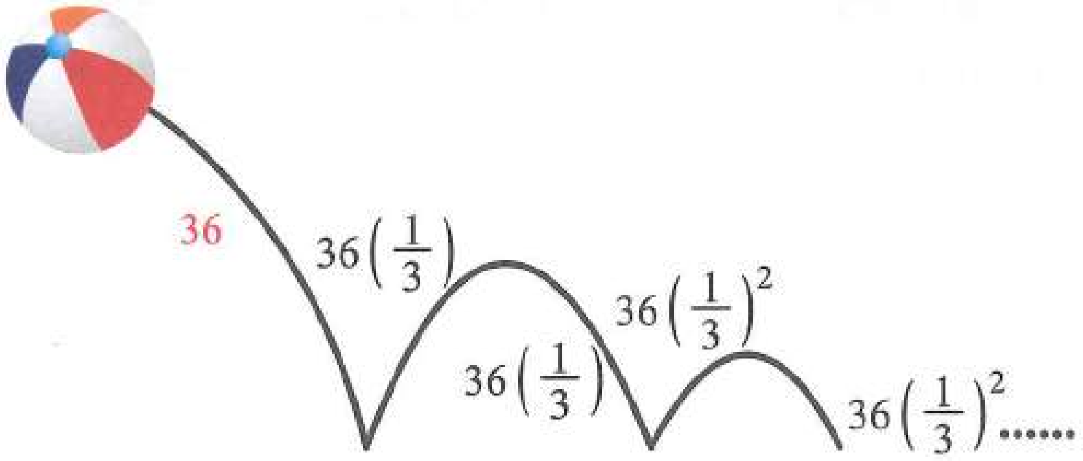
a) $$S_5 = \frac{120\left[1-\left(\frac{2}{3}\right)^5\right]}{1-\frac{2}{3}} \to S_n = \frac{a\left(1-r^n\right)}{1-r}$$
$$=312{,}59\,\text{m}$$

$\therefore$ He earns $312{,}59 \times \text{R}40 = \text{R}12503{,}60$

b) $$S_\infty = \frac{a}{1-r}$$
$$\therefore S_\infty = \frac{120}{1-\frac{2}{3}}$$
$$=360\,\text{m} \to \text{this means that he can drill to a maximum depth of } 360\,\text{m}$$

$\therefore$ The maximum he can earn is $360 \times \text{R}40 = \text{R}14\,400$

Example 3: A beach ball is dropped from a height of 36m. Each time it rebounds, it bounces one third of the height of the previous bounce. What is the total distance the ball travels before coming to rest?


36
$36\left(\frac{1}{3}\right)$
$36\left(\frac{1}{3}\right)$
$36\left(\frac{1}{3}\right)^2$
$36\left(\frac{1}{3}\right)^2$......

series $\rightarrow 36+2(36)\left(\frac{1}{3}\right)+2(36)\left(\frac{1}{3}\right)^2+2(36)\left(\frac{1}{3}\right)^3+\ldots$

Each time the ball bounces it travels up a certain distance. It then has to travel back the same distance down to the ground. This is the reason for multiplying each term by two.

On the first drop, the ball only travels 36m once and is therefore not part of the series. The first term of the series is the term following 36.

$$s_\infty=\frac{a}{1-r}$$

$$\therefore s_\infty=\frac{2(36)\left(\frac{1}{3}\right)}{1-\frac{1}{3}}=36$$

$\therefore$ The total distance the ball travels is: $36+36=72\,\text{m}$

Example 4: The radius of the first circle of an infinite sequence of circles is 10cm. The radius of each of the following circles is two thirds of the radius of the previous circle. What is the total area of all the circles formed?

series for radius $\rightarrow 10+10\left(\frac{2}{3}\right)+10\left(\frac{2}{3}\right)^2+\ldots$

Area of a circle $=\pi r^2$

$\therefore$ series for area $\rightarrow \pi\left[10^2+10^2\left(\frac{2}{3}\right)^2+10^2\left(\frac{2}{3}\right)^4+\ldots\right]$

$$\therefore s_\infty=\frac{a}{1-r}$$

$$\therefore s_\infty=\frac{10^2}{1-\left(\frac{2}{3}\right)^2}$$

$$\therefore s_\infty=180$$

$\therefore$ Total area $=180\pi\ \text{cm}^2 \rightarrow$ always put in units

PERCENTAGES IN SEQUENCES AND SERIES

When dealing with problems involving sequences and series you will often have to work with percentages.

Calculating percent of:

Calculate 30% of 200

30% of 200 is: $200 \times 0,3 = 60$

Adding on a percentage:

Add 14% VAT onto R50

Adding 14% VAT onto R50: $\text{R}50(1,14) = \text{R}57$

Decreasing a value by a percentage:

An investment of R800 loses 30%. What is the investment worth?
Decreasing by 30% is the same as calculating 70% of the investment.

70% of R800 is: $800 \times 0,7 = \text{R}560$

$\therefore$ The investment will be worth R560.

Percentage of a number:	**Add on a percentage to a number**
[number×(0,percent asked for)]	**[number × (1,percent asked for)]**

If the percentage is less than 10, for example 5, then the original number would be multiplied by 0,05 or 1,05.

Example: From 2002 onwards Bob received R250 as a new year's gift every year. He invested his money at 8% p.a on a compound interest rate. How much will Bob have in his account at the end of 2006, just before he receives his new year's gift?

Interest is earned for 5 years on the first deposit, 4 years on the second deposit and so on. Therefore, the series goes in reverse. The first term is the interest earned on the last deposit made and the last term is the interest earned on the first deposit made.

$$\text{series} \rightarrow \overset{2006}{250(1,08)} + \overset{2005}{250(1,08)^2} + \overset{2004}{250(1,08)^3} + \overset{2003}{250(1,08)^4} + \overset{2002}{250(1,08)^5} \rightarrow r = 1,08$$

$$S_n = \frac{a\left[r^n - 1\right]}{r - 1}$$

$$\therefore S_5 = \frac{250(1,08)\left[(1,08)^5 - 1\right]}{(1,08) - 1}$$

$$\therefore S_5 = 1583,98$$

$\therefore$ He has a total of R1583,98 at the end of 2006.

Complete Exercise 5 on pages 20 and 21 and the Mixed Exercise on pages 21 and 22.

Sequences (progressions)

Exercise 1: (answers on page 340)

1) In the following sequences determine the:

a) 12th term of $5;12;19;...$

b) 11th term of $-3;6;-12;...$

c) 8th term of $24;6;\frac{3}{2};...$

d) 21st term of $12;\frac{17}{2};5;...$

e) 18th term of $4;4+2x;4+4x;...$

f) 25th term of $a;ab;ab^2;...$

2) Which term of the sequence:

a) $5;11;17;...$ is equal to 101?

b) $10;\frac{21}{2};11;...$ is equal to $\frac{41}{2}$?

c) $5x;20x;80x;...$ is equal to $20480x$?

d) $21;-7;\frac{7}{3};...$ is equal to $\frac{7}{243}$?

3) Determine an expression for the nth term of the following sequence if the:

a) 4th term is 24 and the 7th term is 192 in a geometric sequence.

b) 2nd term is 1 and the 9th term is -34 in an arithmetic sequence.

c) 6th term is 13 and the 14th term is 33 in an arithmetic sequence.

d) 5th term is 9 and the 9th term is 729 and the sequence contains alternating positive and negative terms.

4) If $(4-x)$, 3 and $\sqrt{2x+3}$ are the first three terms of an arithmetic sequence:

a) Determine the value of x.

b) Which term is equal to –23?

5) If $q;2q+2;5q+3$ form an arithmetic sequence:

a) Determine the value of q.

b) Determine the value of the 15th term.

6) If $2p-1;\ p+1;\ p-1$ form a geometric sequence and $p \neq 0$:

a) What are the first three terms?

b) Which term is equal to $\frac{128}{243}$?

7) If $8;2x;2y$ forms an arithmetic sequence and $2x;2y;36$ forms a geometric sequence determine the values of x and y.

Arithmetic and Geometric Series

Exercise 2: (answers on page 340)

1) Evaluate the sum of each of the series below:

a) $1+2+3+...$ to 52 terms.

b) $25+20+15+...-55$.

c) $3-6+12-...$ to 16 terms.

d) $2{,}125+4{,}25+8{,}5+...+544$.

e) $0{,}5+1+1{,}5+...+40{,}5$.

f) $5+13+21+...$ to 28 terms

2) In an arithmetic sequence $T_1=(1-x)^2$ and $T_2=1+x^2$:

a) Write down an expression for the common difference in terms of x.

b) Determine the value(s) of x if $S_{10}=310$.

3) Using your knowledge of series:

a) Prove that the sum of the first n, odd natural numbers is n^2.

b) Prove that the sum of the first n, even natural numbers is n^2+n.

4) The sum of the first 5 terms of an arithmetic sequence is 0 and the fifth term is 8. Determine the common difference and the first term.

5) In a geometric series the first term is 36, the last term is 16 and the sum of the series is 76. Determine the common ratio and the number of terms in the sequence.

6) The sum of the first 4 terms of an arithmetic sequence is 24. The 7th term exceeds the 3rd term by 24. Determine the sum of the first 12 terms.

Sigma Notation

Exercise 3: (answers on page 341)

1) Evaluate the following:

a) $\sum_{k=1}^{12} 5+2(k-3)$

c) $\sum_{r=7}^{15} -12+(r-2)(-2)$

b) $\sum_{k=5}^{12} 3(2)^{k-5}$

d) $\sum_{k=0}^{8} 2(3)(3)^{k}$

2) Determine the value of *n* for which:

a) $\sum_{k=1}^{n} 3+(k-2)(4)=594$.

b) $\sum_{k=7}^{n} 2k-3$ is equal to the sum of the first 6 terms of the sequence $-24,\ 48,\ -96$.

3) What is the greatest number of terms for which:

a) $\sum_{k=1}^{n}(2k+1)$ will have a value less than 68?

b) $\sum_{x=2}^{n}\left[3(2)^{x-1}\right]$ will have a value less than 89?

Infinite Geometric Sequences

Exercise 4: (answers on page 341)

1) Determine the sum of each of the following infinite geometric sequences below:

a) $5+\frac{5}{2}+\frac{5}{4}+\ldots$

c) $20+2+0{,}2+\ldots$

b) $6x-2x+\frac{2x}{3}-\ldots$

d) $\sum_{x=1}^{\infty} 2^{1-x}$

2) Convert the following recurring decimals to common fractions using a sum to infinity:

a) $3{,}\dot{2}\dot{8}$.

b) $1{,}\dot{5}6\dot{3}$

3) Determine the value(s) of *x*, for which the following series would converge:

a) $3+3(x+1)+3(x+1)^2+\ldots$

b) $2x+\left(-x^2-3x\right)+2x\left(\frac{x+3}{-2}\right)^2+\ldots$

Applications of Sequences and Series

Exercise 5: (answers on page 341)

1) A plant in a greenhouse grows to 2cm in the first month. Thereafter, its growth is two thirds of the previous month's growth. (Give all answers to the nearest millimetre).

a) What is the plant's growth in the 4th month?

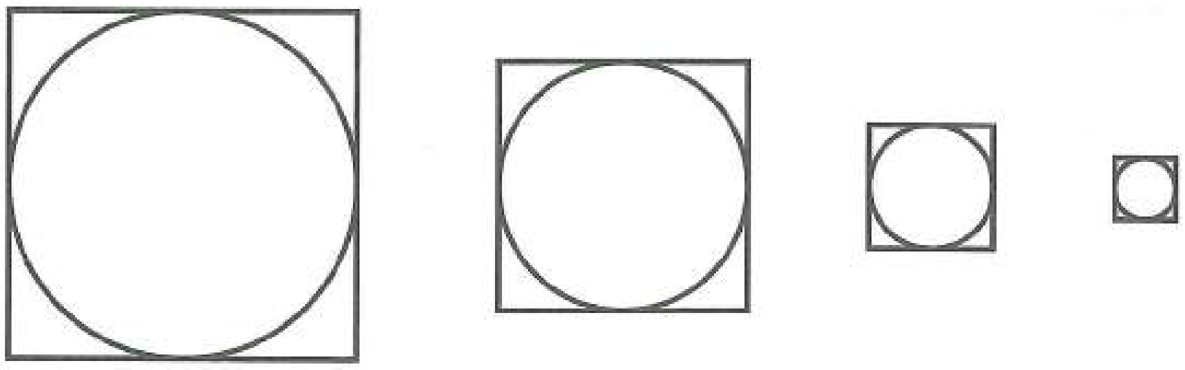
b) What is the maximum height that the plant will reach?

2) The highest bungee jump site in the world is located on the Bloukrans River Bridge in South Africa. On average a person will fall 180m. After the initial fall a person will bounce up two thirds the distance of the previous fall.

a) What is the maximum height of a person's first bounce?

b) Determine the total vertical distance a person would travel before coming to rest.

3) An infinite sequence of squares is drawn with circles inscribed in each square as shown below. Each square is half the size of the square before it. The area of the first square is 64cm^2.



a) Determine the circumference of the 5th circle.
b) Determine the sum of the circumferences of the first 4 circles.
c) Determine the sum to infinity of the circumferences of the circles.

4) On the 31st of January 2011 Masimo started a job with a salary of R12 000 per month. Her contract states that on the 31st of January in each subsequent year she will receive an increase of 14%. How much will Masimo have earned by the 31st December 2016, assuming she gets paid on the last day of each month?

Mixed Exercise: (answers on pages 341 and 342)

1) Evaluate $3+9+7+13+11+17+15+21+19+25+\ldots$ to 100 terms.

2) Determine the value(s) of x in the sequence $4;x;32$ if the sequence is:

a) Arithmetic
b) Geometric

3) The sum of the first n terms of a sequence is given by $S_n = 3^{n+1} - 3$, determine:

a) The sum of the first 5 terms.
b) The sum of the first 4 terms.
c) The 5th term.

4) Given the sequence $6;9;12;18;18;36;\ldots$:

a) Determine the value of $T_{40} - T_{41}$.
b) Show that every term of the sequence is divisible by 3.

5) In the table below each term in sequence C is obtained by square rooting the sum of the squares of each term in sequence A and B.

n	1	2	3	4	5
Sequence A	4	6	8	10	
Sequence B	3	8	15	24	
Sequence C	5	10	17		37

a) Fill in the missing values in the table.
b) Determine a formula for the nth term of sequence A in its simplest form.
c) Determine a formula for the nth term of sequence B.
d) Determine a formula for the nth term of sequence C.

6) A man receives a R4800 Christmas bonus in his first year of work. Every year his bonus increases by a fixed amount.

a) What is the annual increase if he receives a total of R143 500 in bonuses over a 20 year period?
b) Each year he spends R2 600 on Christmas presents. The amount he spends increases by R400 per year. How many years will it take for his Christmas expenditure to exceed his bonus?

7) Given the series $1+3x+9x^2+\ldots$

a) Determine the possible values of x so that the series is convergent.
b) Determine the value of x if $1+3x+9x^2+\ldots=\frac{2}{3}$.

8) The first term of a geometric sequence is 9. The ratio of the sum of the first 8 terms to the sum of the first 4 terms is 97:81. Determine the first 3 terms of the sequence if the sequence contains positive and negative terms.

9) **A quadratic pattern has a second term equal to 15, a third term equal to 28 and a fifth term equal to 66.**

   a) Calculate the second difference of the quadratic pattern.

   b) Hence, calculate the first term of the pattern.

10) **A hot dog vendor walks from his house to the end of his block to sell hot dogs. Each day he walks an additional block further from his house. Assuming each block is 0,75km, determine:**

   a) The total distance that the hot dog vendor travels in 10 days.

   b) The distance that he travelled on the 10th day.

11) **Felicia draws a circle with a radius of 3m. She then stands in the centre and takes a step which takes her half way to the end of the circle. Each consecutive step is half the length of the previous one.**

   a) How far did Felicia move on her first step?

   b) How far did Felicia move on her nth step?

   c) Will Felicia ever get out of the circle?

12) **The first term of a non-trivial arithmetic sequence is 3. The 3rd, 6th and 10th term of the arithmetic sequence form a geometric sequence. Determine the value of the 12th term of the arithmetic sequence.**

13) **Given $S_n = 3^{n+2} - 9$, show that T_n is given by $T_n = 2\left(3^{n+1}\right)$.**

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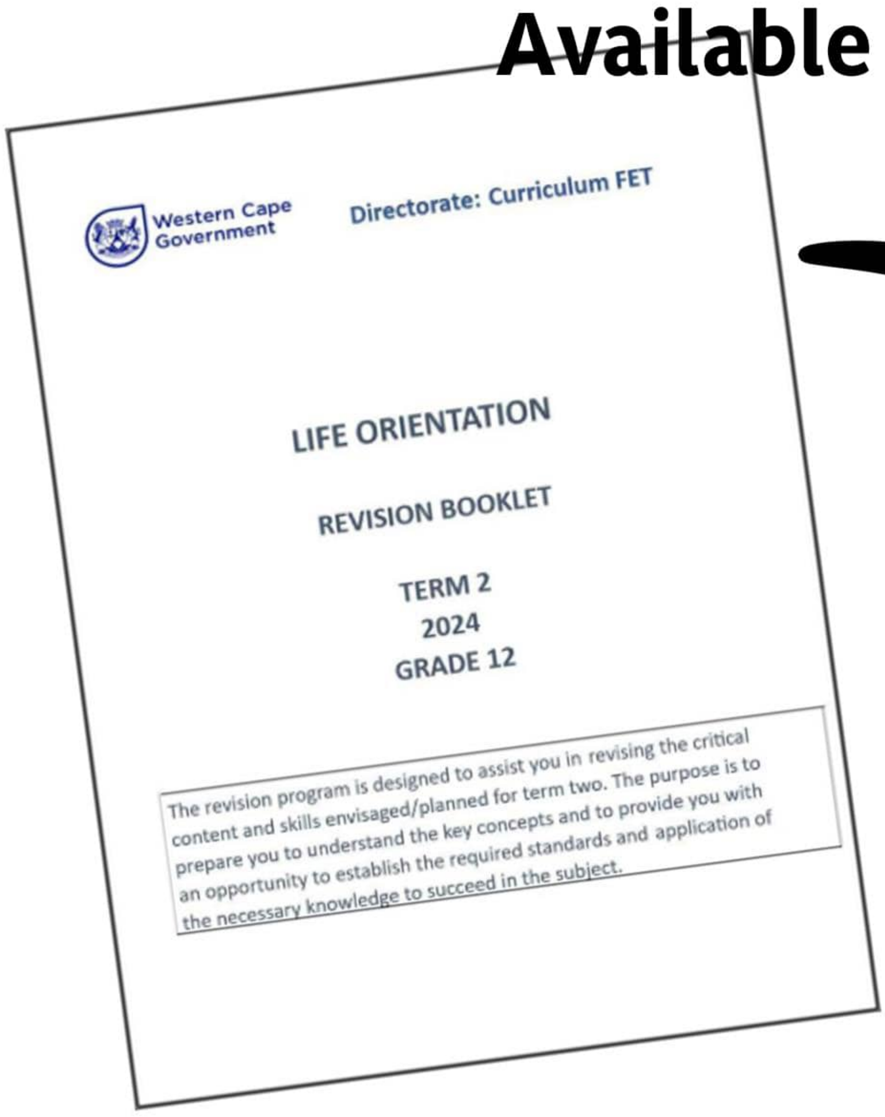

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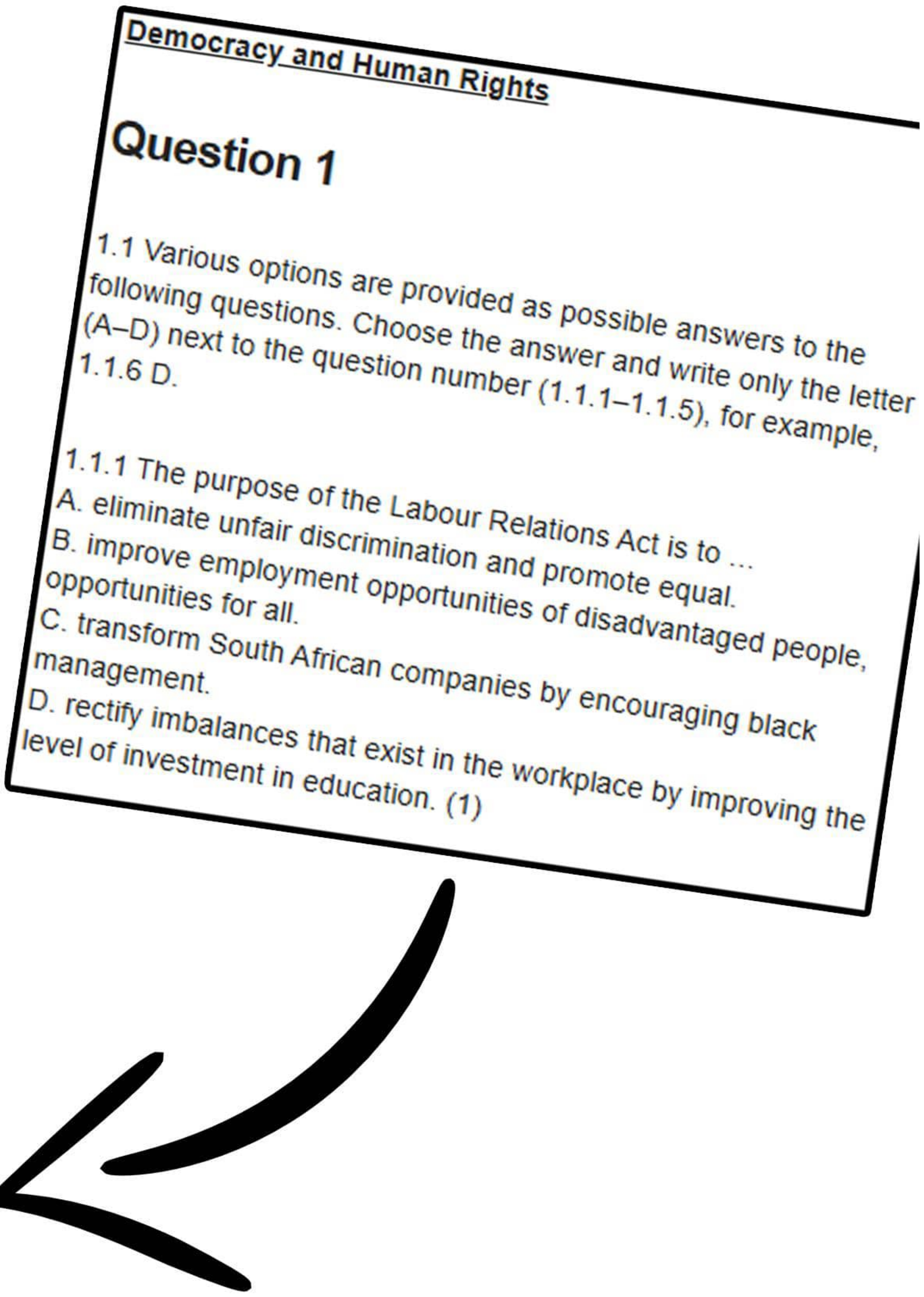
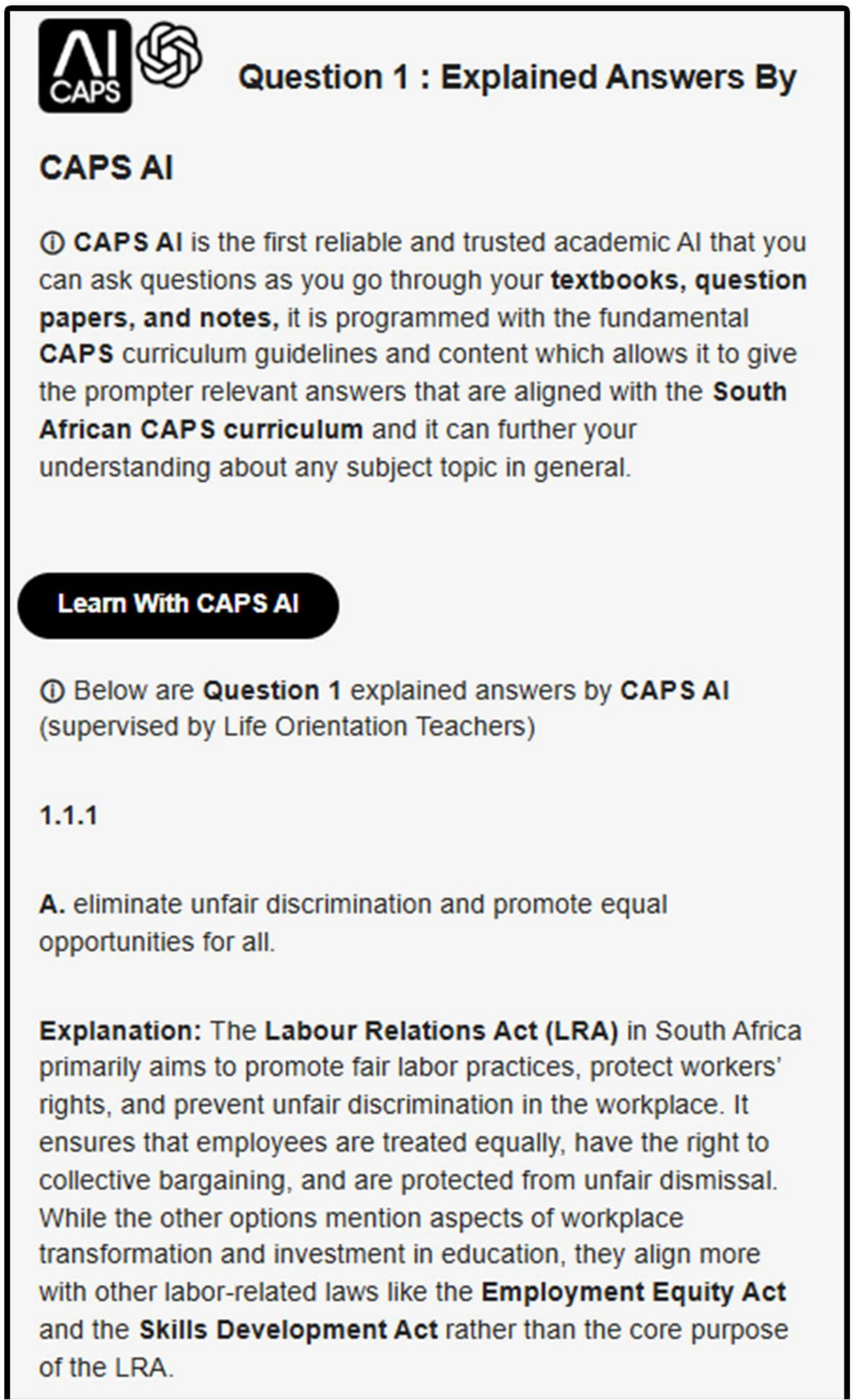


Democracy and Human Rights

Question 1

1.1 Various options are provided as possible answers to the following questions. Choose the answer and write only the letter (A–D) next to the question number (1.1.1–1.1.5), for example, 1.1.6 D.

1.1.1 The purpose of the Labour Relations Act is to …
A. eliminate unfair discrimination and promote equal opportunities for all.
B. improve employment opportunities of disadvantaged people.
C. transform South African companies by encouraging black management.
D. rectify imbalances that exist in the workplace by improving the level of investment in education. (1)



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1.1.1

A. eliminate unfair discrimination and promote equal opportunities for all.

Explanation: The **Labour Relations Act (LRA)** in South Africa primarily aims to promote fair labor practices, protect workers' rights, and prevent unfair discrimination in the workplace. It ensures that employees are treated equally, have the right to collective bargaining, and are protected from unfair dismissal. While the other options mention aspects of workplace transformation and investment in education, they align more with other labor-related laws like the **Employment Equity Act** and the **Skills Development Act** rather than the core purpose of the LRA.

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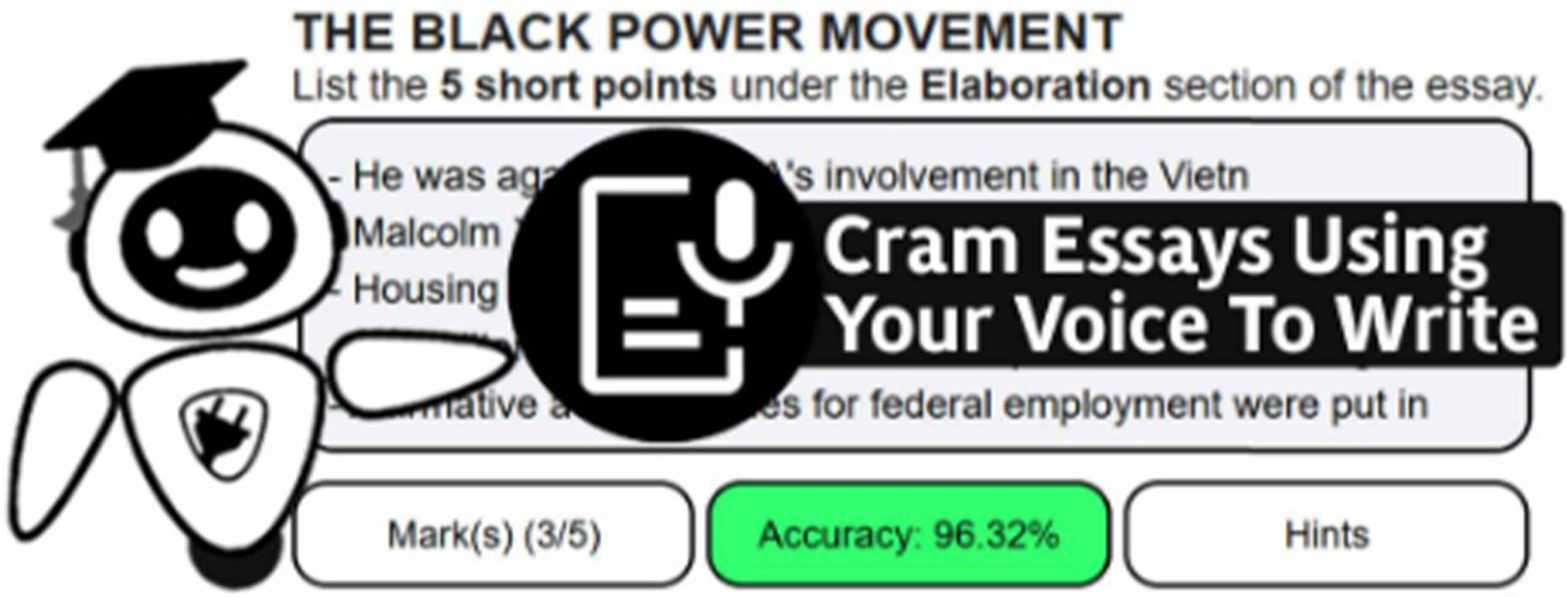

THE BLACK POWER MOVEMENT

List the **5 short points** under the **Elaboration** section of the essay.

- He was ag A's involvement in the Vietn
- Malcolm
- Housing
- mative a es for federal employment were put in

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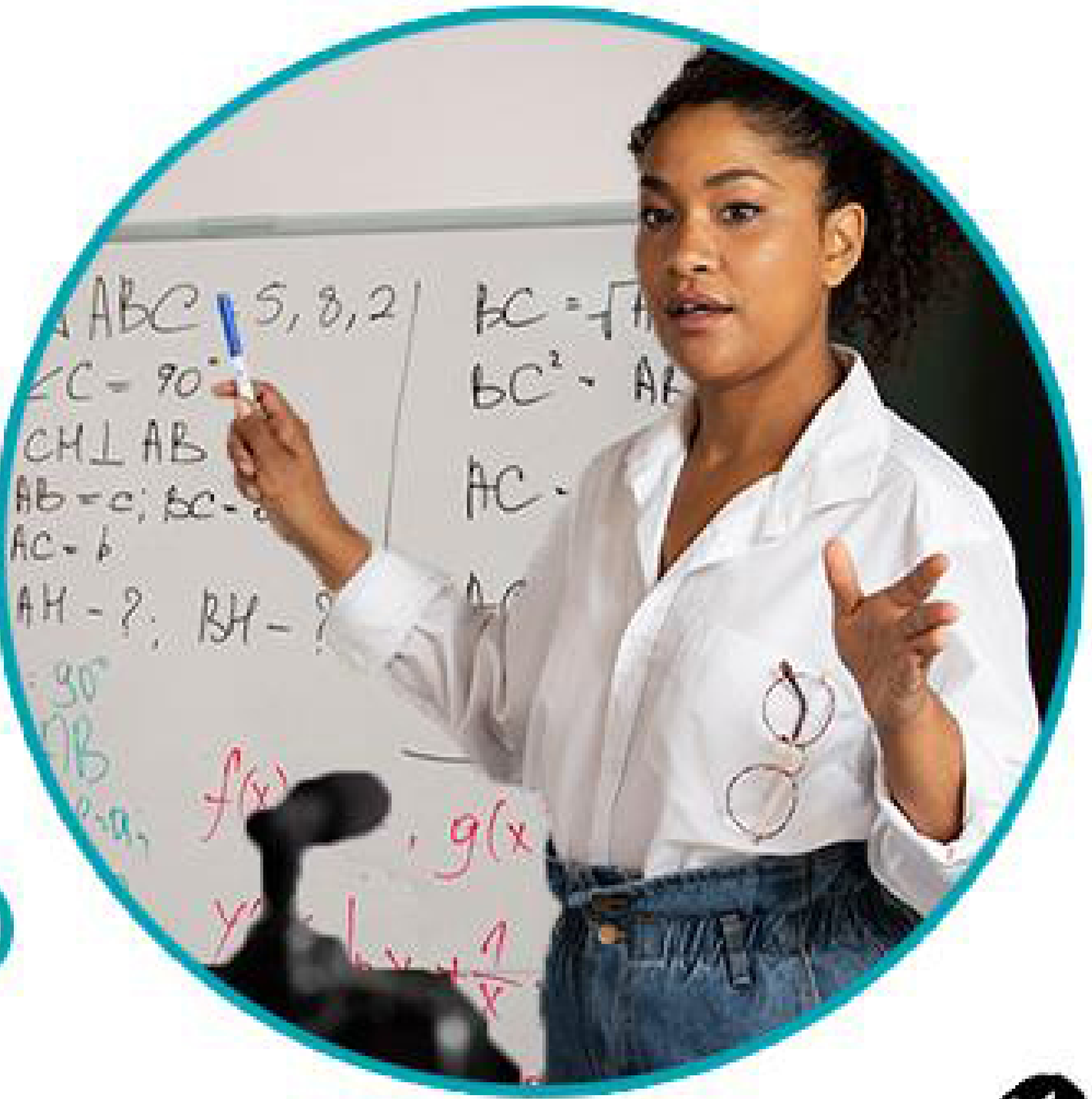

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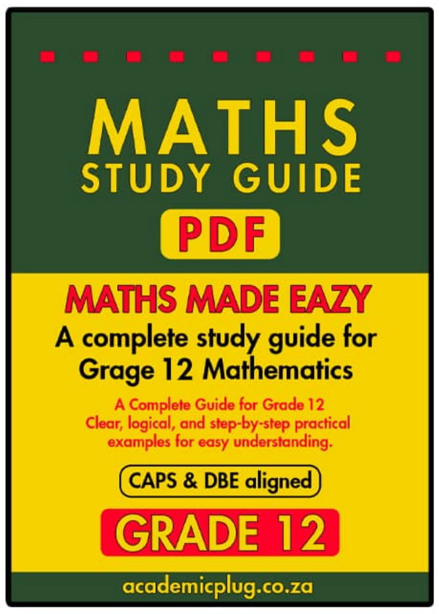
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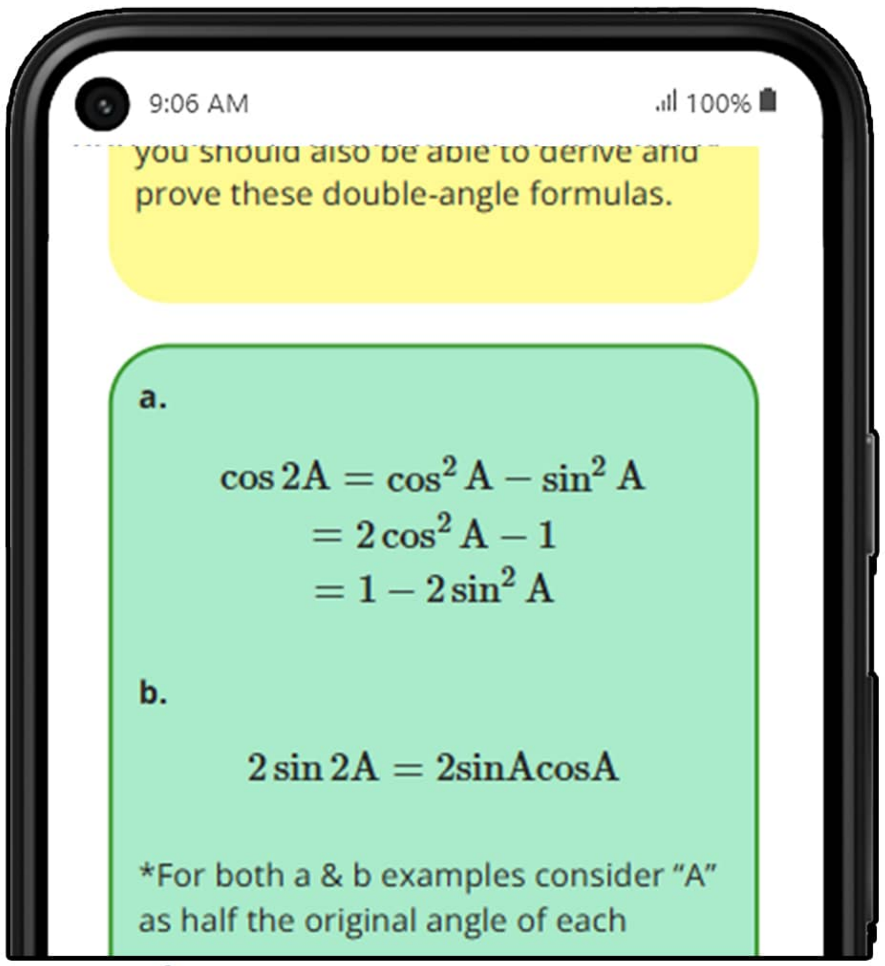

you should also be able to derive and prove these double-angle formulas.

a.

$$\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A\\ &= 2\cos^2 A - 1\\ &= 1 - 2\sin^2 A\end{aligned}$$

b.

$$2\sin 2A = 2\sin A\cos A$$

*For both a & b examples consider "A" as half the original angle of each

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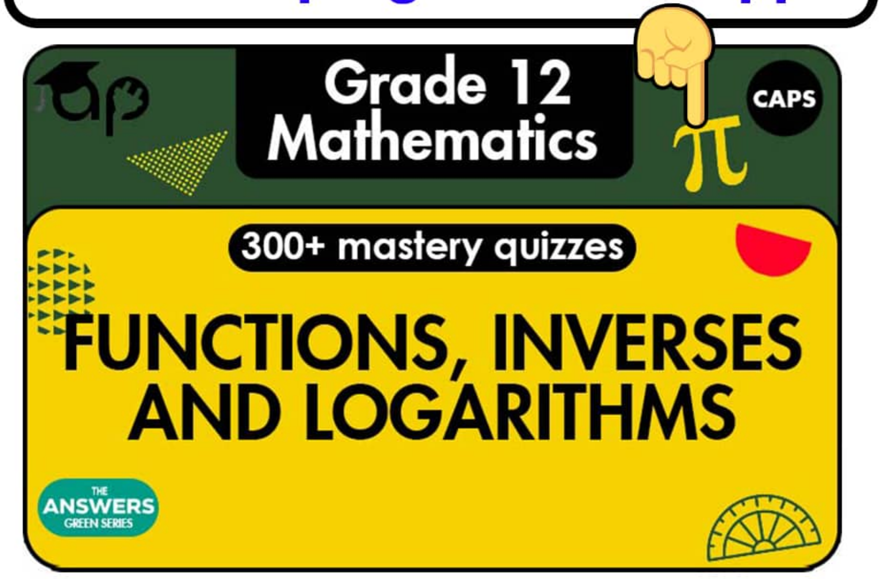
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CHAPTER 2
FUNCTIONS, INVERSES AND LOGARITHMS

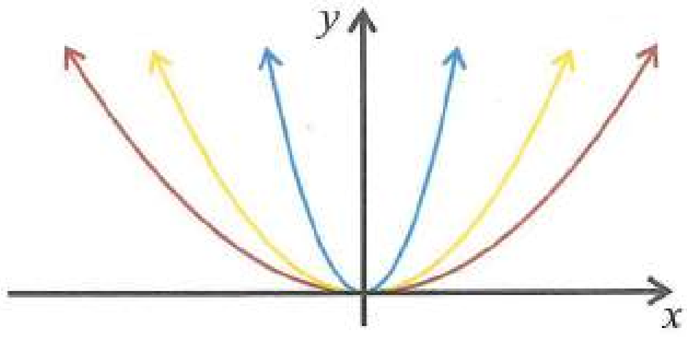
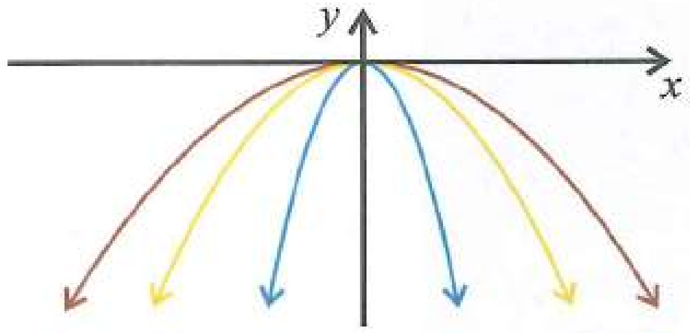
WHAT YOU WILL LEARN IN THIS CHAPTER

PAGE	CONTENT	CAPS	ASSESSMENTS YOU WILL BE ABLE TO:
24	**A. Revision and Summary of Grade 11 Functions**	1	• Introduce a more formal definition of a function and extend Grade 11 work on the relationships between variables in terms of numerical, graphical, verbal and symbolic representations of functions and convert flexibly between these representations. • Work with the inverses of prescribed functions and be aware of the fact that, in the case of many-to-one functions, the domain has to be restricted if the inverse is to be a function. • Deal with problem solving and graph work involving the prescribed functions (including the logarithmic function). • Demonstrate an understanding of the definition of a logarithm and any laws needed to solve real life problems.
24	• The Parabola	4	
25	• The Hyperbola		
26	• The Exponential Function		
27	**B. Functions**		
27	• One-to-One Functions		
27	• Many-to-One Functions		
27	• Vertical Line Test		
27	• Horizontal Line Test		
29	**C. Inverse Graphs**		
29	• Notation		
34	• Restricting the Domain and Range		
37	**D. The Logarithmic Function and Laws of Logarithms**		
38	• The Logarithmic Function		
38	• Steps to Sketching the Logarithmic Function		
43	• Laws of Logarithms		

A. REVISION AND SUMMARY OF GRADE 11 FUNCTIONS

THE PARABOLA

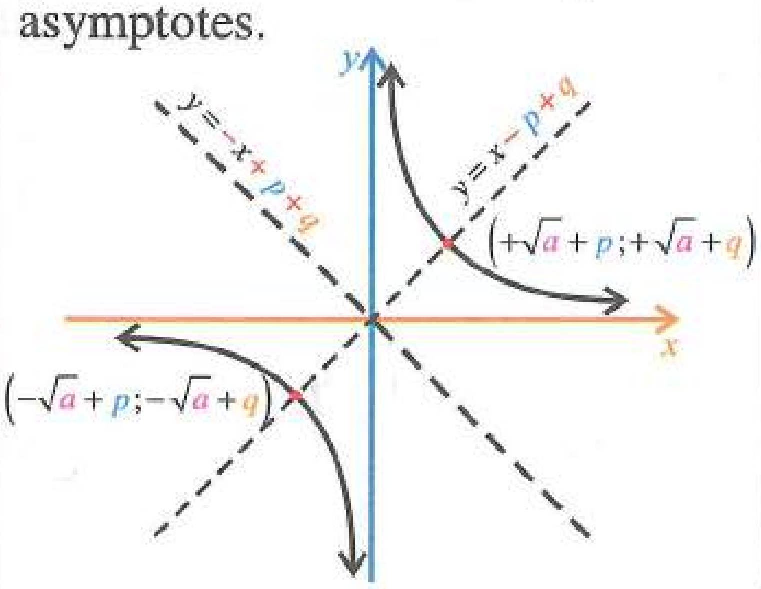
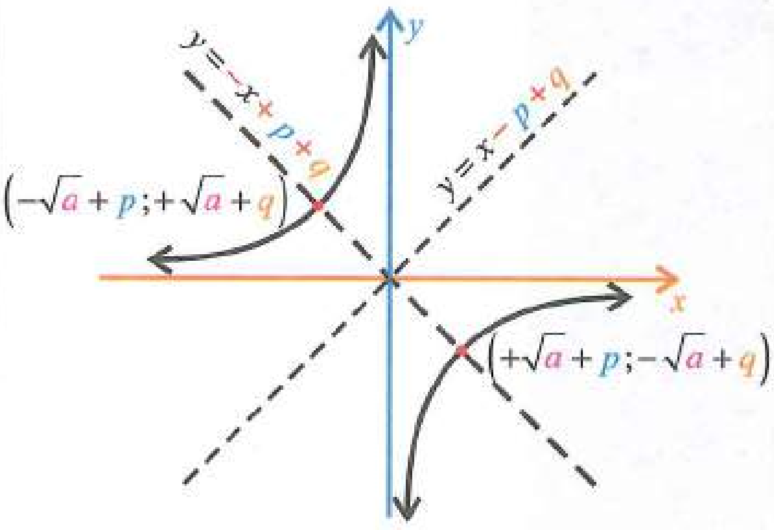
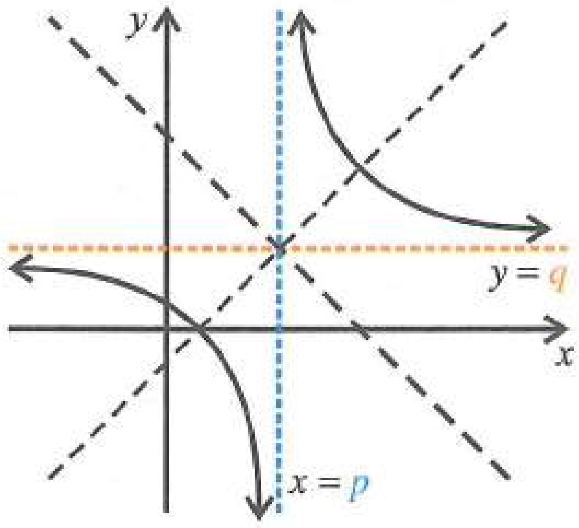
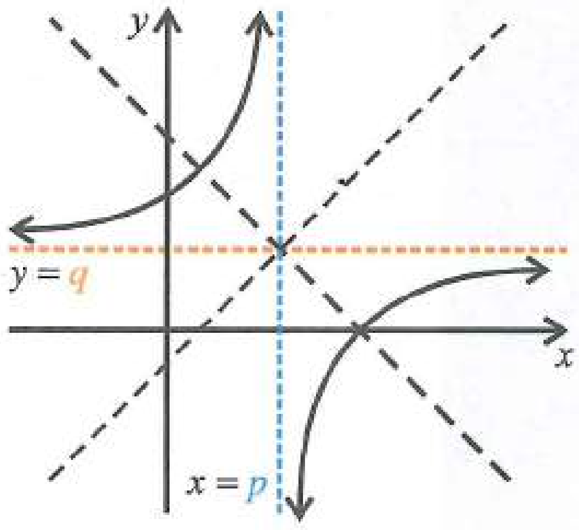
In Grade 11 you were introduced to the parabola in the forms shown in the table below. The table below summarises the properties of the parabola which you learnt in Grade 11.

<table>
<tr><th rowspan="2">Forms</th><td colspan="2">$y=a(x-p)^2+q$</td><td colspan="2">$y=ax^2+bx+c$</td><td colspan="2">$y=a(x-\mathbf{R}_1)(x-\mathbf{R}_2)$</td></tr>
<tr><td colspan="2">In this form p is the x-value of the turning point and q is y-value of the turning point. The parabola is symmetrical about its turning point.</td><td colspan="2">In this form c is the y-intercept. The value of b is responsible for horizontal translations. The parabola is symmetrical about its turning point. The x-value of the turning point and axis of symmetry can be found using the equation $x=\frac{-b}{2a}$.</td><td colspan="2">In this form R_1 and R_2 are the x-intercepts of the parabola. Remember that the x-value of the turning point of a parabola lies halfway between the x-intercepts. This is due to the symmetry of the parabola about its turning point.</td></tr>
<tr><th></th><td colspan="3">$a>0$</td><td colspan="3">$a<0$</td></tr>
<tr><th rowspan="2">Shape
(a)</th><td colspan="3">Concave up (happy face) ☺</td><td colspan="3">Concave down (sad face) ☹</td></tr>
<tr><td colspan="3">As the value of a increases from 0 the graph is compressed and gets narrower. This is demonstrated in the graphs below where a is increased from 0,1 to 0,2 and then to 1.
 </td><td colspan="3">As the value of a decreases from 0 the graph is compressed and gets narrower. This is demonstrated in the graphs below where a is decreased from –0,1 to –0,2 and then to –1.
 </td></tr>
<tr><th>Vertical Shift
(q or c)</th><td colspan="6">The value of q or c results in a vertical shift. If the value of q or c is greater than 0 the graph will be shifted up. If q or c is less than 0 the graph will be shifted down.</td></tr>
<tr><th>Domain and Range</th><td colspan="6">The domain of a parabola is:
$x\in\mathbb{R} \rightarrow$ also written as $x\in(-\infty;\infty)$
The range of a parabola is:
• $y\geq q$ if $a>0 \rightarrow$ also written as $y\in[q;\infty)$
• $y\leq q$ if $a<0 \rightarrow$ also written as $y\in(-\infty;q]$</td></tr>
</table>

THE HYPERBOLA

In Grade 11 you were introduced to the hyperbola in the form of $y=\dfrac{a}{x-p}+q$.

It was shown that a determines the shape of a hyperbola as well as the quadrants which the hyperbola lies in. The value of q resulted in a vertical shift (up or down) and the line $y=q$ is the horizontal asymptote. The value of p resulted in a horizontal shift (left or right) and the line $x=p$ is the vertical asymptote. The properties of a hyperbola that you learnt in Grade 11 are summarised below:

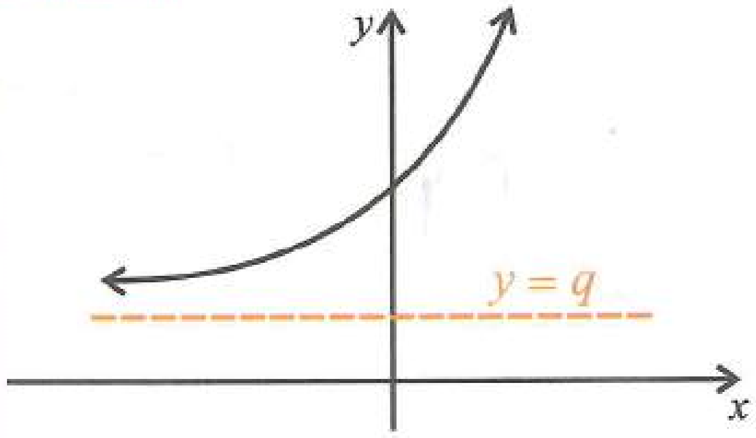
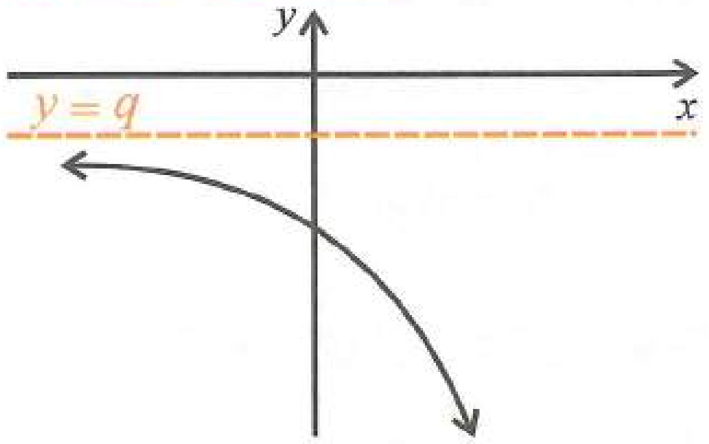
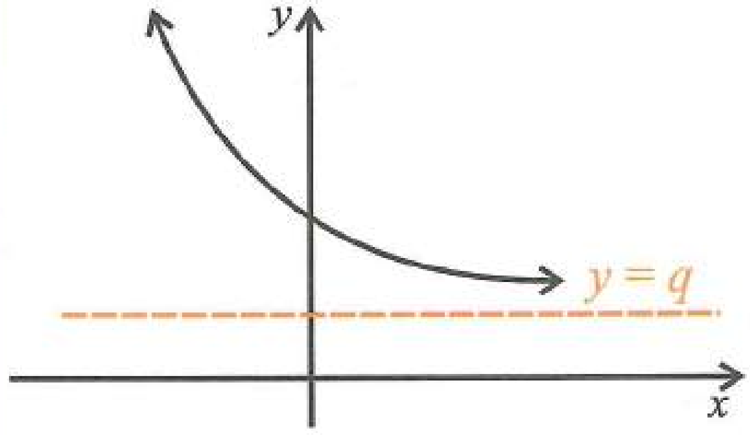
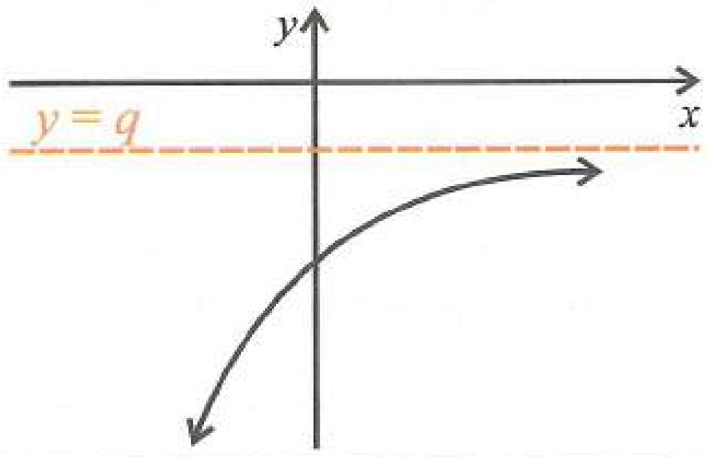
<table>
<tr><th></th><th>$a>0$</th><th>$a<0$</th></tr>
<tr><td rowspan="4">Shape
(a)</td><td>The graph lies in the 1st and 3rd quadrants formed by the asymptotes.

</td><td>The graph lies in the 2nd and 4th quadrants formed by the asymptotes.

</td></tr>
<tr><td colspan="2">Note that in the above graphs p and q are equal to 0. The graphs below shows examples where p and q are greater than 0.</td></tr>
<tr><td>
</td><td>
</td></tr>
<tr></tr>
<tr><td>Axes of Symmetry</td><td colspan="2">The hyperbola is symmetrical about the lines:
$y=x-p+q$ and $y=-x+p+q$.</td></tr>
<tr><td rowspan="2">Points of intersection with the Axes of Symmetry</td><td>If $a>0$ the axis of symmetry $y=x-p+q$ intersects the hyperbola at the points :
• $\left(+\sqrt{a}+p;\sqrt{a}+q\right)$
• $\left(-\sqrt{a}+p;-\sqrt{a}+q\right)$</td><td>If $a<0$ the axis of symmetry $y=-x+p+q$ intersects the hyperbola at the points :
• $\left(-\sqrt{a}+p;+\sqrt{a}+q\right)$
• $\left(+\sqrt{a}+p;-\sqrt{a}+q\right)$</td></tr>
<tr><td colspan="2">The sign of a is not taken into account when calculating the points of intersection.</td></tr>
<tr><td>Domain and Range</td><td colspan="2">The domain of a hyperbola is:
$x\in\mathbb{R},\ x\neq p \rightarrow$ also written as $x\in(-\infty;\infty),\ x\neq p$
The range of a hyperbola is:
$y\in\mathbb{R},\ x\neq q \rightarrow$ also written as $y\in(-\infty;\infty),\ y\neq q$</td></tr>
</table>

THE EXPONENTIAL FUNCTION

In Grade 11 you were introduced to the exponential function in the form of:

$$y = ab^{x-p} + q \text{ where } b > 0 \text{ and } b \neq 0$$

It was shown that a determines the shape of the graph as well as on which side of the asymptote the exponential function lies. Together with a, the value of b determines whether the function is increasing or decreasing.
The value of q resulted in a vertical translation (up or down) and the line $y = q$ is the horizontal asymptote. The value of p resulted in a horizontal translation. The properties of the exponential function that you learnt in Grade 11 are summarised below:

<table>
<tr><td rowspan="4">Shape
(<i>a</i> and <i>b</i>)</td><td>$a > 0$ and $b > 1$</td><td>$a < 0$ and $b > 1$</td></tr>
<tr><td>The graph lies above the horizontal asymptote and is an increasing function.
 </td><td>The graph lies below the horizontal asymptote and is a decreasing function.
 </td></tr>
<tr><td>$a > 0$ and $0 < b < 1$</td><td>$a < 0$ and $0 < b < 1$</td></tr>
<tr><td>The graph lies above the horizontal asymptote and is a decreasing function.
 </td><td>The graph lies below the horizontal asymptote and is an increasing function.
 </td></tr>
<tr><td>Domain and Range</td><td colspan="2">The domain of an exponential function is:
$x \in \mathbb{R} \rightarrow$ also written as $x \in (-\infty ; \infty)$
The range of an exponential function is:
• $y > q$ if $a > 0 \rightarrow$ also written as $y \in (q ; \infty)$
• $y < q$ if $a < 0 \rightarrow$ also written as $y \in (-\infty ; q)$</td></tr>
</table>

B. FUNCTIONS

A function *f*, is defined as a relationship between values, where each input value maps to one output value.
In other words, for an equation to be called a function, there can only be one *y*-value for a particular *x*-value.

There are two types of functions:

1. **One-to-One Functions**
2. **Many-to-One Functions**

ONE-TO-ONE FUNCTIONS

A one-to-one function is a function where there is a single *y*-value for a particular *x*-value.

MANY-TO-ONE FUNCTIONS

A function cannot have more than one *y* value to each *x* value. However, a function can have more than one *x* value for a particular *y* value. These are known as many to-one functions.

VERTICAL LINE TEST

To test if a graph is a function, use the vertical line test. If a vertical line (a line parallel to the *y*-axis) touches the graph more than once at any point, the graph is not a function. You don't have to draw a line, just hold a ruler parallel to the *y*-axis and move it along the graph. If the ruler touches the graph more than once for a single *x* value, anywhere on the graph, then the graph is not a function. In the case of the graph not being a function, it is said to be a relation.

HORIZONTAL LINE TEST

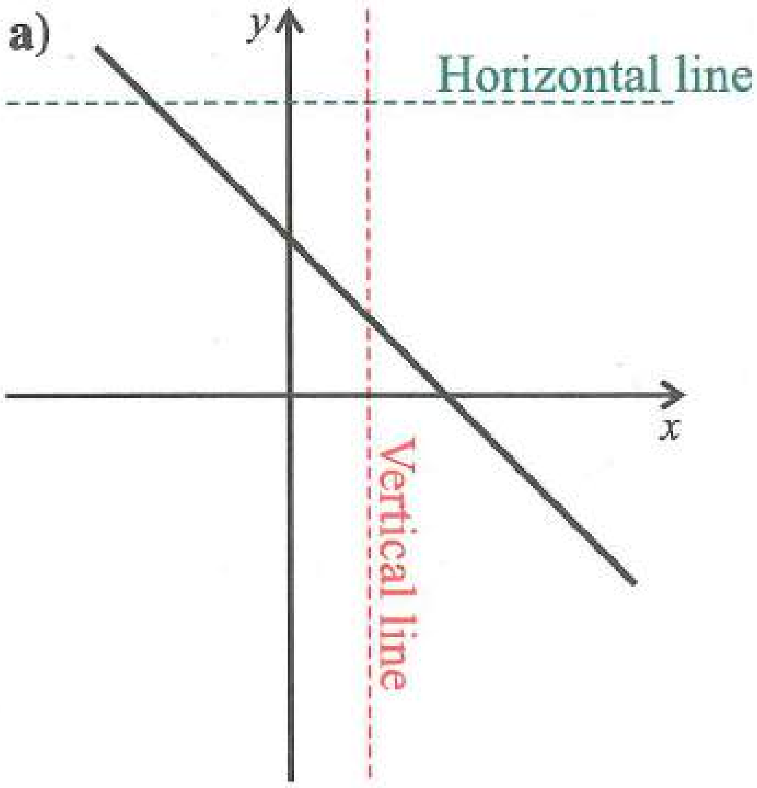
If a graph passes the vertical line test, it is a function. **The horizontal line test can be used to determine what type of function the graph represents.**

If a horizontal line (a line parallel to the *x*-axis) is drawn and moved along the graph and it **touches the graph more than once** at any point, it is a **many-to-one function** (many *x*-values to a single *y*-value). Otherwise, it is a one-to-one function.

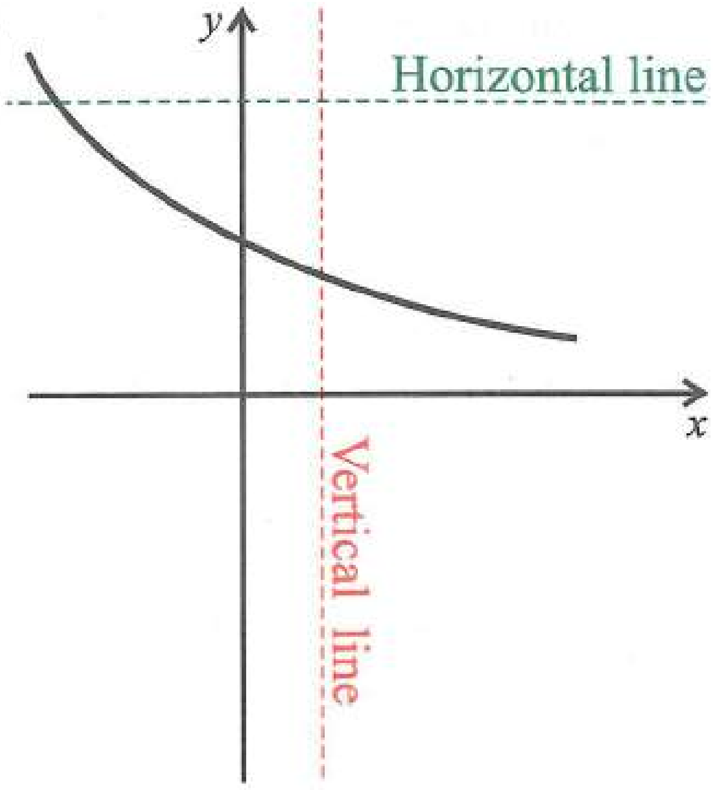
The horizontal line test does not determine whether a graph is a function, only the type of function. The vertical line test must always be used first. Only if this test shows that the graph is a function can the horizontal line test be used to determine the type of function. If a graph is not a function it is known as a relation.

Example: **Determine if the following graphs are functions. If the graph is a function, state what type of function it is. Give reasons for all of your answers.**

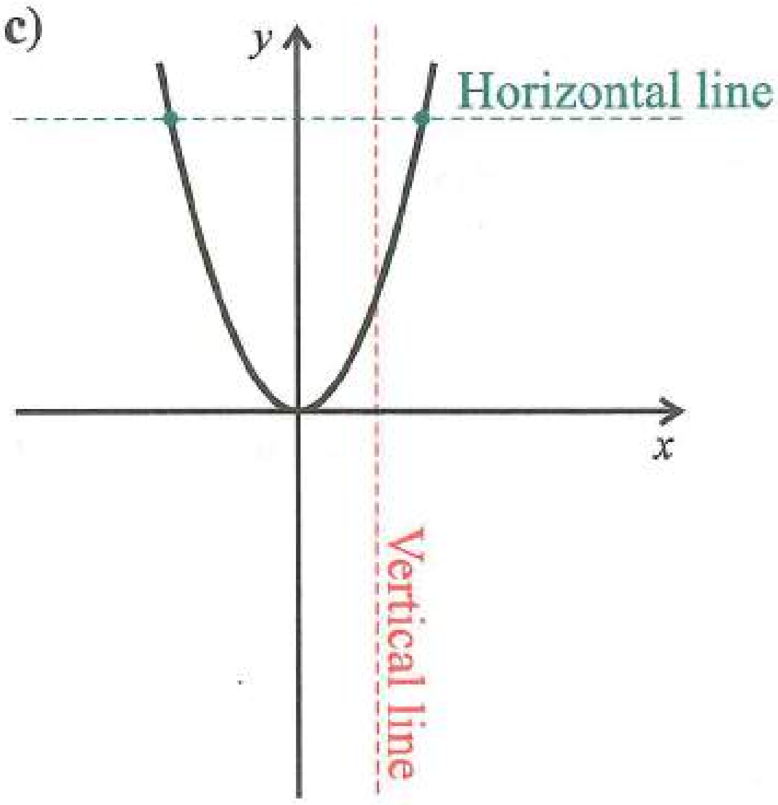
a)



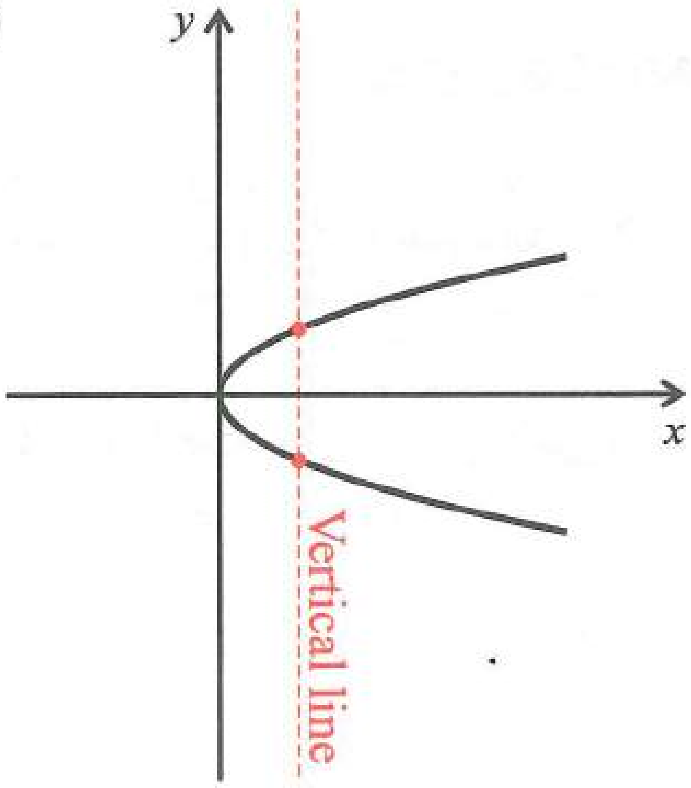
b)



c)



d)



a) One-to-One function: There is one y-value for any x-value (vertical and horizontal lines only touch the graph once).

b) One-to-One function: There is one y-value for any x-value (vertical and horizontal lines only touch the graph once).

c) Many-to-One function: There is more than one x-value for the same y-value (vertical line cuts the graph once for every x-value but a horizontal line cuts the graph more than once for different y-values).

d) Not a function: There is more than one y-value for a single x-value (a vertical line cuts the graph more than once at a single x-value). This means that the graph is a one-to-many relation.

Complete Exercise 1 on page 44.

C. INVERSE GRAPHS

In order to determine the inverse of a graph, swop every x and y-coordinate with each other, so if there is a point (1;5) the inverse would be (5;1). This is done for every point on the graph. In order to express this in the form of an equation, follow the same principle and swop the positions of x and y.

This is the same as a reflection about the line $y = x$.

Example 1: Determine the inverse function of $y = -3x + 4$.

$y = -3x + 4$

In order to find the inverse of a function, swap the x and y values with each other.

$\therefore$ The inverse is: $x = -3y + 4 \rightarrow$ never leave the equation in this form

$\therefore\ -3y = x - 4$

$\therefore\ y = -\frac{1}{3}x + \frac{4}{3} \rightarrow$ always put the equation in the form of $y = \ldots$

NOTATION

In most cases, the equation of a graph is given in the form of $f(x) = \ldots$
The general way of expressing the inverse is in the form of $f^{-1}(x)$.
Therefore, the inverse function of $f(x)$ is $f^{-1}(x)$.

Example 2: Determine the inverse function of $f(x) = -3x + 4$ and write it in the form of $f^{-1}(x) = \ldots$

$f(x) = -3x + 4$

$\therefore\ y = -3x + 4$

In order to find the inverse of a function, swop the x and y-values with each other.

$\therefore$ The inverse is: $x = -3y + 4 \rightarrow$ never leave the equation in this form

$\therefore\ -3y = x - 4$

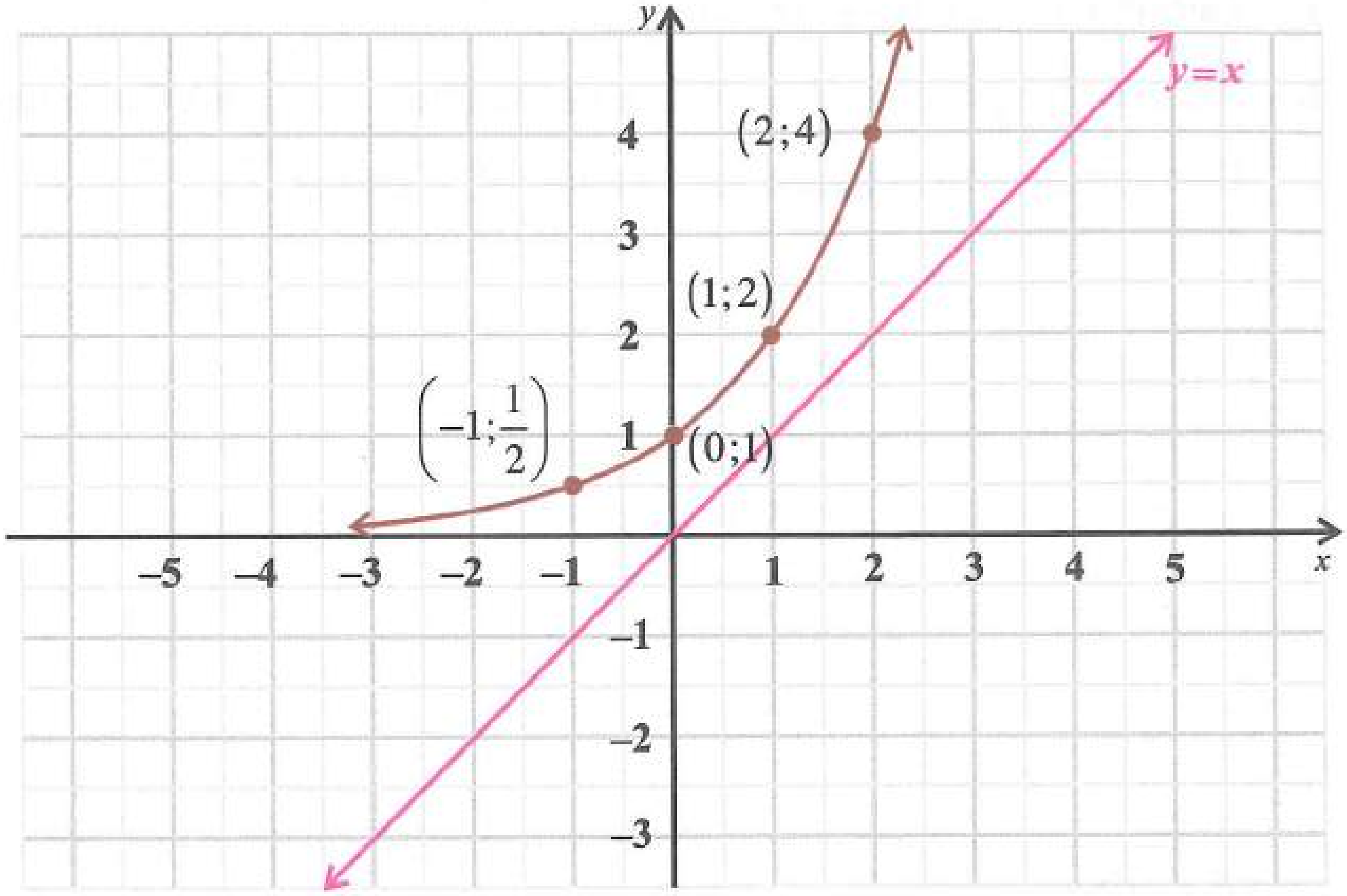
$\therefore\ y = -\frac{1}{3}x + \frac{4}{3} \rightarrow$ now write the equation in the form $f^{-1}(x) = \ldots$

$\therefore\ f^{-1}(x) = -\frac{1}{3}x + \frac{4}{3}$

The notation $f^{-1}(x) = \ldots$ can only be used to represent the inverse of a one-to-one function and must not be used for inverses of many-to-one functions. This is because the inverse of a many-to-one function is not a function but a one-to-many relation. This will be discussed in more detail further on in this chapter.

The inverse function $f^{-1}(x)$ is not the same as the reciprocal $\frac{1}{f(x)}$.

Example 3: The graph of $f(x) = 2^x$ and $y = x$ are sketched below:

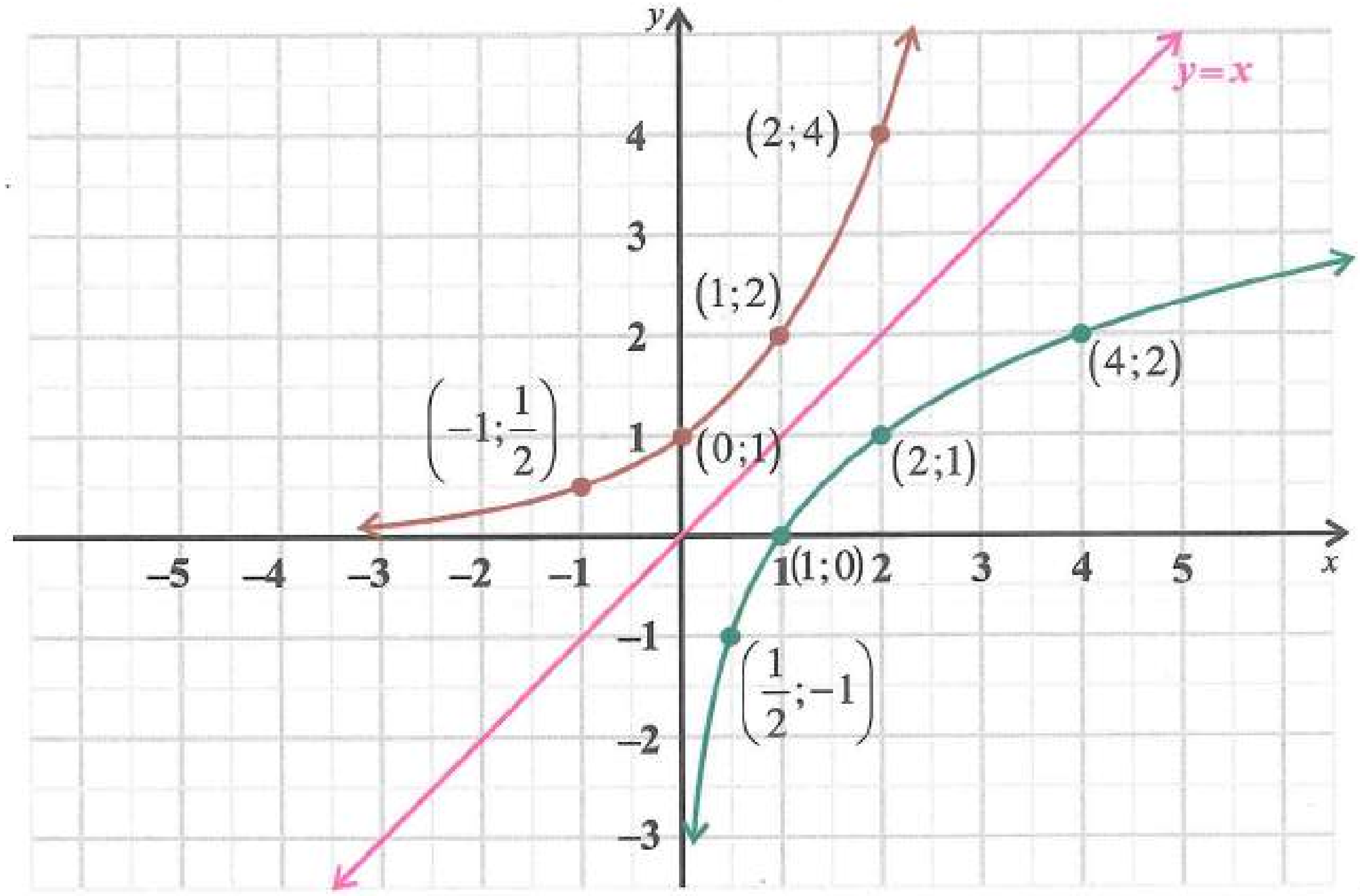


a) Without doing any calculations sketch the graph of $f^{-1}(x)$.

b) State the domain and range of $f(x)$.

c) State the domain and range of $f^{-1}(x)$.

a) In order to sketch the graph of $f^{-1}(x)$ every x and y-value has to be swopped. This is shown in the solution below. It can clearly be seen that the inverse graph is a reflection about the line $y = x$.



b) Domain: $x \in \mathbb{R} \rightarrow$ this is also written as $x \in (-\infty;\infty)$

Range: $y > 0 \rightarrow$ this is also written as $y \in (0;\infty)$

c) Domain: $x > 0 \rightarrow$ this is also written as $x \in (0;\infty)$

Range: $y \in \mathbb{R} \rightarrow$ this is also written as $y \in (-\infty;\infty)$

It is important to note that the domain and range of the inverse graph also swop (i.e. are opposite to the original).

Example 4: Given the function f, where $f(x)=3x-1$:

- **a) Determine the inverse function f^{-1}.**
- **b) Sketch the graphs of f, f^{-1} and the line $y=x$ on the same set of axes**
- **c) Determine the domain and range of f and f^{-1}.**

a) $f(x)=3x-1$

$\therefore y=3x-1 \rightarrow$ swop the x and y-values to obtain the inverse

$\therefore$ The inverse is: $x=3y-1 \rightarrow$ never leave the equation in this form

$\therefore 3y=x+1 \rightarrow$ divide both sides by 3 to make y the subject of the formula

$$\therefore y=\frac{1}{3}x+\frac{1}{3}$$

$$\therefore f^{-1}(x)=\frac{1}{3}x+\frac{1}{3}$$

b) Using the table method:

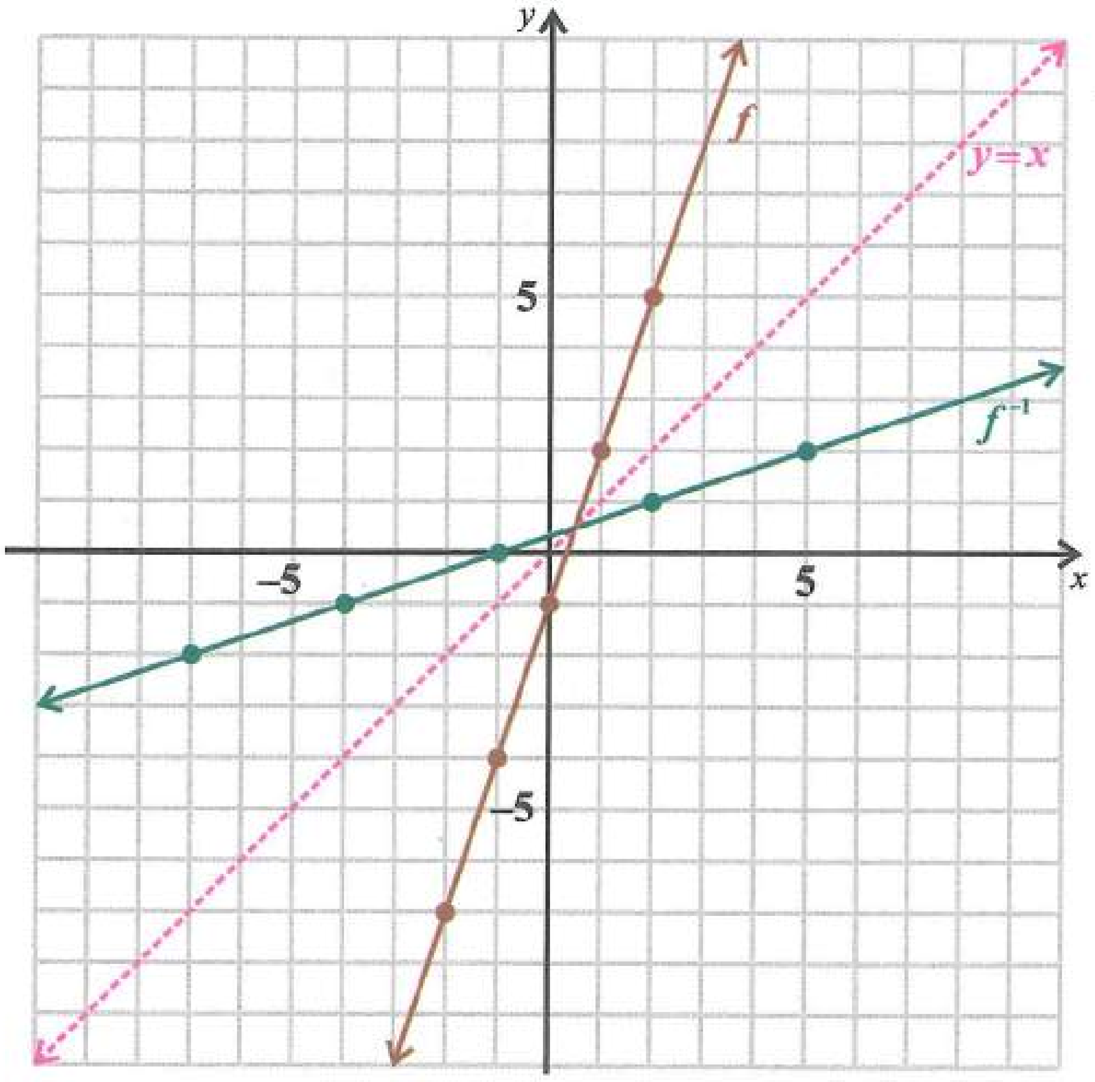
$f(x)$

x	-2	-1	0	1	2
y	-7	-4	-1	2	5

In order to sketch the inverse draw up a new table with all of the x and y-values swopped.

$f^{-1}(x)$

x	-7	-4	-1	2	5
y	-2	-1	0	1	2



c) $f(x)$

Domain: $x \in \mathbb{R}$ and Range: $y \in \mathbb{R}$

$f^{-1}(x)$

Domain: $x \in \mathbb{R}$ and Range: $y \in \mathbb{R}$

Example 5: **Given the function *f*, where $f(x) = 2x^2$:**

a) **What type of function is $f(x)$?**

b) **Is the inverse of *f* a function? Give a reason for your answer.**

c) **Determine the inverse of *f*.**

d) **Sketch the graphs of *f*, its inverse and the line $y = x$ on the same set of axes.**

e) **Determine the domain and range of *f* and its inverse.**

a) The graph of f is a parabola, which is a many to one function.

b) No. The inverse of a many-to-one function is a one-to-many relation.

c) $f(x) = 2x^2$

$\therefore y = 2x^2 \rightarrow$ swop the x and y-values to obtain the inverse

$\therefore$ The inverse is: $x = 2y^2 \rightarrow$ never leave the equation in this form

$\therefore y^2 = \dfrac{x}{2} \rightarrow$ square root both sides

$\therefore y = \pm\sqrt{\dfrac{x}{2}}\,;\, x \geq 0 \rightarrow$ remember the $\pm$ when square rooting and put in restrictions

The notation $f^{-1}(x) = \ldots$ cannot be used to write the inverse of a parabola, as the inverse is not a function.

d) Using the table method:

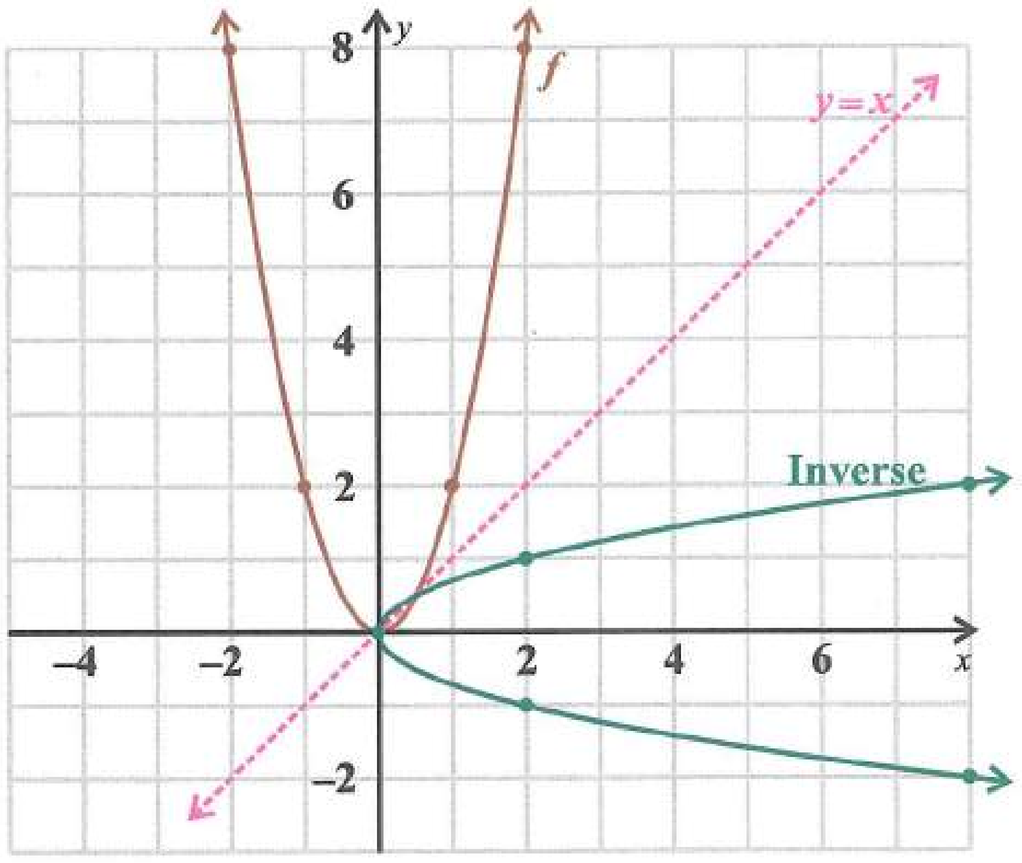
$f(x)$:

x	−2	−1	0	1	2
y	8	2	0	2	8

In order to sketch the inverse draw up a new table with all of the *x* and *y*-values swopped.

Inverse:

x	8	2	0	2	8
y	−2	−1	0	1	2



e) Domain of f: $x \in \mathbb{R}$

Range of f: $y \in [0;\infty)$

Domain of the inverse: $x \in [0;\infty)$

Range of the inverse: $y \in \mathbb{R}$

Example 6: The graph of $f(x)=b^x+q$ has an asymptote of $y=-2$ and passes through the point $(3;6)$.

a) Determine the values of b and q.
b) Determine the coordinates of the x and y-intercepts.
c) Determine the equation of the asymptote of $f^{-1}(x)$.
d) Determine the coordinates of the x and y-intercepts of $f^{-1}(x)$.

a) $q=-2 \rightarrow$ the line $y=q$ is the equation of the asymptote

$\therefore f(x)=b^x-2$

sub $(3;6)$

$\therefore 6=b^3-2$

$\therefore 8=b^3 \rightarrow$ cube root both sides

$\therefore b=2$

b) x-int: let $y=0$

$\therefore 0=2^x-2$

$\therefore 2^1=2^x$

$\therefore x=1$

$\therefore x\text{-int}:(1;0)$

y-int: let $x=0$

$\therefore y=2^0-2$

$\therefore y=1-2 \rightarrow$ any number raised to the power of 0 is equal to 1

$\therefore y=-1$

$\therefore y\text{-int}:(0;-1)$

c) The asymptote of the inverse is: $x=-2 \rightarrow$ the x and y values of the inverse swop

d) $x\text{-int}:(-1;0)$

$y\text{-int}:(0;1)$

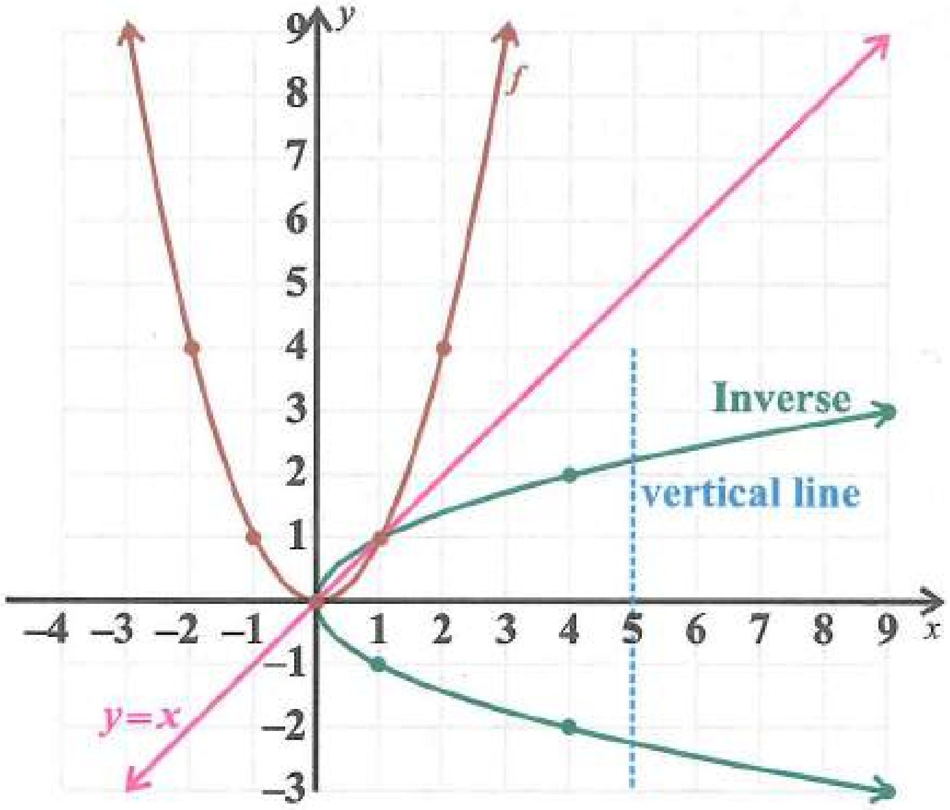
RESTRICTING THE DOMAIN AND RANGE

You have seen that if a function is a many-to-one function (many x-values to a single y-value) the inverse will be a one-to-many relation (many y-values to a single x-value).

This means that the inverse of any many-to-one function will never be a function. However, the domain of a many-to-one function can be restricted so that its inverse is a function.

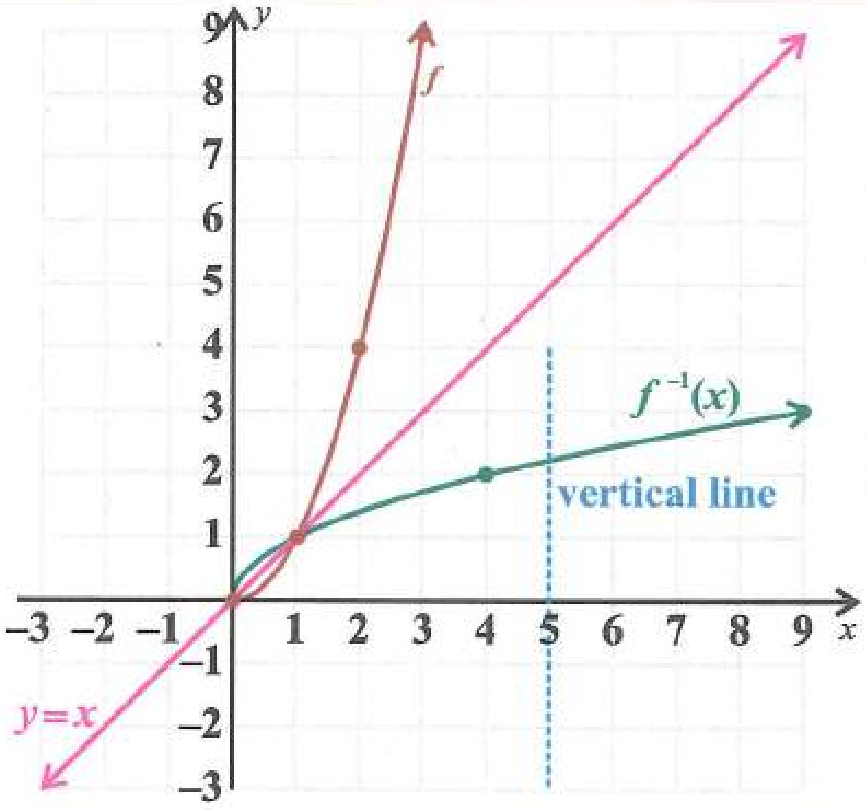
Only the inverse of a one-to-one function is itself a function.

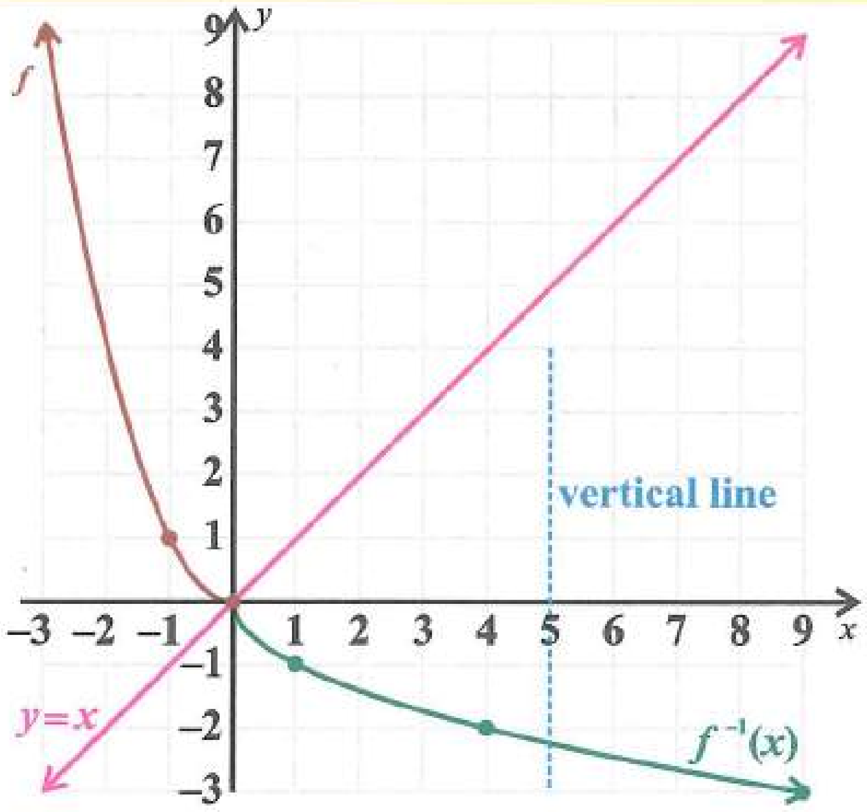
Illustrative Example: $f(x)=x^2$, $y=x$ **and the inverse of f are sketched below:**



As you have seen previously and from the vertical line test, the inverse of f is not a function. In order for the inverse to be a function, the domain of the original function (f) has to be restricted so that one arm of the parabola is 'cut off '. In order to do this, you have to restrict the domain from the turning point. Therefore, there are two different restriction that can be placed on the domain so that the inverse is a function.

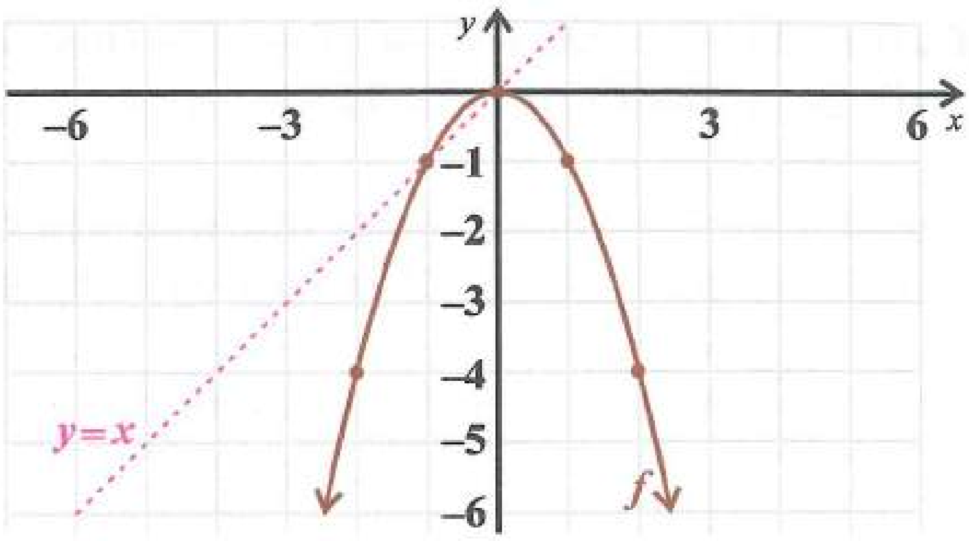
The restriction is either: $x\le 0$ or $x\ge 0 \rightarrow$ also written as $x\in(-\infty;0]$ or $x\in[0;\infty)$.





Domain of f:	Domain of f^{-1}:	Domain of f:	Domain of f^{-1}:
$x\ge 0\rightarrow x\in[0;\infty)$	$x\ge 0\rightarrow x\in[0;\infty)$	$x\le 0\rightarrow x\in(-\infty;0]$	$x\ge 0\rightarrow x\in[0;\infty)$
Range of f:	Range of f^{-1}:	Range of f:	Range of f^{-1}:
$y\ge 0\rightarrow y\in[0;\infty)$	$y\ge 0\rightarrow y\in[0;\infty)$	$y\ge 0\rightarrow y\in[0;\infty)$	$y\le 0\rightarrow y\in(-\infty;0]$

Example 1: **The graphs of $f(x)=-x^2$ and $y=x$ are sketched below:**



a) Determine the domain and range of *f*.

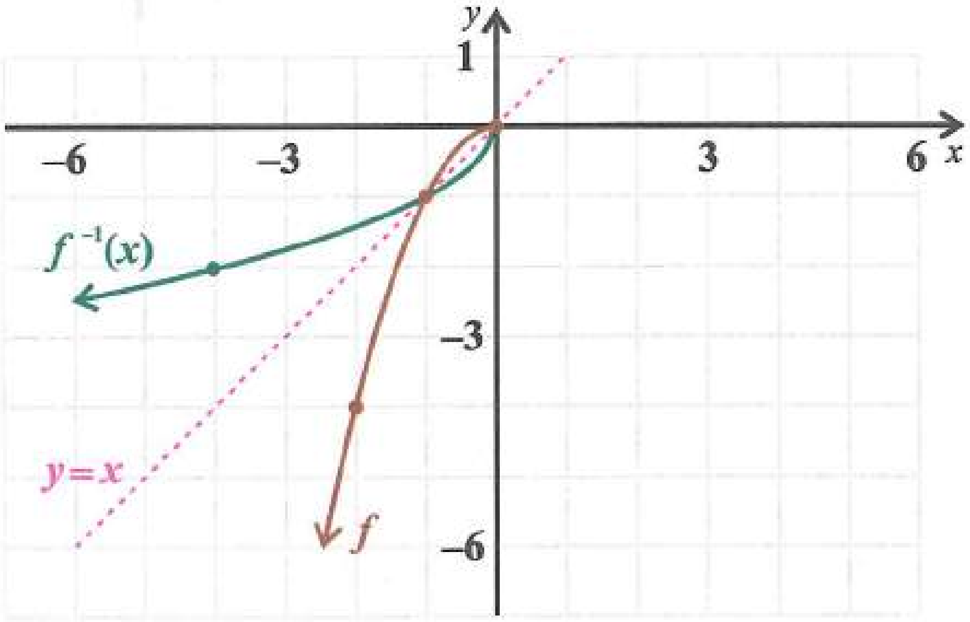
b) Restrict the domain of *f* so that its inverse is a function.

c) Draw a sketch graph of the inverse of *f* according to your restriction above. State the domain and range of the inverse function.

a) The domain is: $x\in\mathbb{R}$ and the range is: $y\in(-\infty;0]$.

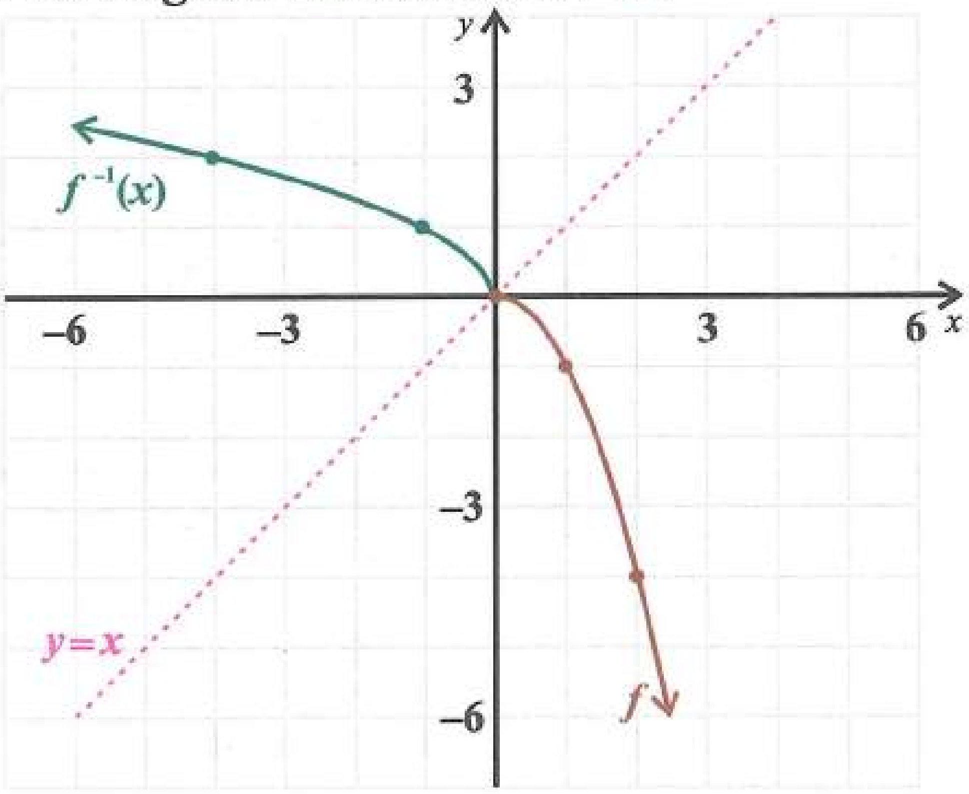
b) Either: $x\le 0$ or $x\ge 0 \rightarrow$ also written as $x\in(-\infty;0]$ or $x\in[0;\infty)$

c) If the restriction on the original function is $x\le 0$:



The domain of $f^{-1}(x)$ is: $x\in(-\infty;0]$ and the range is: $y\in(-\infty;0]$.

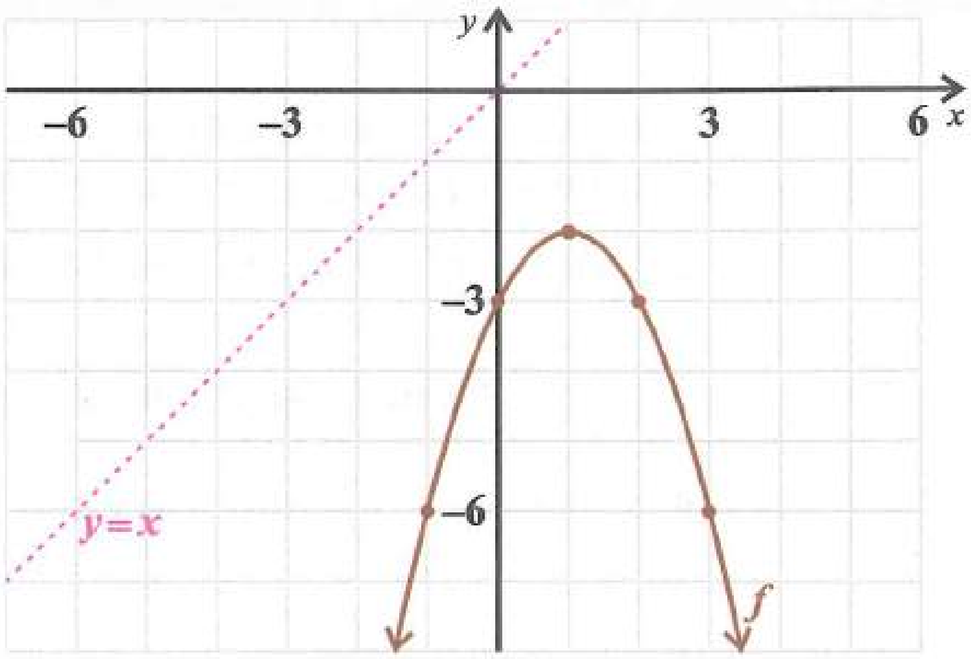
If the restriction on the original function is $x\ge 0$:



The domain of $f^{-1}(x)$ is: $x\in(-\infty;0]$ and the range is: $y\in[0;\infty)$.

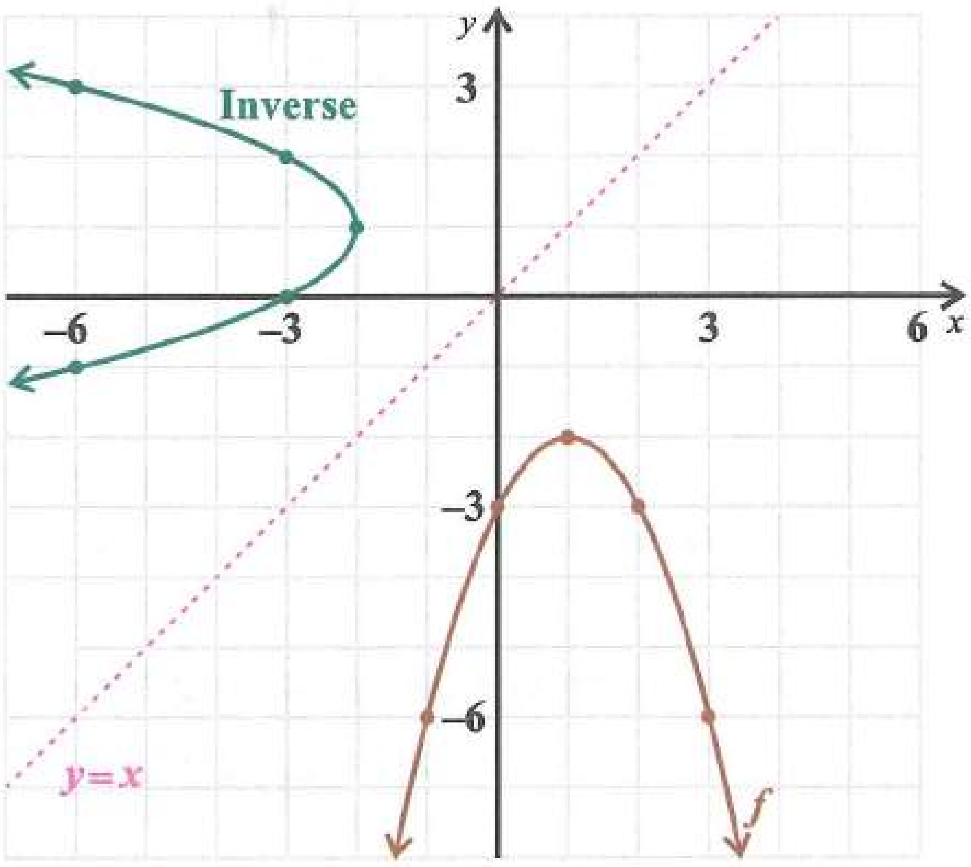
A restriction in the domain of the original function corresponds to a restriction in the range of the inverse.

Example 2: **The graphs of $f(x) = -(x-1)^2 - 2$ and $y = x$ are sketched below:**



- a) **Draw a sketch graph of the inverse of f.**
- b) **Determine the domain and range of f.**
- c) **Restrict the domain of f so that its inverse is a function.**

a)



b) The domain is: $x \in \mathbb{R}$ and the range is: $y \in (-\infty; -2]$.

c) In order for the inverse to be a function, the domain of the original function has to be restricted so that one arm of the parabola is 'cut off '. In order to do this to a parabola, you have to restrict the domain from the turning point.
$\therefore$ The restriction is either: $x \leq 1$ or $x \geq 1$

Complete Exercise 2 on pages 44 and 45.

D. THE LOGARITHMIC FUNCTION AND LAWS OF LOGARITHMS

A logarithm is a way of writing an exponential equation with the exponent as the subject of the equation. The definition of the logarithm is shown below:

In the equation $x = b^y$ where $b > 0$ and $b \neq 1$, b is referred to as the base and y is the exponent.

In general, converting from exponential form to logarithmic form is as follows:

$$x = b^y \text{ then } y = \log_b x \text{ where } b > 0 \text{ and } b \neq 1 \text{ and } x > 0.$$

Example 1: Write the following in exponential form:

a) $\log_2 32 = 5$ b) $\log_3 81 = 4$

c) $\log_a c = b$ d) $\log_{10} x = b$

a) $\log_2 32 = 5$

$\therefore 2^5 = 32$

b) $\log_3 81 = 4$

$\therefore 3^4 = 81$

c) $\log_a c = b$

$\therefore a^b = c$

d) $\log_{10} x = b$

$10^b = x$

If no base to the log is given you may assume it is 10. This means that example d) above, could also have been written as $\log x = b$.

Example 2: Solve for a in the following equations without the use of a calculator:

a) $\log_a 81 = 4$ b) $\log_3 9 = a$

a) $\log_a 81 = 4$

$\therefore a^4 = 81 \rightarrow$ take the $\sqrt[4]{}$ of both sides

$\therefore a = \pm 3 \rightarrow$ but $a > 0$

$\therefore a = 3$

b) $\log_3 9 = a$

$\therefore 3^a = 9 \rightarrow$ change 9 to a prime base

$\therefore 3^a = 3^2$

$\therefore a = 2$

Example 3: If $75(1{,}025)^{n-1} = 300$, solve for n correct to two decimal places.

$75(1{,}025)^{n-1} = 300 \rightarrow$ divide both sides by 75

$\therefore (1{,}025)^{n-1} = 4 \rightarrow$ use logs

$\therefore n - 1 = \log_{(1{,}025)}(4)$

$\therefore n = \log_{(1{,}025)}(4) + 1$

$\therefore n = 57{,}14$

Complete Exercise 3 on page 45.

THE LOGARITHMIC FUNCTION

The logarithmic function is the inverse of the exponential function. In order to determine the equation of the inverse of an exponential function you use the definition of a logarithm.

Since the exponential function is a one-to-one function, its inverse, the logarithmic function is also a one-to-one function.

Example 1: Given the function $y=b^x$, determine the equation of its inverse in the form $y = \ldots$

$y=b^x$

In order to determine the equation of the inverse swop the x and y-values:

$\therefore x=b^y \rightarrow$ write in logarithmic form

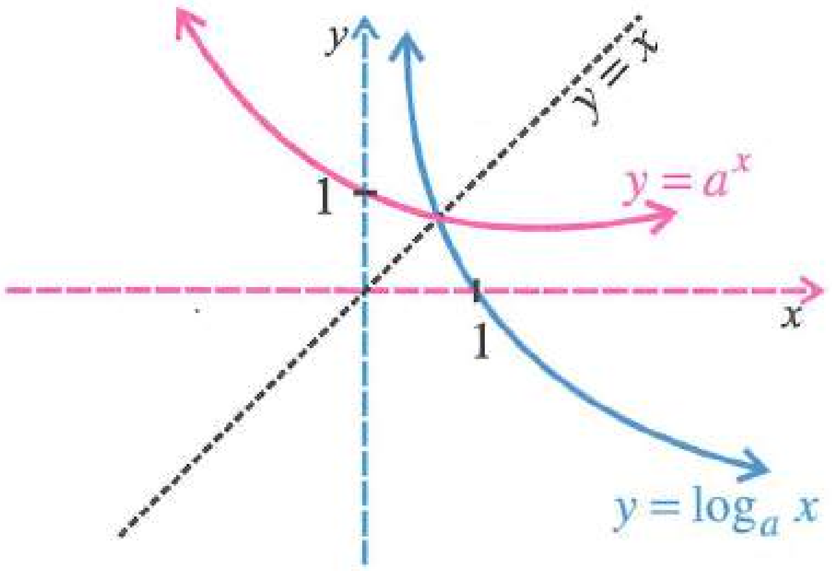
$\therefore y=\log_b x$

$x>0$ as any positive number raised to an exponent can never be negative.

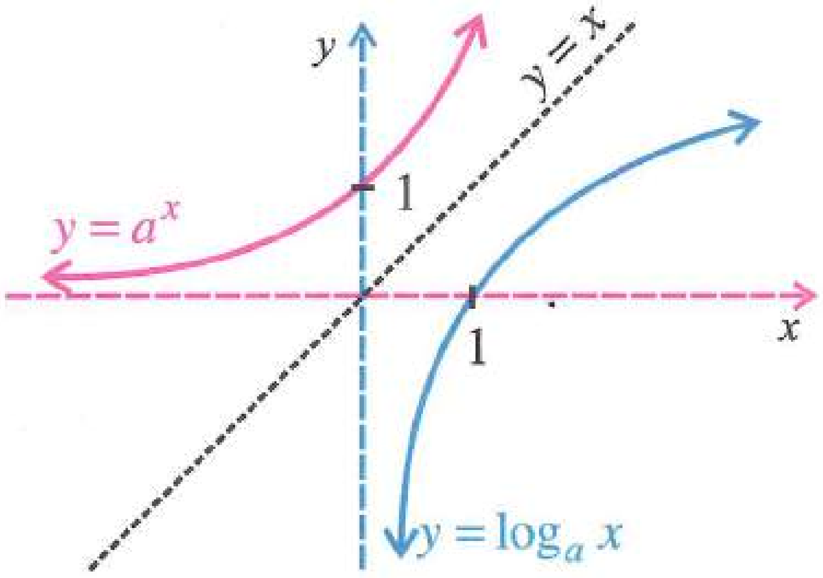
STEPS TO SKETCHING THE LOGARITHMIC FUNCTION

1. **Test for shape:**

$0<b<1 \rightarrow$ decreasing function



$b>1 \rightarrow$ increasing function



2. **Find the x-intercept by setting y equal to 0:**

   $x-\text{int}: y=0$

   $\therefore 0=\log_a x \rightarrow$ put into exponential form

   $\therefore x=a^0 \rightarrow$ any number raised to the power of 0 is equal to 1

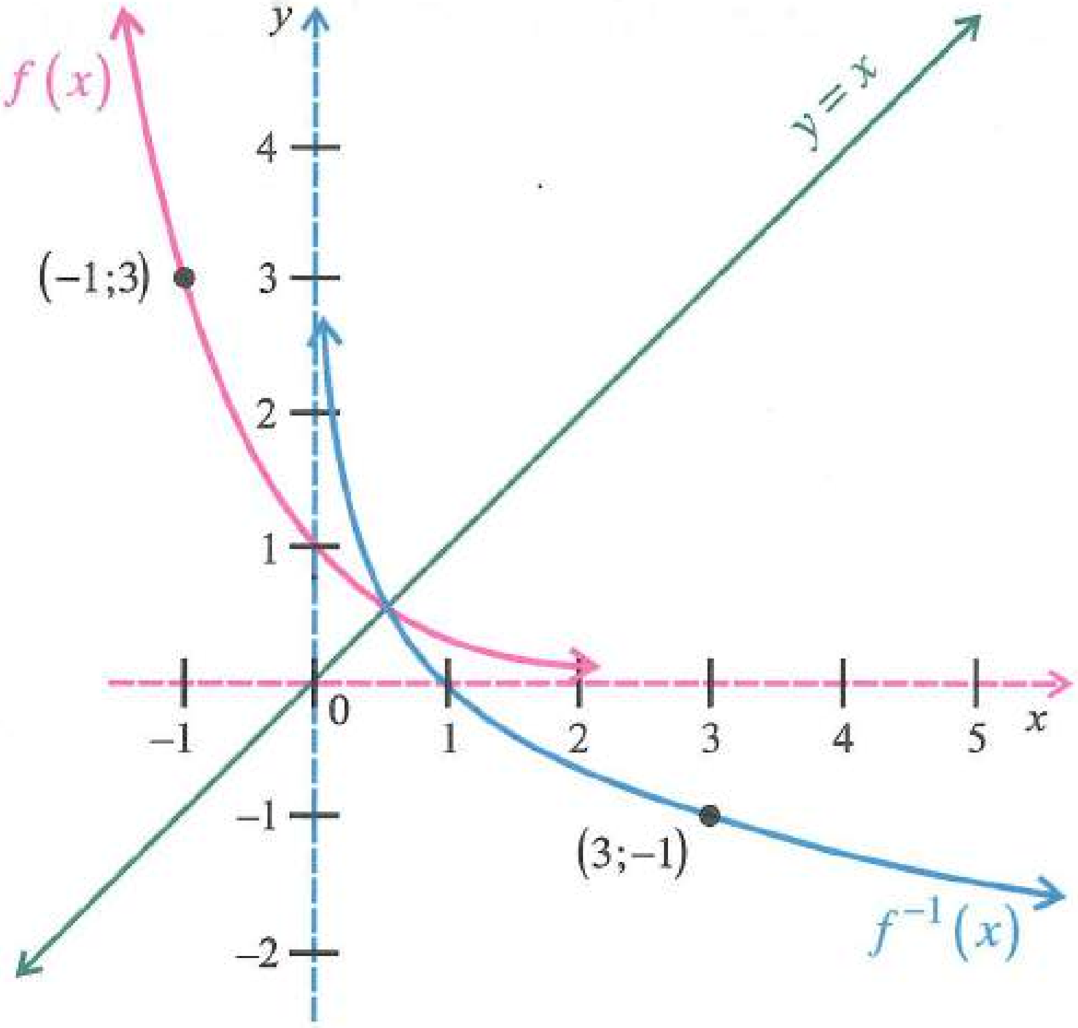
   $\therefore x=1 \rightarrow$ this is always the case (if there are no vertical or horizontal translations)

3. **Substitute an x-value to get another point on the graph and the correct shape.**

In this form, the logarithmic graph is asymptotic to the y-axis. This means that there are no y-intercepts.

Example 2: **Sketch the graphs of $f(x)=3^{-x}$, $f^{-1}(x)$ and $y=x$ on the same set of axes and state the domain and range of $f(x)$ and $f^{-1}(x)$.**

$f(x)=3^{-x}=\left(\frac{1}{3}\right)^{x}$	$f^{-1}(x): x=3^{-y}=\left(\frac{1}{3}\right)^{y}$
$0<b<1$ $\therefore$ decreasing function y-int: $(0;1)$ substitute a point in for x eg: $x=-1$ $\therefore$ point is $(-1;3)$	$y=-\log_3 x$ or $y=\log_{\frac{1}{3}} x$ **Note that $-\log_b x=\log_{\frac{1}{b}} x$** $0<b<1$ $\therefore$ decreasing function substitute a point in for x eg: $x=3$ $\therefore$ point is $(3;-1)$



$f(x)$:

Domain: $x\in\mathbb{R}$

Range: $y\in(0;\infty)$

$f^{-1}(x)$:

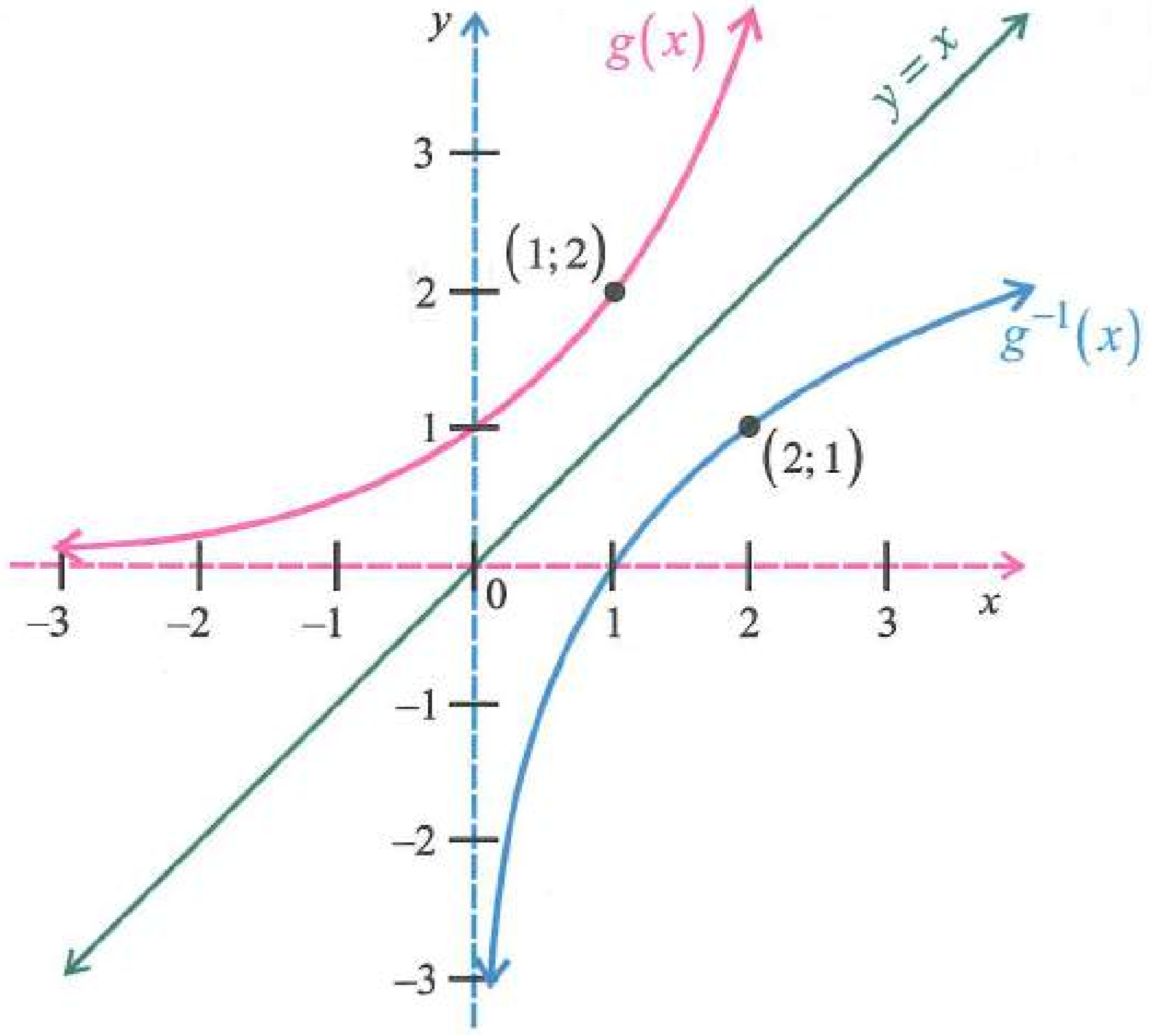
Domain: $x\in(0;\infty)$

Range: $y\in\mathbb{R}$

$y=\log_b x$ is symmetrical to $y=a^x$ about the line $y=x$
$y=\log_b x$ is symmetrical to $y=-\log_b x$ about the x-axis

Example 3: **Sketch the graphs of $g(x)=2^x$, $g^{-1}(x)$ and $y=x$ on the same set of axes and state the domain and range of $g(x)$ and $g^{-1}(x)$.**

$g(x)=2^x$	$g^{-1}(x): x=2^y$
$b>1$ $\therefore$ increasing function y-int: $(0;1)$ substitute in an x-value eg: $x=1$ $\therefore$ point is $(1;2)$	$y=\log_2 x$ $b>1$ $\therefore$ increasing function substitute in an x-value eg: $x=2$ $\therefore$ point is $(2;1)$



$g(x)$:

Domain: $x \in \mathbb{R}$

Range: $y \in (0;\infty)$

$g^{-1}(x)$:

Domain: $x \in (0;\infty)$

Range: $y \in \mathbb{R}$

Example 4: **If the graph of $f(x) = a^x$ and $a > 0$, determine:**

a) The value of a if the graph passes through the point $\left(2;\frac{25}{16}\right)$.

b) Determine the equation of f^{-1}.

c) Determine the equation of $h(x)$, if h is the reflection of the graph of f through the y-axis.

d) Determine the equation of $k(x)$, if k is the reflection of the graph of f through the x-axis.

e) Determine the equation of $p(x)$, if p is obtained by translating the graph of f two units to the right.

f) Sketch the graphs of f, f^{-1}, h, k and p on the same set of axes.

g) Determine the domain and range of each of the functions that you sketched above.

h) For which values of x is $f^{-1} \geq 0$?

a) $f(x) = a^x$

Sub $\left(2;\frac{25}{16}\right)$

$\therefore \frac{25}{16} = a^2$

$\therefore a = \pm\frac{5}{4} \rightarrow$ the negative is not applicable as $a > 0$

$a = \frac{5}{4}$

b) $y = \left(\frac{5}{4}\right)^x \rightarrow$ swop the x and y-values to obtain the inverse

$\therefore x = \left(\frac{5}{4}\right)^y$

$\therefore y = \log_{\frac{5}{4}} x$

$\therefore f^{-1}(x) = \log_{\frac{5}{4}} x$

c) A reflection about the y-axis causes all of the values of x to change signs.

$\therefore h(x) = f(-x) = \left(\frac{5}{4}\right)^{-x} = \left(\frac{4}{5}\right)^x$

$\therefore h(x) = \left(\frac{4}{5}\right)^x$

d) A reflection about the x-axis causes all of the values of y to change signs.

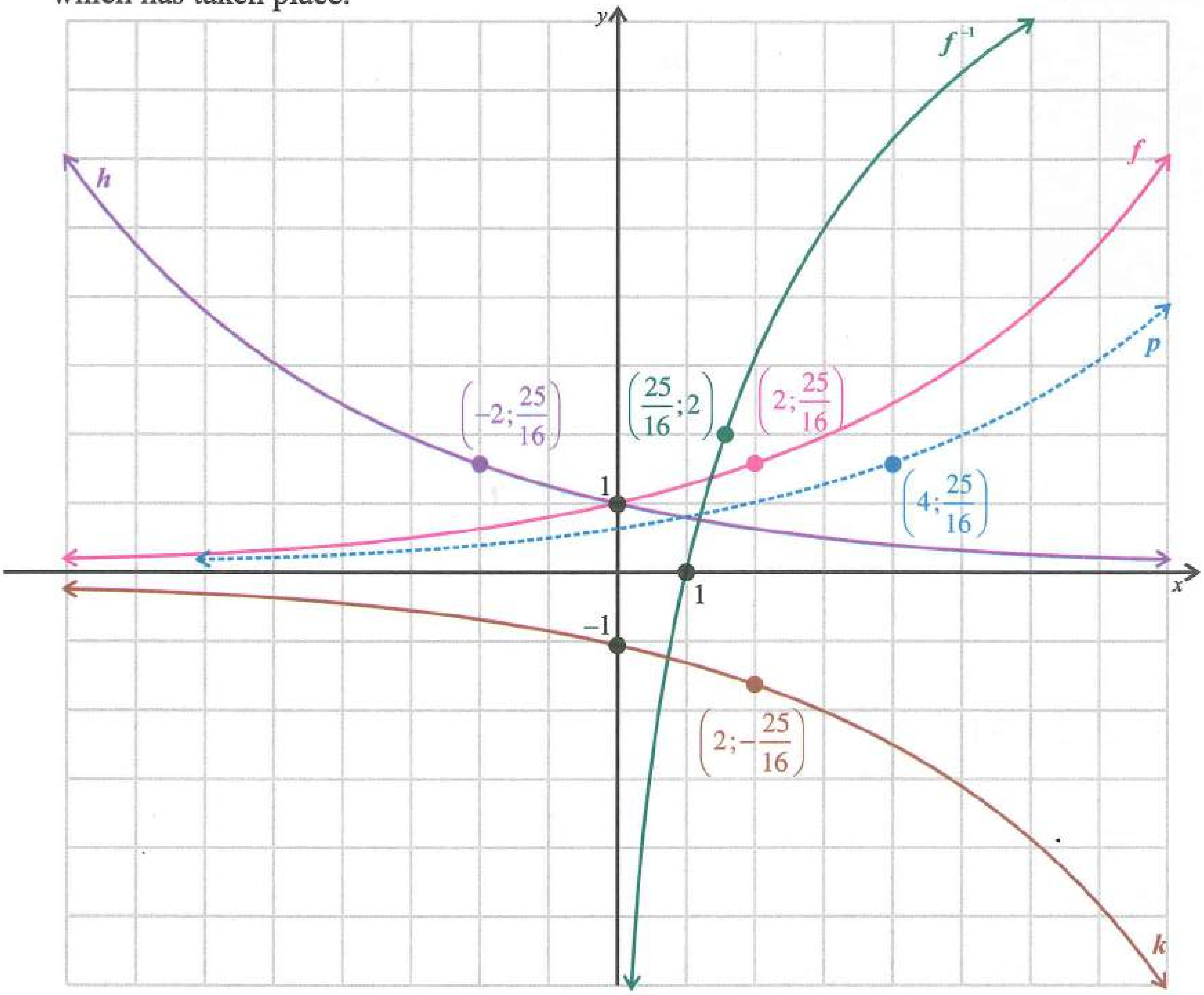
$\therefore k(x) = -f(x) = -\frac{5^x}{4}$

$\therefore k(x) = -\left(\frac{5}{4}\right)^x$

e) $p(x) = f(x-2) = \left(\frac{5}{4}\right)^{x-2}$ → right shift

$p(x) = \left(\frac{5}{4}\right)^{x-2}$

f) Take note of how each point on f changes with each type of reflection or translation which has taken place.



g)

Function	Domain	Range
$f(x)$	$x \in \mathbb{R}$	$y \in (0;\infty)$
f^{-1}	$x \in (0;\infty)$	$y \in \mathbb{R}$
$h(x)$	$x \in \mathbb{R}$	$y \in (0;\infty)$
$k(x)$	$x \in \mathbb{R}$	$y \in (-\infty;0)$
$p(x)$	$x \in \mathbb{R}$	$y \in (0;\infty)$

h) $x \geq 1$ → also written as $x \in [1;\infty)$

Do not confuse $y = b^x$ with $y = x^b$ where b is a positive (constant) real number. A graph is only an exponential function if the exponent is a variable.

Complete Exercise 4 on pages 45-47.

LAWS OF LOGARITHMS

There are four logarithmic laws that will be applied in the context of real-life problems related to finance, growth and decay. These laws are shown below:

Law 1	$\log_b(A \times B) = \log_b A + \log_b B$
Law 2	$\log_b\left(\frac{A}{B}\right) = \log_b A - \log_b B$
Law 3	$\log_b A^n = n\log_b A$
Law 4	$\log_b A = \frac{\log A}{\log B}$

Example 1: Express $\log 4 + \log 9 - \log 12$ as the log of a single number.

$$\log 4 + \log 9 - \log 12$$
$$= \log(4 \times 9) - \log 12 \rightarrow \text{law 1}$$
$$= \log\frac{36}{12} \rightarrow \text{law 2}$$
$$= \log 3$$

Example 2: Simplify $\dfrac{\log_x 3}{\log_x \frac{1}{27}}$.

$$\frac{\log_x 3}{\log_x \frac{1}{27}}$$

There are no laws for dividing logs!

$\therefore$ Try get the same logs on the 'top' and the 'bottom' to cancel with each other.

$$= \frac{\log_x 3}{\log_x \frac{1}{3^3}}$$
$$= \frac{\log_x 3}{\log_x 3^{-3}} \rightarrow \text{use law 3}$$
$$= \frac{\log_x 3}{-3\log_x 3}$$
$$= -\frac{1}{3}$$

Complete Exercise 5 on page 47 and the Mixed Exercise on pages 48 and 49.

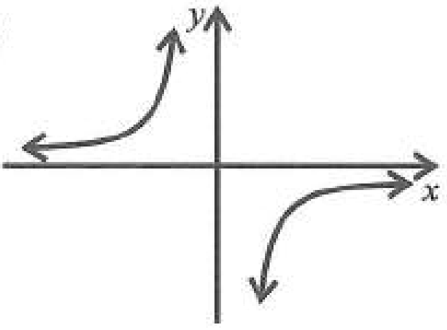
Chapter 2 Functions, Inverses and Logarithms (Exercises)

Classifying Functions and Inverse Functions

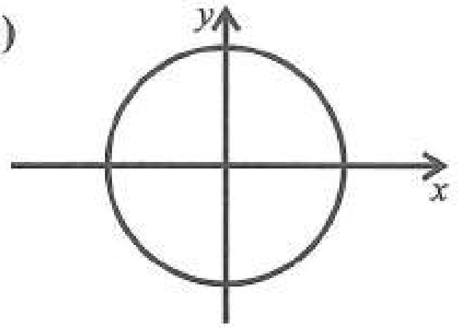
Exercise 1: (answers on page 343)

1) **State whether or not the following graphs are functions. If the graph is a function, state which type of function it represents.**

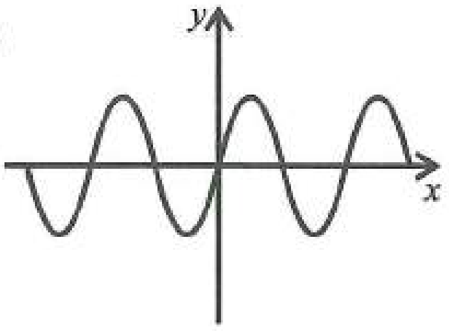
a)



b)



c)



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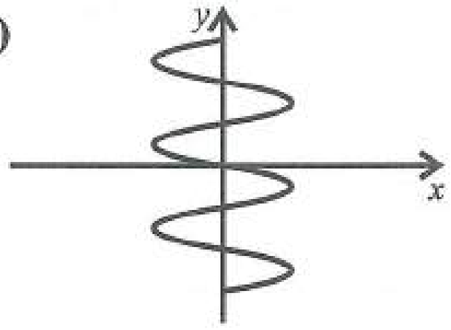
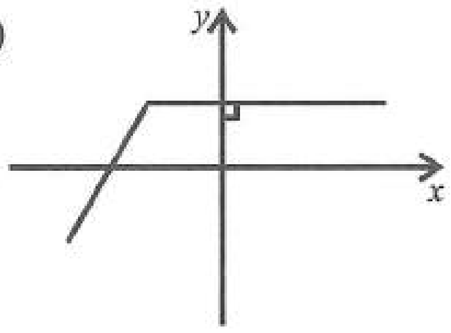
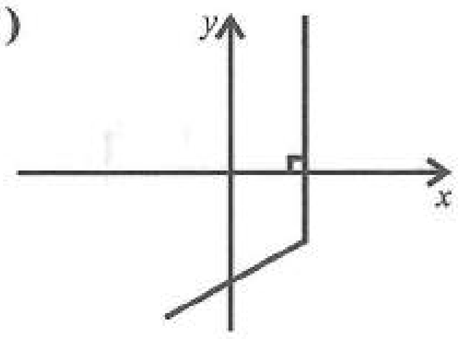


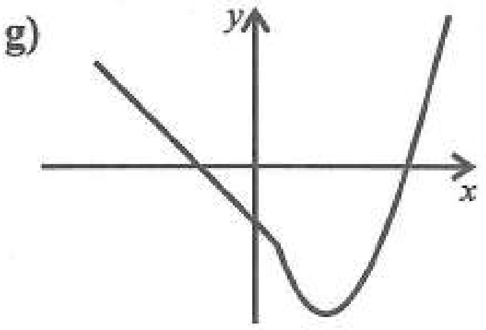
e)



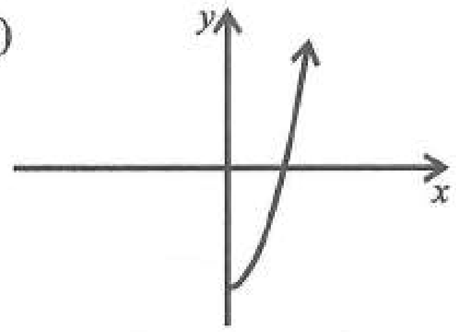
f)



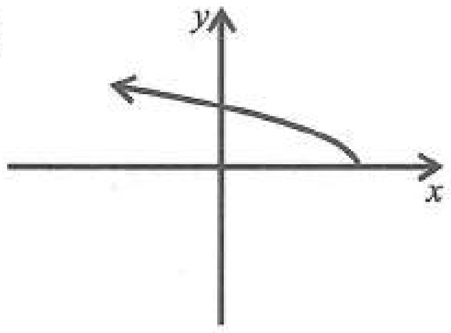
g)



h)



i)



Exercise 2: (answers on page 343)

1) **If $g(x) = 2x^2 - 8$:**

   a) Draw a sketch graph of $g(x)$.

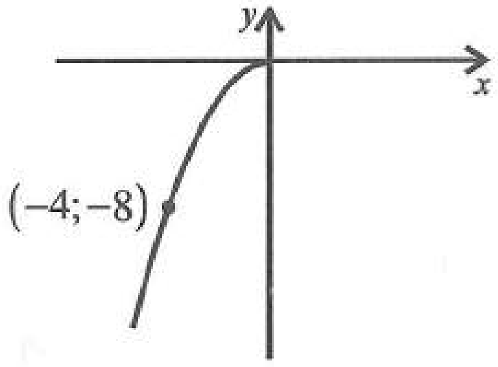
   b) State the domain and range of $g(x)$.

   c) Determine the equation of the inverse of g in the form $g^{-1}(x) = \ldots$

   d) Sketch the graph of $g^{-1}(x)$, showing all intercepts with the axes.

   e) State the domain and range of $g^{-1}(x)$.

2) **The graph of $h(x) = ax^2$, $x \leq 0$ is sketched below and the point (–4 ;–8) lies on the graph of h.**

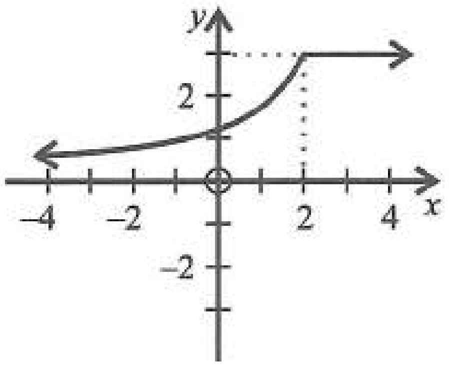


   a) Determine the value of a.
   b) Determine the equation of the inverse of $h(x)$ in the form $h^{-1}(x) = \ldots$
   c) Write down the range of h^{-1}.
   d) Sketch the graph of h^{-1} showing one point on the graph other than the origin.
   e) The graph of $h(x)$ is reflected about the line $y = x$ and is then reflected across the x-axis. Determine the equation of the new function in the form $y = \ldots$

3) If $f(x)=\frac{1}{2}x^2$, determine:

- a) The inverse of f in the form $y=...$
- b) The value of the inverse of f when $x=8$.
- c) The domain and range of the inverse of f.

4) Using the sketch of $f(x)$ below answer the following questions:



- a) Explain what type of function f is ?
- b) What is the range of f ?
- c) Will the inverse of f still be a function?
- d) If the range of f is not changed, what restriction could be placed on the domain of f in order to ensure that the inverse would always be a function?

5) Given that $g(x)=4(x+5)^2+3$, restrict the domain so that the inverse will always be a function.

Logarithms

Exercise 3: (answers on page 343 and 344)

1) Express the following in exponential form.

- a) $2=\log_5 25$
- b) $3=\log_{28} 21952$
- c) $2=\log_{21} 441$
- d) $5=\log_7 16807$
- e) $x=\log_5 7$
- f) $a=\log_x 9$

2) Express the following in exponential form.

- a) $\log_2 8=3$
- b) $\log_3 81=4$
- c) $\log_5 5=1$
- d) $\log_3 x=5$
- e) $\log 3=0,477$
- f) $\log_6 (1296)^{-1}=-4$
- g) $\log_x 64=2$
- h) $\log 1000=3$
- i) $\log x=5$

3) Solve for x, without the use of a calculator:

- a) $x=\log_2 8$
- b) $x=\log_3 3$
- c) $x=\log_8 2$
- d) $x=\log_3 1$
- e) $\log_5 x=1$
- f) $\log_2 x=8$
- g) $\log_{22} x=0$
- h) $\log_{\frac{1}{2}} x=-3$

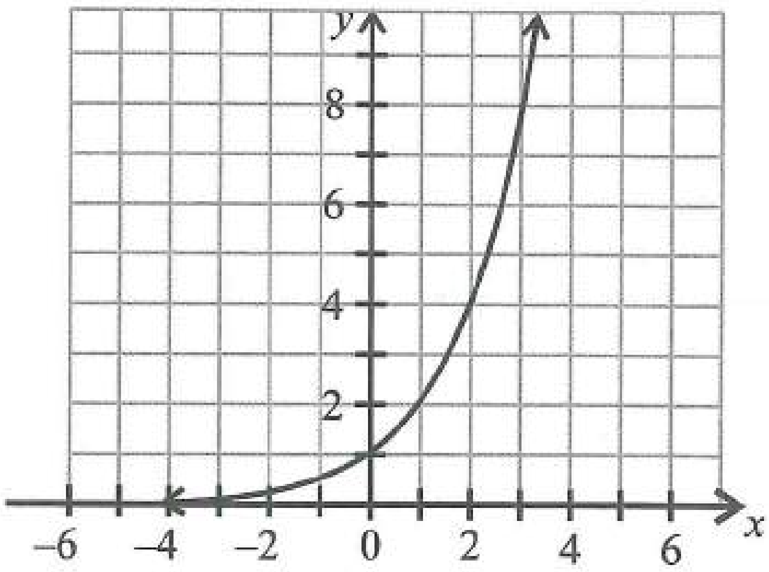
The logarithmic Function

Exercise 4: (answers on page 344)

1) Given that $h(x)=\left(\frac{1}{2}\right)^x$:

- a) Draw a sketch graph of h, clearly showing the y-intercept and one other point.
- b) What is the range of h?
- c) The graph of $f(x)$ is the inverse of h. Determine the equation of f in the form $y=...$
- d) On the same set of axes draw a sketch graph of f and show two points on the graph.
- e) If the graph of g is the mirror of h in the x-axis, determine the equation of g.
- f) If p is the graph of g translated 2 units down and 4 units left, determine the equation of p.

2) The graph of $f(x)=(2)^x$ is sketched below:

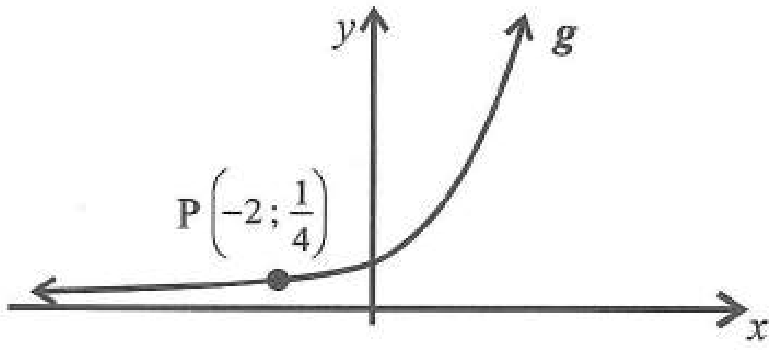


a) Write the inverse in the form $f^{-1}(x)=\ldots$

b) Sketch the graphs of $f^{-1}(x)$ and $f^{-1}(x-3)$ on the same set of axes.

c) For what values of x is $f^{-1}(x-3)\le 3$

3) The graph of $g(x)=a^x$ is sketched below. The point $\mathrm{P}\left(-2;\frac{1}{4}\right)$ lies on the graph of g:

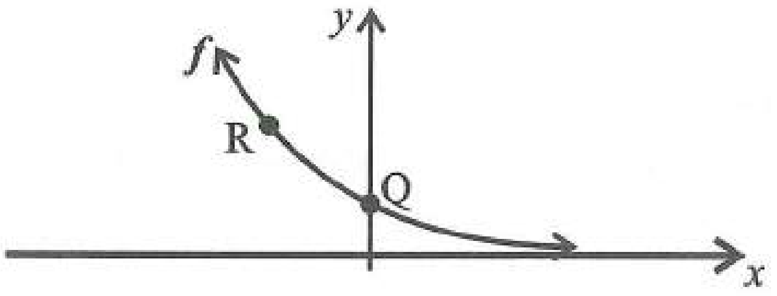


a) Determine the value of a.

b) If h is the inverse of g, determine the equation of h in the form $y=\ldots$

c) Sketch the graph of h and use your graph determine the values of x for which $\log_2 x>-2$.

4) The graph of $f(x)=\left(\frac{1}{2}\right)^x$ is sketched below. Points $\mathrm{Q}(x;y)$ and $\mathrm{R}(b;4)$ lie on f:



a) Determine the coordinates of Q.

b) Determine the value of b.

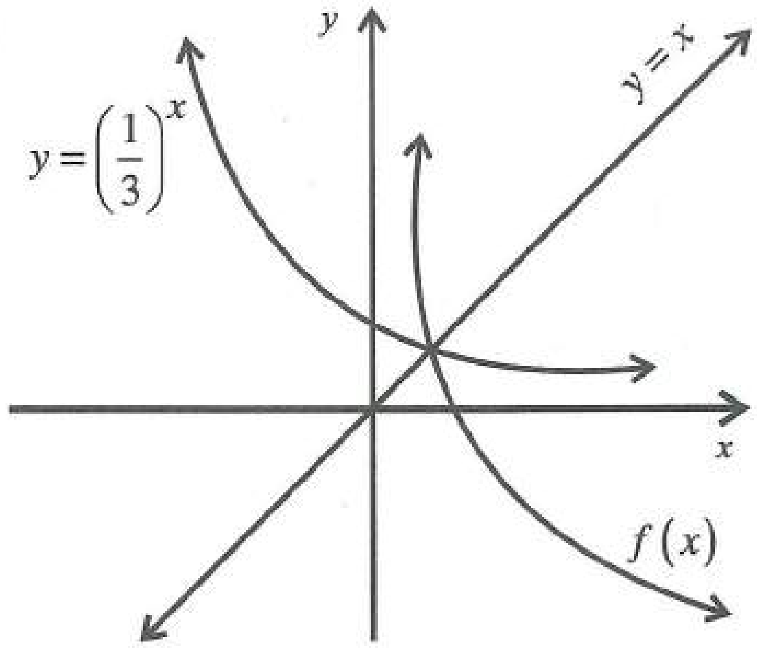
c) Determine the equation of the inverse of f in the form $y=\ldots$

d) If $g(x)$ is the image of f^{-1} under a reflection about the x-axis, determine the equation of g.

e) $p(x)$ is the result of translating f 4 units up. Write down the equation of $p(x)$.

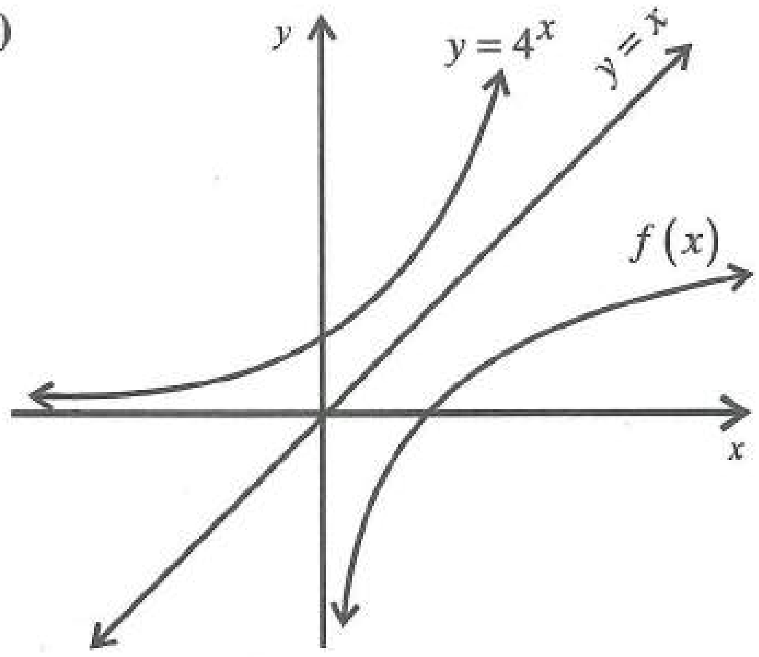
5) Determine the equations of $f(x)$ for the given symmetrical curves below:

a)

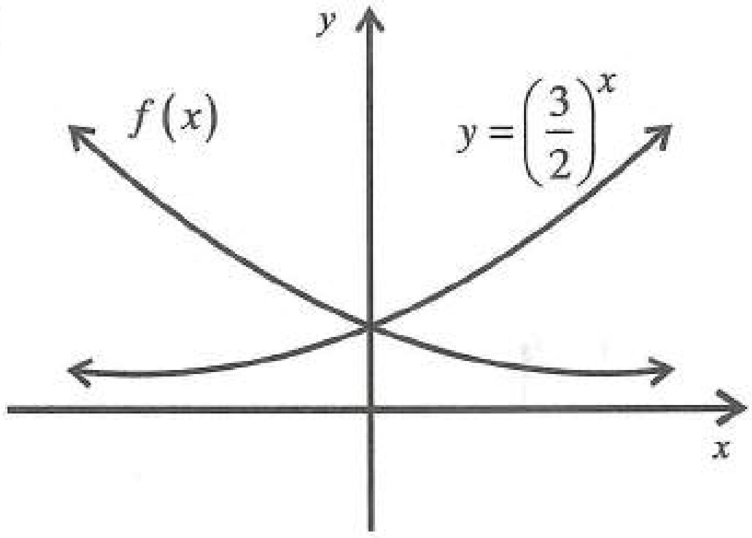


y
$y=\left(\frac{1}{3}\right)^x$
$y=x$
x
$f(x)$

c)



b)



y
$f(x)$
$y=\left(\frac{3}{2}\right)^x$
x

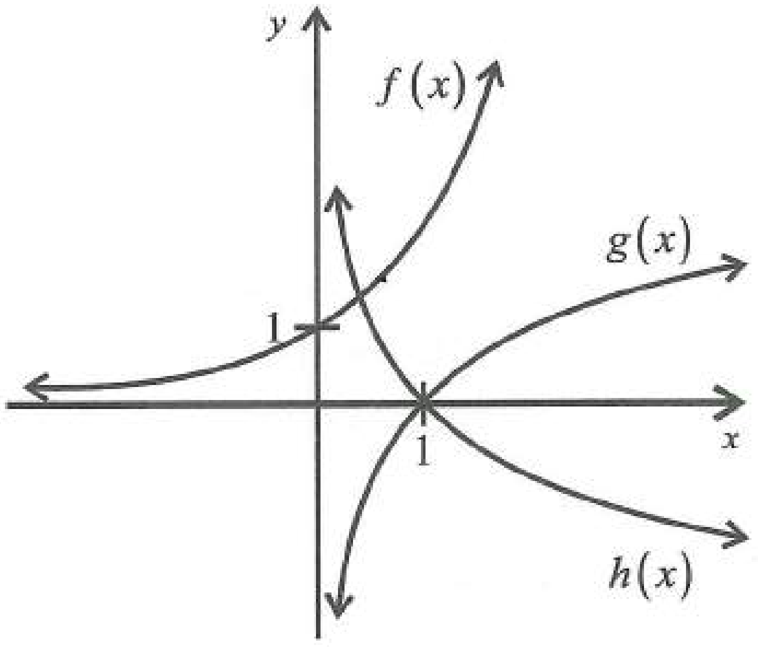
6) The following graphs are shown in the sketch alongside:

- $f(x)=3^x$
- $g(x)$ the reflection of f along the line $y=x$
- $h(x)$ the reflection of g about the x-axis

a) Determine the equations of g and h.

b) Determine the values of x for which $3^x > 0$.

c) Determine the values of x for which $\log_{\frac{1}{3}} x \le 0$.



y
$f(x)$
$g(x)$
1
1
x
$h(x)$

Laws of Logarithms

Exercise 5: (answers on page 345)

1) Express the following as a logarithm of a single number or term:

a) $\log 4-\log 10-\log 4$

c) $\log_2 6+3\log_2 2$

b) $5\log 2+2\log 5$

d) $\log(x+4)^2-\log\left(x^2-16\right)$

2) Expand the following using log laws. Assume that all variables are greater than 0.

a) $\log abc$

c) $\log\frac{xy}{z}$

b) $\log 10ab$

d) $\log\left[\frac{x+1}{x-1}\right]$

Mixed Exercise: (answers on pages 345 and 346)

1) Answer the following questions:

a) For which values of p will $f(x) = x^2 + 4x + 4\ ;\ x \geq p$ be a one-to-one function.

b) If $\left(6p^2\ ;8\right)$ and $(3p\ ;5)$ are elements of a relation. For which values of p will the relation not be a function?

c) If x is a rational number, then $f(x) = 5$. If x is an irrational number then $f(x) = \frac{2}{3}$. Determine the value of $3f\left[f\left(\sqrt{5}\right)\right]$.

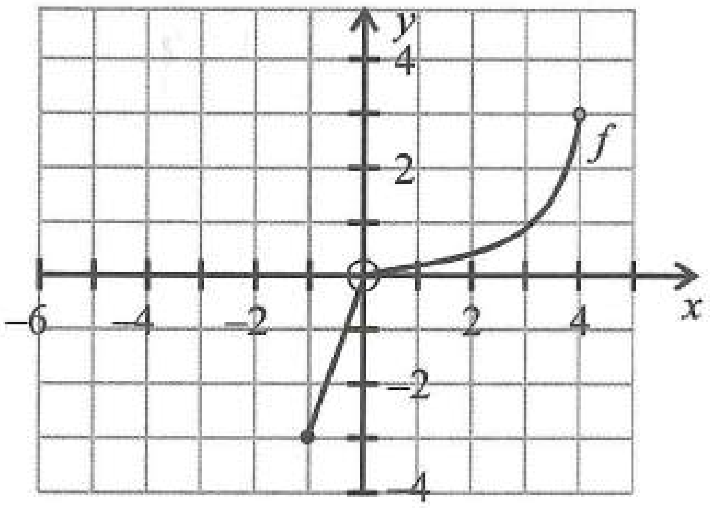
2) Consider the function $f(x) = 2x^2$ and $p(x) = \left(\frac{1}{2}\right)^x$.

a) Restrict the domain of f so that the inverse of f will also be a function.

b) Write down the equation of the inverse of p in the form $p^{-1}(x) = \ldots$

c) If the inverse of another function is $g^{-1}(x) = 2x - 4$. Determine the equation of $g(x)$.

3) The graph of $f(x)$ is shown below:



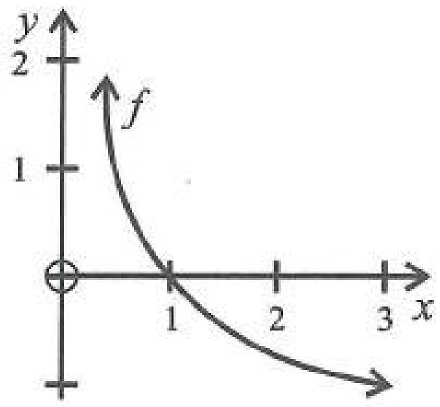
a) State the domain and range of f.

b) Draw the graph of $f^{-1}(x)$.

c) Explain why $f^{-1}(x)$ is a function.

4) Given the functions $f(x) = \dfrac{x}{x-2}$ and $g(x) = \dfrac{2x}{x-1}$ prove that $g(x) = f^{-1}(x)$.

5) The graph of $f(x) = -\log_3 x$ is shown below:

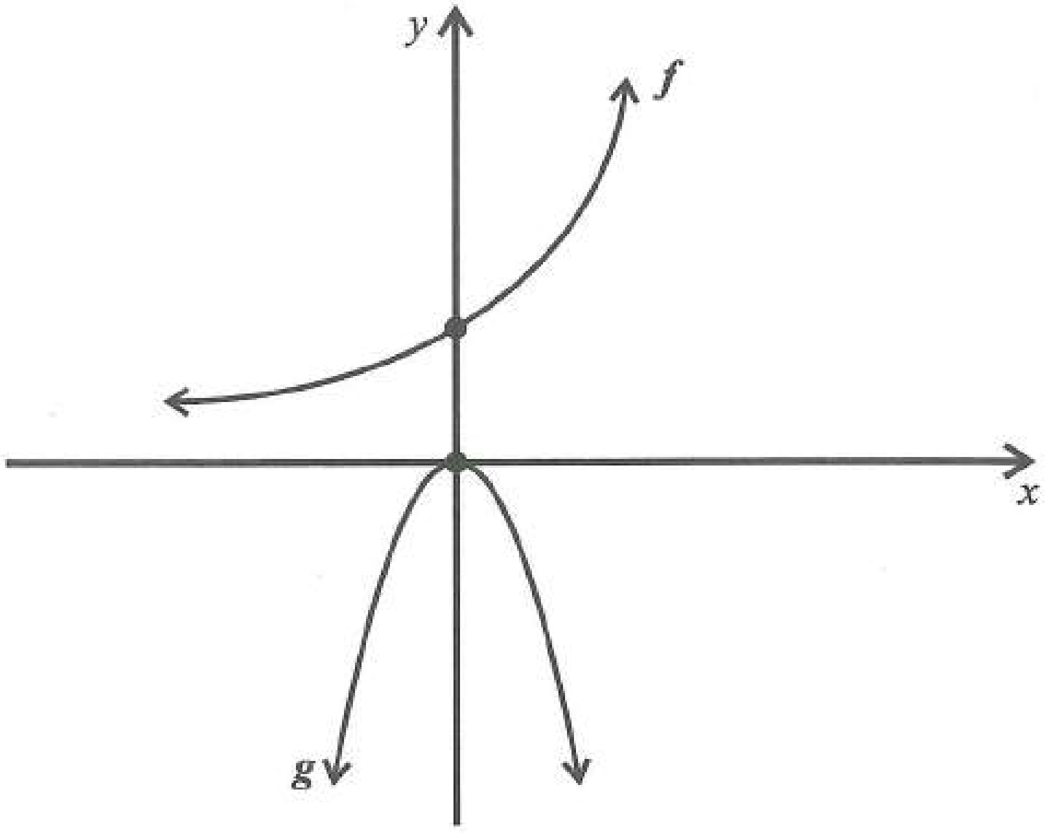


a) Write down the equation of $f^{-1}(x)$ in the form $y = \ldots$

b) Write down the equation of the asymptote of $f^{-1}(x)$.

c) Using the graph explain how you would sketch the graph of $\log_3 x$

6) Sketched below are the graphs of $g(x) = -x^2$ and $f(x) = 3^x$.



a) Write down the equation of the inverse of the graph of $f(x) = 3^x$ in the form $f^{-1}(x) = \ldots$

b) Draw a sketch graph of $f^{-1}(x)$ showing all intercepts with the axes.

c) State the domain and range of $f^{-1}(x)$.

d) Is the inverse graph of $g(x)$ a function? Explain your answer.

e) What restriction could be placed on the graph of $g(x)$ so that its inverse would be a function?

f) Draw the inverse function of g.

g) Describe the transformation of f, that would enable you to sketch the graph of $h(x) = -\log_3 x$.

h) Describe the transformation of f, that would enable you to sketch the graph of $p(x) = \left(\frac{1}{3}\right)^x + 1$.

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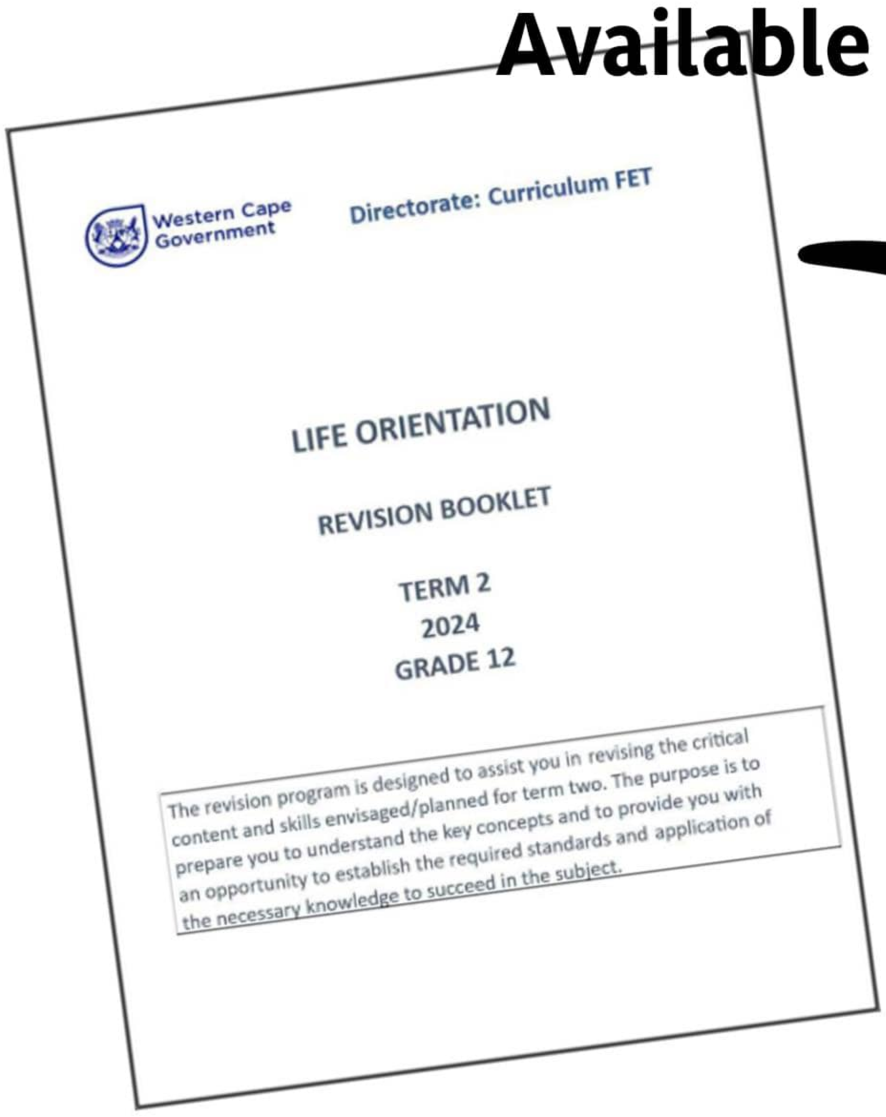

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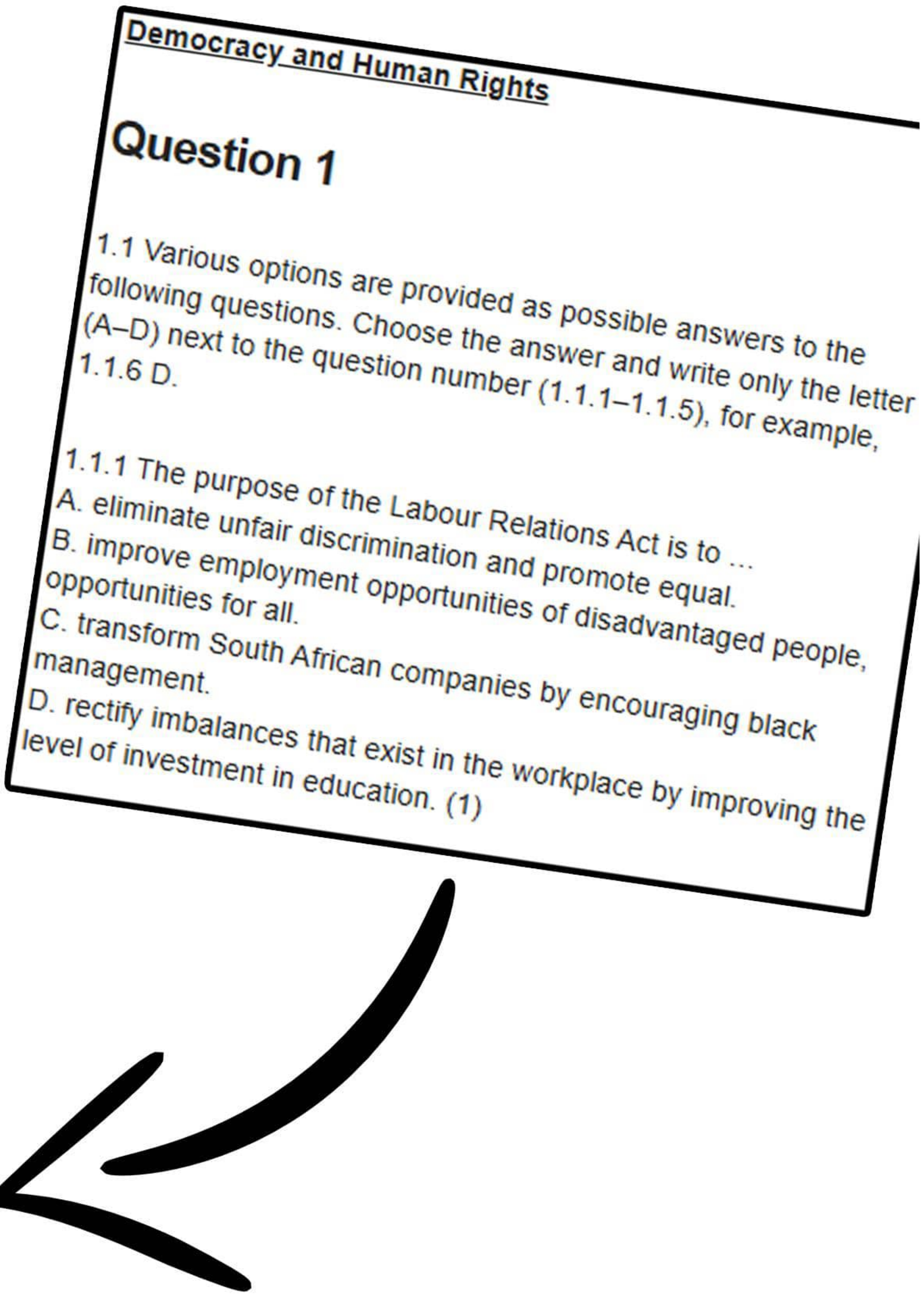
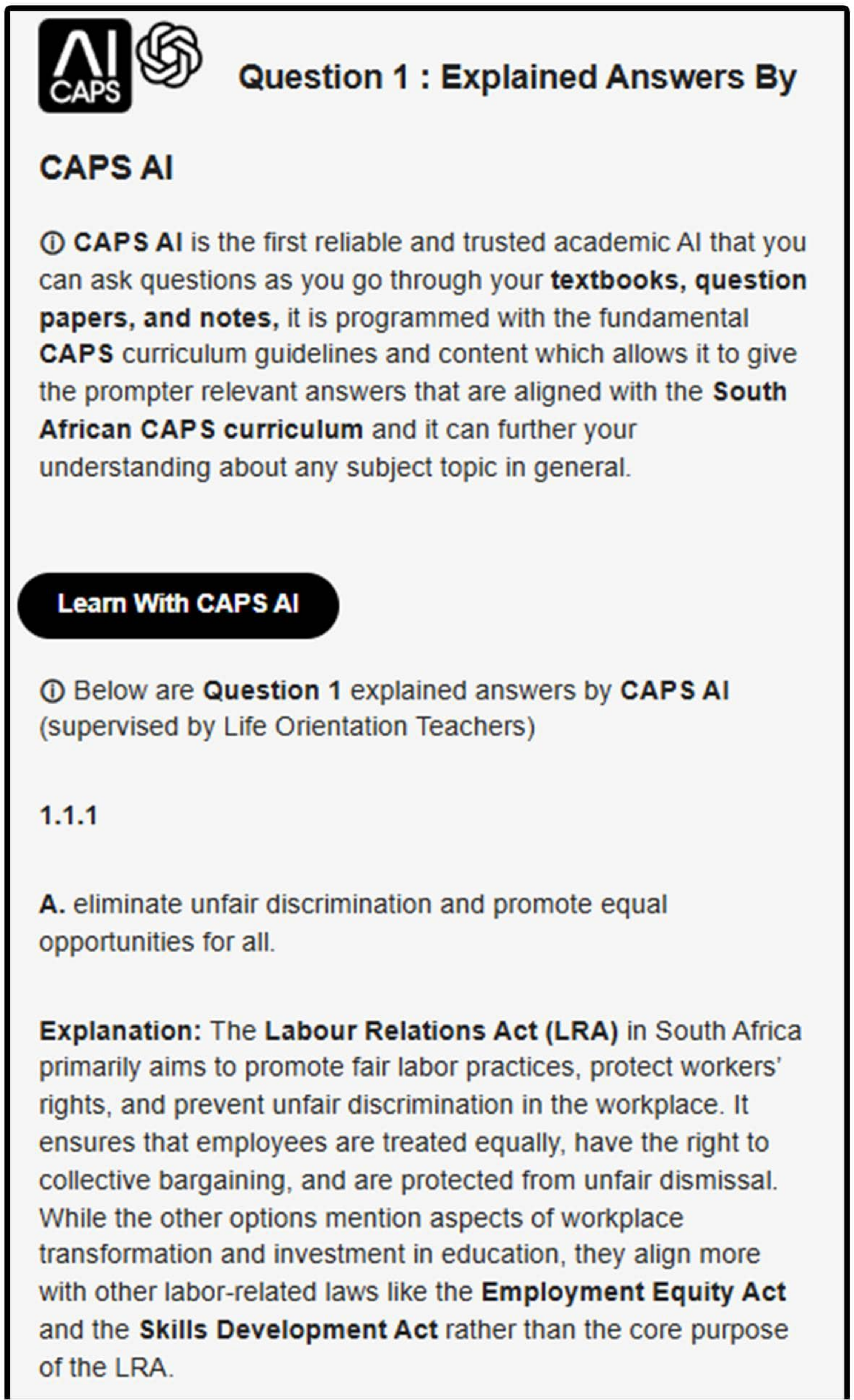


Democracy and Human Rights

Question 1

1.1 Various options are provided as possible answers to the following questions. Choose the answer and write only the letter (A–D) next to the question number (1.1.1–1.1.5), for example, 1.1.6 D.

1.1.1 The purpose of the Labour Relations Act is to …
A. eliminate unfair discrimination and promote equal opportunities for all.
B. improve employment opportunities of disadvantaged people, management.
C. transform South African companies by encouraging black management.
D. rectify imbalances that exist in the workplace by improving the level of investment in education. (1)



Question 1 : Explained Answers By CAPS AI

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ⓘ Below are **Question 1** explained answers by **CAPS AI** (supervised by Life Orientation Teachers)

1.1.1

A. eliminate unfair discrimination and promote equal opportunities for all.

Explanation: The **Labour Relations Act (LRA)** in South Africa primarily aims to promote fair labor practices, protect workers' rights, and prevent unfair discrimination in the workplace. It ensures that employees are treated equally, have the right to collective bargaining, and are protected from unfair dismissal. While the other options mention aspects of workplace transformation and investment in education, they align more with other labor-related laws like the **Employment Equity Act** and the **Skills Development Act** rather than the core purpose of the LRA.

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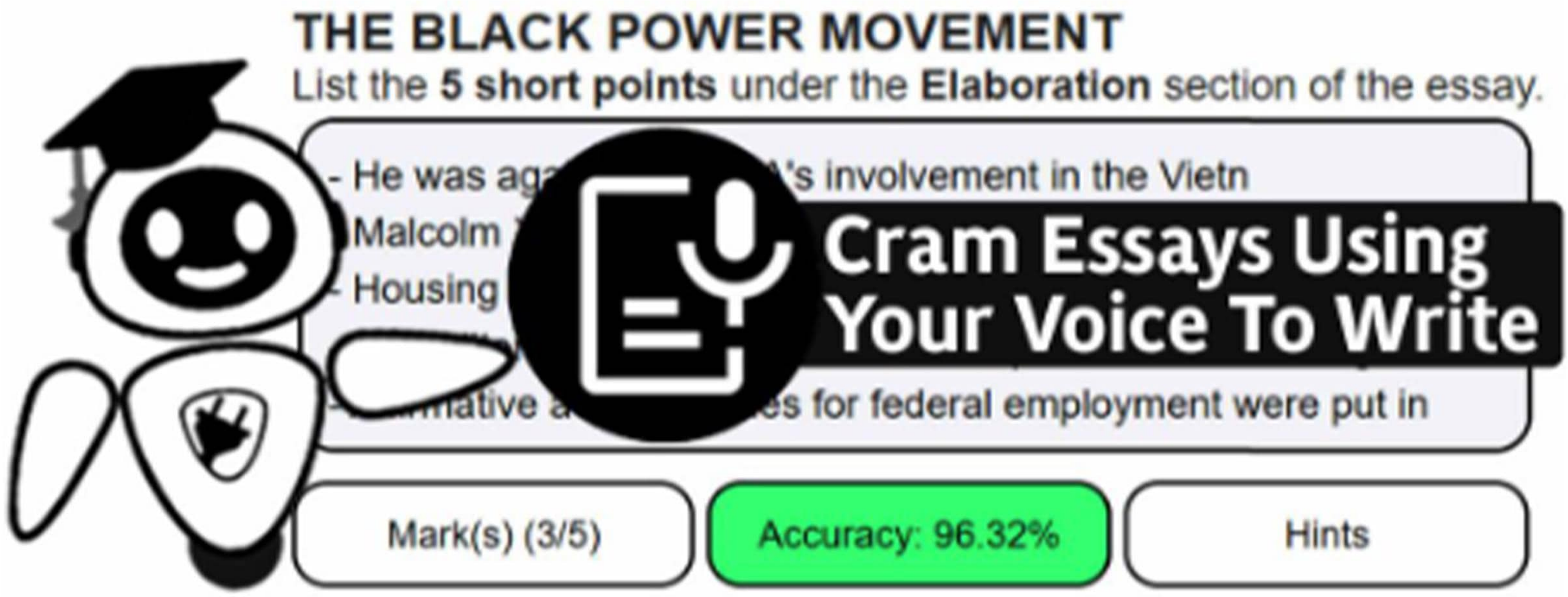

THE BLACK POWER MOVEMENT

List the **5 short points** under the **Elaboration** section of the essay.

- He was ag... 's involvement in the Vietn
- Malcolm
- Housing
- ...mative a... es for federal employment were put in

Mark(s) (3/5) | Accuracy: 96.32% | Hints

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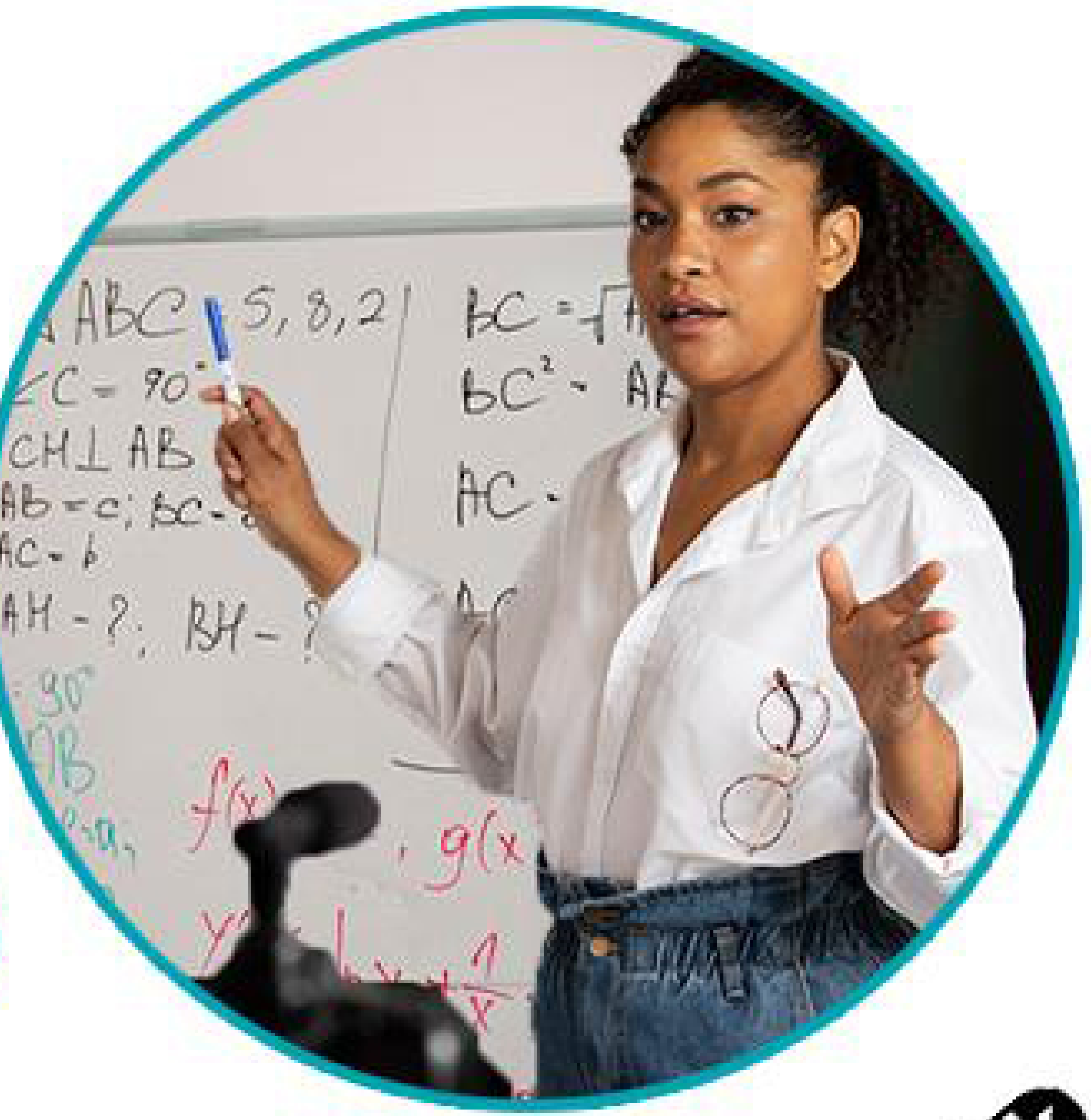

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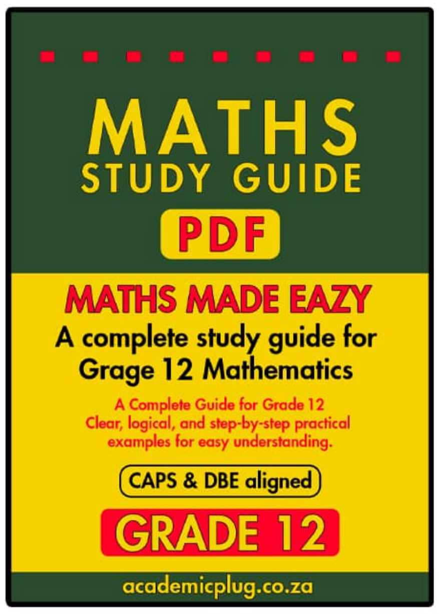

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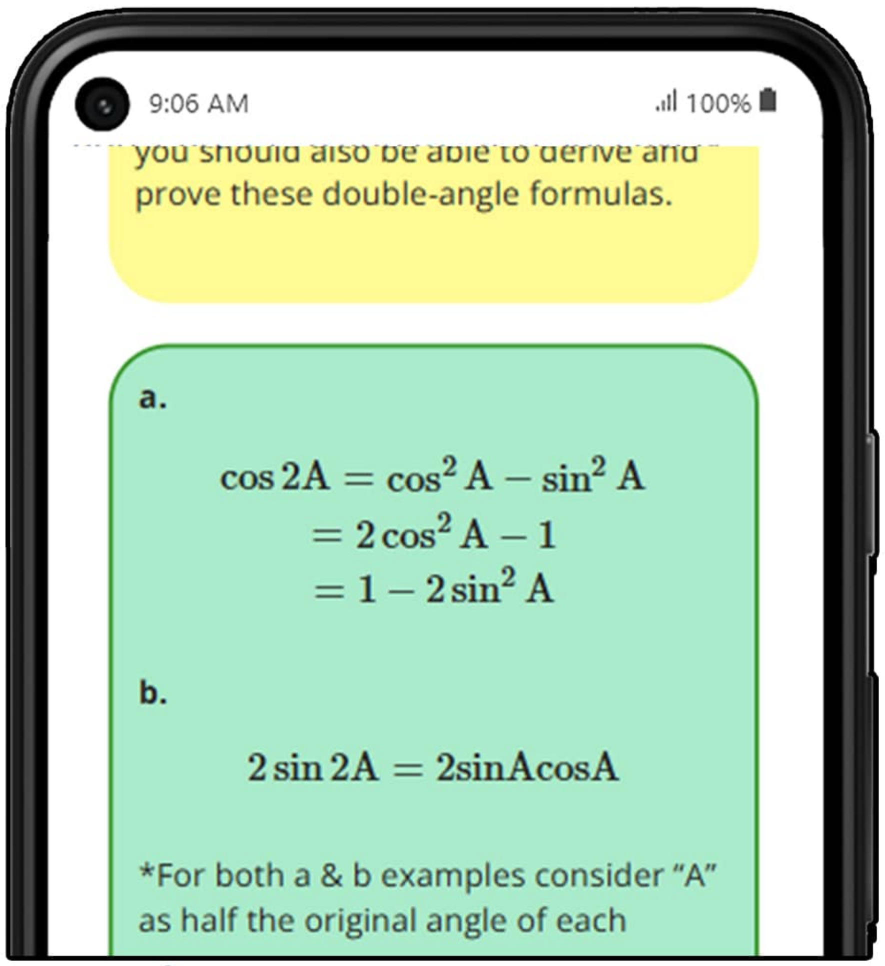

9:06 AM 100%

you should also be able to derive and prove these double-angle formulas.

a.

$$\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A \\ &= 2\cos^2 A - 1 \\ &= 1 - 2\sin^2 A\end{aligned}$$

b.

$$2\sin 2A = 2\sin A\cos A$$

*For both a & b examples consider "A" as half the original angle of each

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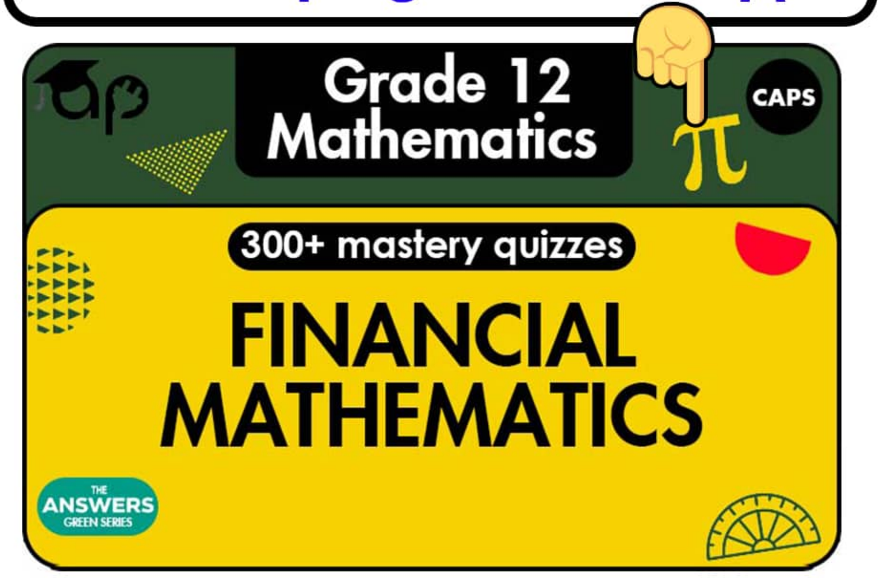

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CHAPTER 3
FINANCIAL MATHEMATICS

WHAT YOU WILL LEARN IN THIS CHAPTER

PAGE	CONTENT	CAPS	ASSESSMENTS YOU WILL BE ABLE TO:
51	**A. Grade 11 Revision and the use of Logarithms**	3	• Calculate the value of *n* in problems involving financial mathematics • Apply your knowledge of sequences and series to solve annuity problems • Critically analyse different loan an investment options
52	• Nominal and Effective Interest Rates		
53	• Time Lines		
54	• Using Logarithms to Determine the Time Period		
55	**B. Future Value and Present Value**		
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A. GRADE 11 REVISION AND THE USE OF LOGARITHMS

GRADE 11 REVISION

In Grade 12 it is essential that you fully understand the Grade 11 concepts of simple and compound growth and decay (interest and depreciation), as well as the concept of nominal and effective interest rates.

Recap of Grade 11 Formulas:

Simple Interest: $A=P(1+in)$ **Compound Interest:** $A=P(1+i)^n$

Simple Decay: $A=P(1-in)$ **Reducing Balance :** $A=P(1-i)^n$

Example 1: How long must any amount of money be invested for, in order for it to double at an interest rate of 8,5% p.a simple interest.
Give your answer in years and months.

Simple interest:

$A=P(1+in)$

Let the initial amount invested be x. Therefore the final amount will be $2x$:

$\therefore 2x=x(1+0,085n) \rightarrow$ interest must be in a decimal

$\therefore 2=1+0,085n \rightarrow$ both sides were divided by x

$\therefore 1=0,085n$

$\therefore n=11,76...$ years $\rightarrow$ multiply 0,76... by 12 to obtain the number of months in 0,76 years

$\therefore$ It would take 11 years and 10 months before the money doubles.

> **It is important to note that in the above example, although $0,76\times 12=9,12$ months, after 11 years and 9 months the money still has not doubled. This is the reason for rounding up to 10 months.**

Example 2: A computer server valued at R24 000 depreciates at 18% p.a. on a reducing balance. Determine the value of the server after 2 years.

Reducing Balance:

$A=P(1-i)^n$

$=24000(1-0,18)^2 \rightarrow$ interest must be in a decimal

$=\text{R}16137,60$

NOMINAL AND EFFECTIVE INTEREST RATES

The nominal interest rate is the rate that is quoted in the question. The effective interest rate is the equivalent annual interest rate that you would receive once compounding has been taken into account.

Example: Given an interest rate of 12% p.a. compounded monthly:

a) Determine the effective interest rate.

b) Using the effective interest rate, convert an interest rate of 12% p.a. compounded monthly to an interest rate of 12% p.a. compounded quarterly.

a) $1+i_{eff}=(1+i_{nom})^n$

$$\therefore 1+i_{eff}=\left(1+\frac{0,12}{12}\right)^{12}$$

$$\therefore i_{eff}=\left(1+\frac{0,12}{12}\right)^{12}-1$$

$$\therefore i_{eff}=0,1268=12,68\%$$

b) $1+i_{eff}=(1+i_{nom})^n$

$$\therefore 1+0,1268=\left(1+\frac{i_{nom}}{4}\right)^4 \rightarrow \text{quarterly}$$

$$\therefore 1+\frac{i_{nom}}{4}=\sqrt[4]{1+0,1268} \rightarrow 4^{\text{th}} \text{ root both sides}$$

$$\therefore \frac{i_{nom}}{4}=\sqrt[4]{1+0,1268}-1$$

$$\therefore i_{nom}=0,1212$$

$$\therefore i_{nom}=12,12\% \text{ p.a. compounded quarterly}$$

In Grade 12 you will be required to convert interest rates from one compounding period to a different compounding period as you did in the question above. The formula below provides a quick method of doing this.

Formula:

$$(1+i_{new})^m=(1+i_{nom})^n$$

Where: i_{new} → **interest rate with new compounding period**

i_{nom} → **interest rate given in question (original compounding rate)**

n → **Number of time periods at the original compounding rate**

m → **Number of time periods at the new compounding rate**

Example: Convert an interest rate of 12% p.a. compounded monthly to an interest rate compounded quarterly.

$$(1+i_{new})^m=(1+i_{nom})^n$$

$$\therefore \left(1+\frac{i_{new}}{4}\right)^4=\left(1+\frac{0,12}{12}\right)^{12} \rightarrow \text{quarterly to monthly}$$

$$\therefore 1+\frac{i_{new}}{4}=\sqrt[4]{\left(1+\frac{0,12}{12}\right)^{12}} \rightarrow 4^{\text{th}} \text{ root both sides}$$

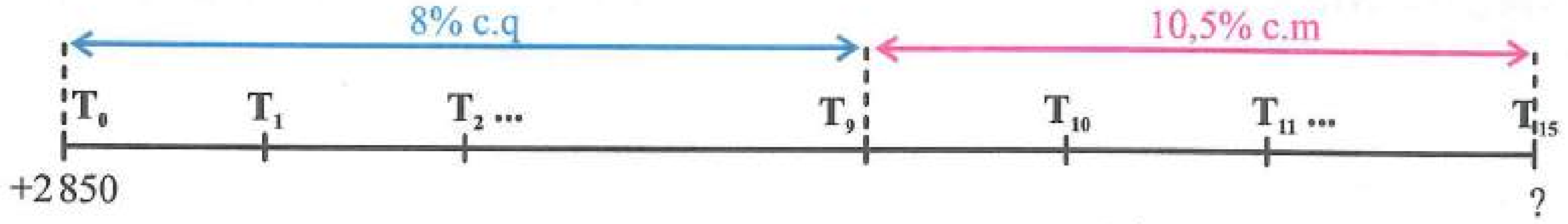
$$\therefore \frac{i_{new}}{4}=\sqrt[4]{\left(1+\frac{0,12}{12}\right)^{12}}-1$$

$$\therefore i_{new}=0,1212=12,12\% \text{ p.a. compounded quarterly}$$

TIME LINES

It is useful to draw a timeline for problems where there is a change to the interest rate as well as problems where there are additional withdrawals and/or deposits which are not made on a regular basis.

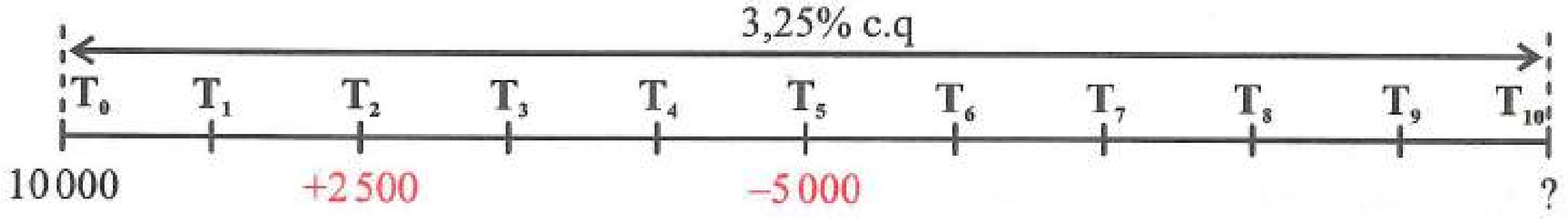
Example 1: R2 850 is invested at 8% p.a. compounded quarterly for 15 years. After 9 years the interest rate changes to 10,5% p.a. compounded monthly. How much is the investment worth after 15 years?



$$\therefore A = 2850\left(1+\frac{0,08}{4}\right)^{9\times4}\left(1+\frac{0,105}{12}\right)^{6\times12} \rightarrow \text{10,5\% c.m is earned for 6 years}$$

$$= R10885,95$$

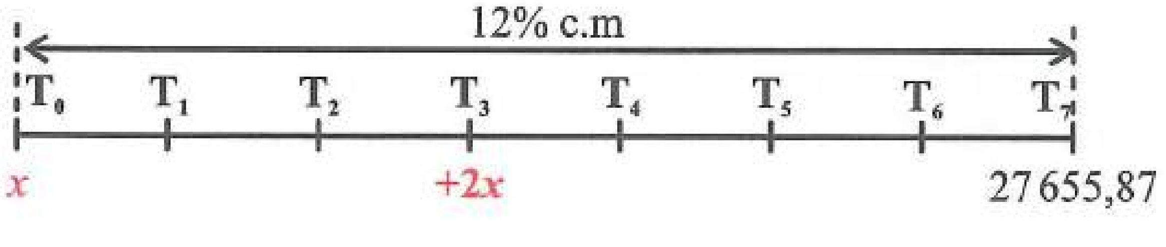
Example 2: Ted invests R10 000 into an account that offers an interest rate of 3,25% p.a. compounded quarterly. After 2 years he deposits an additional R2 500 into the account and 3 years later withdraws R5 000. How much will he have in his account after 10 years?



$$A = 10000\left(1+\frac{0,0325}{4}\right)^{10\times4} + 2500\left(1+\frac{0,0325}{4}\right)^{8\times4} - 5000\left(1+\frac{0,0325}{4}\right)^{5\times4}.$$

$$= R11\,182,68$$

Example 3: x Rand is invested into an account offering an interest rate of 12% p.a. compounded monthly. 3 years later $2x$ Rand is deposited into the account. After 7 years there is R27 655,87 in the account. Determine the value of x.



$$27\,655,87 = x\left(1+\frac{0,12}{12}\right)^{7\times12} + 2x\left(1+\frac{0,12}{12}\right)^{4\times12} \rightarrow \text{take out a HCF of } x$$

$$\therefore 27\,655,87 = x\left[\left(1+\frac{0,12}{12}\right)^{7\times12} + 2\left(1+\frac{0,12}{12}\right)^{4\times12}\right]$$

$$\therefore x = \frac{27\,655,87}{\left[\left(1+\frac{0,12}{12}\right)^{7\times12} + 2\left(1+\frac{0,12}{12}\right)^{12\times4}\right]}$$

$$\therefore x = R5000$$

USING LOGARITHMS TO DETERMINE THE TIME PERIOD

In Grade 12 you will be expected use logarithms to determine the period of an investment or a loan.

Example 1: **An investment of R25 000 grows to an amount of R55 267,04. If the account offers an interest rate of 12% per annum compounded annually, for how long was the money invested?**

Method 1: Using the definition of the logarithm

$A = P(1+i)^n$

$\therefore 55\,267{,}04 = 25\,000(1+0{,}12)^n \rightarrow$ divide both sides by 25 000

$\therefore 2{,}2106816 = (1+0{,}12)^n \rightarrow$ do not round off during calculations

$\therefore n = \log_{(1+0{,}12)}(2{,}2106816)$

$\therefore n = 7$ years → as interest is compounded annually n is in years

Method 2: Using the laws of logarithms

$A = P(1+i)^n$

$\therefore 55\,267{,}04 = 25\,000(1+0{,}12)^n \rightarrow$ divide both sides by 25 000

$\therefore 2{,}2106816 = (1+0{,}12)^n \rightarrow$ 'log' both sides

$\therefore \log(2{,}2106816) = \log(1+0{,}12)^n \rightarrow$ use log law 3 on page 43

$\therefore \log(2{,}2106816) = n\log(1+0{,}12)$

$$\therefore n = \frac{\log(2{,}2106816)}{\log(1+0{,}12)}$$

$\therefore n = 7$ years → as interest is compounded annually n is in years

Method 1 above is recommended and will be used throughout the remainder of this textbook. However, you may choose whichever method you prefer. As an exercise, complete example 2 using method 2 as well.

Example 2: **A motor vehicle which costs R250 000 depreciates at rate of 3,8% per annum compounded monthly. How long will it take for the vehicle to be worth R206 677,47?**

$A = P(1-i)^n$

$\therefore 206\,677{,}47 = 250\,000\left(1-\frac{0{,}038}{12}\right)^n \rightarrow$ divide both sides by 250 000

$\therefore 0{,}82670988 = \left(1-\frac{0{,}038}{12}\right)^n \rightarrow$ do not round off during calculations

$\therefore n = \log_{\left(1-\frac{0{,}038}{12}\right)}(0{,}82670988)$

$\therefore n = 59{,}999\ldots$ months → as interest is compounded monthly n is in months

$\therefore n = 5$ years

B. FUTURE VALUE AND PRESENT VALUE

In Grade 12 you will learn two new concepts in financial mathematics:

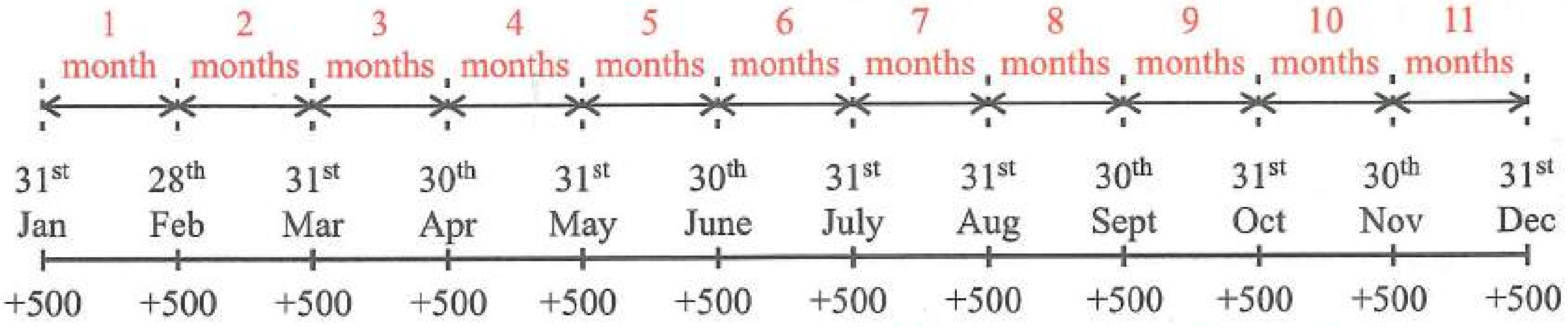
1. **Future Value Calculations**
2. **Present Value Calculations**

1. FUTURE VALUE CALCULATIONS

Future value calculations are used when a certain amount of money is placed into an account on a regular basis in order to save money.
Therefore, future value calculations are used for:

- **Retirement annuities**
- **Savings accounts (where equal deposits are made at regular set intervals)**
- **Sinking funds**

Example 1: Equal payments of R500 are made into a savings account at the end of every month for 1 year. If the account offers an interest rate of 5,5% per annum compounded monthly, how much will have accumulated in the account after 1 year?


1 month
2 months
3 months
4 months
5 months
6 months
7 months
8 months
9 months
10 months
11 months
31st Jan
28th Feb
31st Mar
30th Apr
31st May
30th June
31st July
31st Aug
30th Sept
31st Oct
30th Nov
31st Dec
+500
+500
+500
+500
+500
+500
+500
+500
+500
+500
+500
+500

It is important to realise that the above scenario forms a geometric series:

- The last deposit on the 31st of December will earn no interest
- The 11th deposit on the 30th November will earn interest for 1 month
- The 10th deposit on the 31st of October will earn interest for 2 months
- This pattern continues until you arrive at the 1st deposit which earns interest for 11 months.

The geometric series is represented below starting from the **last deposit**. Interest for each term is calculated using the formula $A = P(1+i)^n$:

$$\underbrace{500}_{\text{last deposit}} + \underbrace{500\left(1+\frac{0,055}{12}\right)^{1}}_{11^{th}\text{ deposit}} + \underbrace{500\left(1+\frac{0,055}{12}\right)^{2}}_{10^{th}\text{ deposit}} + \ldots + \underbrace{500\left(1+\frac{0,055}{12}\right)^{11}}_{1^{st}\text{ deposit}}$$

This forms a geometric series with 12 terms and $r = \left(1+\frac{0,055}{12}\right)$ and $a = 500$.

$$S_n = \frac{a\left[r^n - 1\right]}{r-1}$$

$$\therefore S_{12} = \frac{500\left[\left(1+\frac{0,055}{12}\right)^{12} - 1\right]}{1+\frac{0,055}{12}-1}$$

$$\therefore S_{12} = \frac{500\left[\left(1+\frac{0,055}{12}\right)^{12} - 1\right]}{\frac{0,055}{12}} = \text{R}6153,58$$

DERIVATION OF THE FUTURE VALUE FORMULA

In order to derive a formula which can be used for problems when deposits are made on a regular basis, let us look at the case where x Rand is deposited n times into a savings account at the end of every period offering an interest rate of i. This will form the geometric series shown below starting with the last deposit.

$$x+x(1+i)^1+x(1+i)^2+x(1+i)^3+x(1+i)^4+\ldots+x(1+i)^{n-1}+x(1+i)^n$$

This forms a geometric series with n terms and $r=(1+i)$ and $a=x$.

$$S_n=\frac{a\left[r^n-1\right]}{r-1}$$

$$\therefore S_n=\frac{x\left[(1+i)^n-1\right]}{1+i-1}$$

$$\therefore S_n=\frac{x\left[(1+i)^n-1\right]}{i}$$

The derivation of the future value formula is not examinable.

Formula:

$$F=\frac{x\left[(1+i)^n-1\right]}{i}$$

Where: F → Future value

x → Value of the instalment (usually paid monthly)

i → Interest rate (not in a percentage)

n → Number of payments (deposits) / number of compounding periods

It is important to take note that the above future value formula can only be used if payments (deposits) are made at the end of a period and end after n periods. Unless otherwise stated, you may assume that payments begin at the end of a period.

Example 1: A company will require a new delivery truck in 3 years which is expected to cost R250 000. In order to ensure that the company has enough money to purchase the truck, they start a sinking fund which earns an interest rate of 7,5% per annum compounded monthly. Determine the amount that the company must invest every month in order to ensure that they can purchase the new truck.

$$F=\frac{x\left[(1+i)^n-1\right]}{i}$$

$$\therefore 250000=\frac{x\left[\left(1+\frac{0,075}{12}\right)^{3\times12}-1\right]}{\frac{0,075}{12}}$$

$$\therefore x=250000\div\frac{\left(1+\frac{0,075}{12}\right)^{3\times12}-1}{\frac{0,075}{12}}$$

$$\therefore x=\text{R}6\,214{,}05$$

A sinking fund is a savings account which is set up in order to save money to replace an item sometime in the future.

Do not round off during financial mathematics calculations. You may only round off at the end of each question.

Example 2: A company purchases a new vehicle for R200 000. The vehicle is expected to depreciate at a rate of 24% p.a. on a reducing balance. It is expected that the vehicle will have to be replaced after 5 years. A sinking fund is set up for this purpose. If the replacement cost of the vehicle increases by 18% p.a. compounded annually, calculate:

a) the value of the current vehicle after 5 years
b) the cost of a new vehicle after 5 years
c) the total required value of the sinking fund, if the old vehicle is sold and the proceeds contribute towards the new vehicle
d) the monthly instalments paid into the sinking fund if the interest rate is 15% p.a. compounded monthly and payments start 1 month after the vehicle is initially purchased.

a) Reducing balance:

$$A = P(1-i)^n$$

$$= 200\,000(1-0,24)^5$$

$$= 50\,710,51$$

$\therefore$ The vehicle will be worth R50 710,51 after 5 years

b) Compound interest:

$$A = P(1+i)^n$$

$$= 200\,000(1+0,18)^5$$

$$= 457\,551,55$$

$\therefore$ The cost of a new vehicle will be R457 551,55

c) Sinking fund value = cost of the new vehicle − value of the old vehicle

$$= 457\,551,55 - 50\,710,51$$

$$= 406\,841,04$$

$\therefore$ The value of the sinking fund has to be R406 841,04

d) $F = \dfrac{x\left[(1+i)^n - 1\right]}{i}$

$$\therefore\ 406\,841,04 = \frac{x\left[\left(1+\frac{0,15}{12}\right)^{60} - 1\right]}{\frac{0,15}{12}}$$ → there are 60 deposits made in 5 years

$$\therefore\ 406\,841,04\left(\frac{0,15}{12}\right) = x\left[\left(1+\frac{0,15}{12}\right)^{60} - 1\right]$$

$$\therefore\ x = \frac{406\,841,04\left(\frac{0,15}{12}\right)}{\left[\left(1+\frac{0,15}{12}\right)^{60} - 1\right]}$$

$\therefore\ x = 4\,593,21$

$\therefore$ The monthly instalments are R4 593,21

Example 3: Harry wishes to start saving for his retirement. He estimates that he will need R4 000 000 in order to retire comfortably. He invests R5 000 a month at an interest rate of 13% per annum, compounded monthly. How many years will he have to wait to retire?

$$\text{Investment}: F=\frac{x\left[(1+i)^n-1\right]}{i}$$

$$\therefore 4\,000\,000=\frac{5000\left[\left(1+\frac{0,13}{12}\right)^n-1\right]}{\frac{0,13}{12}}\rightarrow \text{interest is compounded monthly}$$

$$\therefore 4\,000\,000\left(\frac{0,13}{12}\right)=5000\left[\left(1+\frac{0,13}{12}\right)^n-1\right]$$

$$\therefore \frac{130\,000}{3}=5000\left[\left(1+\frac{0,13}{12}\right)^n-1\right]\rightarrow 4\,000\,000\left(\frac{0,13}{12}\right)=\frac{520\,000}{12}=\frac{130\,000}{3}$$

$$\therefore \frac{26}{3}=\left(1+\frac{0,13}{12}\right)^n-1$$

$$\therefore \frac{29}{3}=\left(1+\frac{0,13}{12}\right)^n\rightarrow \text{convert from exponential to log form}$$

$$\therefore n=\log_{\left(1+\frac{0,13}{12}\right)}\left(\frac{29}{3}\right)\rightarrow \text{if } y=x^n \text{ then } n=\log_x(y)$$

$$\therefore n=210,55\rightarrow \text{number of payments}$$

$$\therefore \text{number of years until he can retire}=\frac{210,55}{12}=17,54\approx 18 \text{ years}$$

Example 4: Solomon invests R800 into a savings account every month. If the annual interest rate is 12,5% compounded monthly, how much will he have after 6 years?

$$\text{Investment}: F=\frac{x\left[(1+i)^n-1\right]}{i}$$

$$\therefore F=\frac{800\left[\left(1+\frac{0,125}{12}\right)^{72}-1\right]}{\frac{0,125}{12}}\rightarrow \text{in 6 years 72 payments are made}$$

$$\therefore F=85\ 156,10$$

$\therefore$ After 6 years he will have R 85 156,10

Example 5: **Ndumi starts a 5 year savings plan. He deposits R5 000 into a savings account at the end of every month. The interest rate on the account is 6% per annum compounded quarterly.**

a) Determine the nominal monthly interest rate which Ndumi receives.
b) Determine the amount Ndumi will receive at the end of the 5 year period.

a) $(1+i_{new})^m = (1+i_{nom})^n$

$$\left(1+\frac{i_{new}}{12}\right)^{12} = \left(1+\frac{0,06}{4}\right)^{4} \rightarrow \text{quarterly to monthly}$$

$$\therefore 1+\frac{i_{new}}{12} = \sqrt[12]{\left(1+\frac{0,06}{4}\right)^{4}} \rightarrow 12^{\text{th}} \text{ root both sides}$$

$$\therefore \frac{i_{new}}{12} = \sqrt[12]{\left(1+\frac{0,06}{4}\right)^{4}} - 1$$

$$\therefore i_{new} = 0,0597$$

$$\therefore i_{new} = 5,97\% \text{ p.a. compounded monthly}$$

b) As deposits are being made monthly you have to use the monthly interest rate:

$$\text{Investment}: F = \frac{x\left[(1+i)^n - 1\right]}{i}$$

$$\therefore F = \frac{5000\left[\left(1+\frac{0,0597}{12}\right)^{60} - 1\right]}{\frac{0,0597}{12}} \rightarrow \text{use the monthly rate, 60 payments are made}$$

$$\therefore F = \text{R}348\,581,33$$

$\therefore$ After 5 years he will have R348 581,33

Interest has to be compounded at the same rate as the payments.

Example 6: **Ahmed deposits R2 000 into a savings account at the end of every quarter for 16 years. The account offers an interest rate of 8,5% p.a. compounded quarterly. Due to financial difficulty he is unable to make his last two payments. How much will he have in his account after 16 years?**

In 16 years 64 deposits should be made. However, the last 2 deposits are missed and only 62 deposits are made.

$$\text{Investment}: F=\frac{x\left[(1+i)^n-1\right]}{i}$$

$$\therefore F=\frac{2000\left[\left(1+\frac{0,085}{4}\right)^{62}-1\right]}{\frac{0,085}{4}} \rightarrow \text{2 deposits are missed}$$

$$\therefore F=\text{R}252\,506,9461 \rightarrow \text{this is the amount in the account after 62 deposits}$$

This amount will still accumulate interest for 2 quarters:

$$\therefore A=P(1+i)^n$$

$$\therefore A=252\,506,9461\left(1+\frac{0,085}{4}\right)^{2} \rightarrow \text{2 quarters}$$

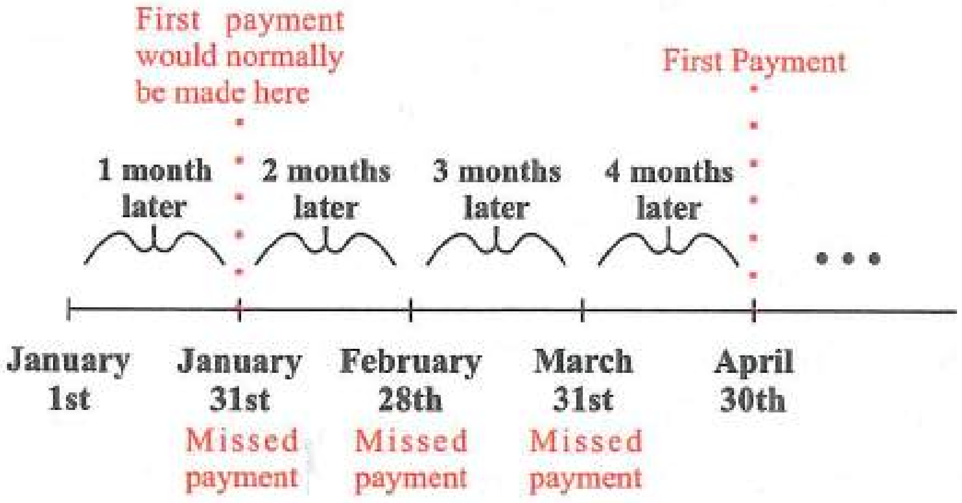
$$\therefore A=\text{R}263\,352,51$$

$\therefore$ He will have R263 352,51 after 16 years

Do not round off during calculations in financial mathematics.

Example 7: At the beginning of January a factory purchases a new delivery truck which they expect to replace in 5 years' time. In order to replace the truck the company will need R200 000 after trading in the current vehicle. What monthly payments must be made into a sinking fund which provides an interest rate of 6,25% p.a. compounded monthly, if payments start 4 months after the original vehicle is initially purchased?

In normal sinking fund calculations, the first payment is made one month after the purchase of the vehicle. In this case payments start 4 months later, which is at the end of April (beginning of May). It can be seen from the timeline below that **3 payments are missed** as payments start when the 4th payment would usually be made.


First payment would normally be made here
First Payment
1 month later
2 months later
3 months later
4 months later
January 1st
January 31st
Missed payment
February 28th
Missed payment
March 31st
Missed payment
April 30th

$$\text{Sinking Fund}: F=\frac{x\left[(1+i)^n-1\right]}{i}$$

$$\therefore 200000=\frac{x\left[\left(1+\frac{0,0625}{12}\right)^{60-3}-1\right]}{\frac{0,0625}{12}}\rightarrow \text{3 payments are missed}$$

$$\therefore\ 200000\left(\frac{0,0625}{12}\right)=x\left[\left(1+\frac{0,0625}{12}\right)^{57}-1\right]$$

$$\therefore x=200000\left(\frac{0,0625}{12}\right)\div\left[\left(1+\frac{0,0625}{12}\right)^{57}-1\right]$$

$$\therefore x=3022,73$$

$\therefore$ The monthly payments are R3022,73 per month

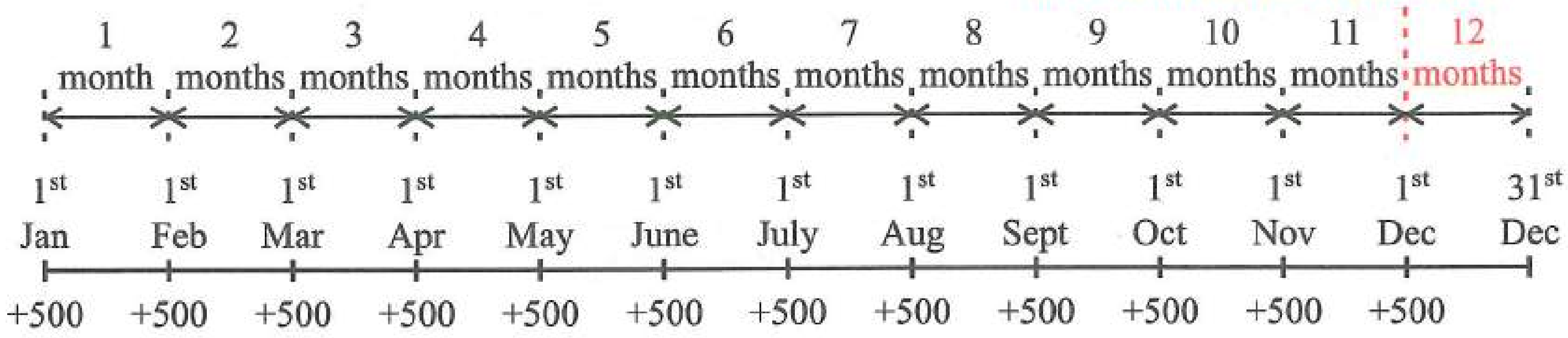
Complete Exercise 1 on page 85.

HANDLING PAYMENTS MADE AT THE BEGINNING OF THE PERIOD

As you have seen, the future value formula only applies to cases where payments are made at the end of a period. Should the first payment be made at the beginning of a period, the formula has to be adjusted to take this change into account.

Example 1: **An amount of R500 is deposited at the beginning of every month starting on the 1st of January and ending on the 1st of December. Determine the amount that will be in the account at the end of the year if the account earns an interest rate of 10% per annum compounded monthly.**

Using $n=12$ will give the amount in the account on the 1st of December


1 month
2 months
3 months
4 months
5 months
6 months
7 months
8 months
9 months
10 months
11 months
12 months
1st Jan
1st Feb
1st Mar
1st Apr
1st May
1st June
1st July
1st Aug
1st Sept
1st Oct
1st Nov
1st Dec
31st Dec
+500
+500
+500
+500
+500
+500
+500
+500
+500
+500
+500
+500

As payments are made at the beginning of the month, using $n = 12$ in the future value formula will provide the amount in the account on the 1st of December. However, you are asked for the value in the account at the end of December.
This means that you have to add one month's interest to the value in the account on the 1st of December.

$$\text{Investment}: F=\frac{x\left[(1+i)^n-1\right]}{i}$$

$$\therefore F=\frac{500\left[\left(1+\frac{0,1}{12}\right)^{12}-1\right]}{\frac{0,1}{12}} \rightarrow 12 \text{ deposits are made}$$

$\therefore F=\text{R}6\,282,784046 \rightarrow$ this is the amount in the account on the 1st of December

This amount will still accumulate interest for 1 month:

$A=P(1+i)^n \rightarrow P$ is the amount in the account on the 1st of December

$$\therefore A=6\,282,784046\left(1+\frac{0,1}{12}\right)^1 \rightarrow 1 \text{ month}$$

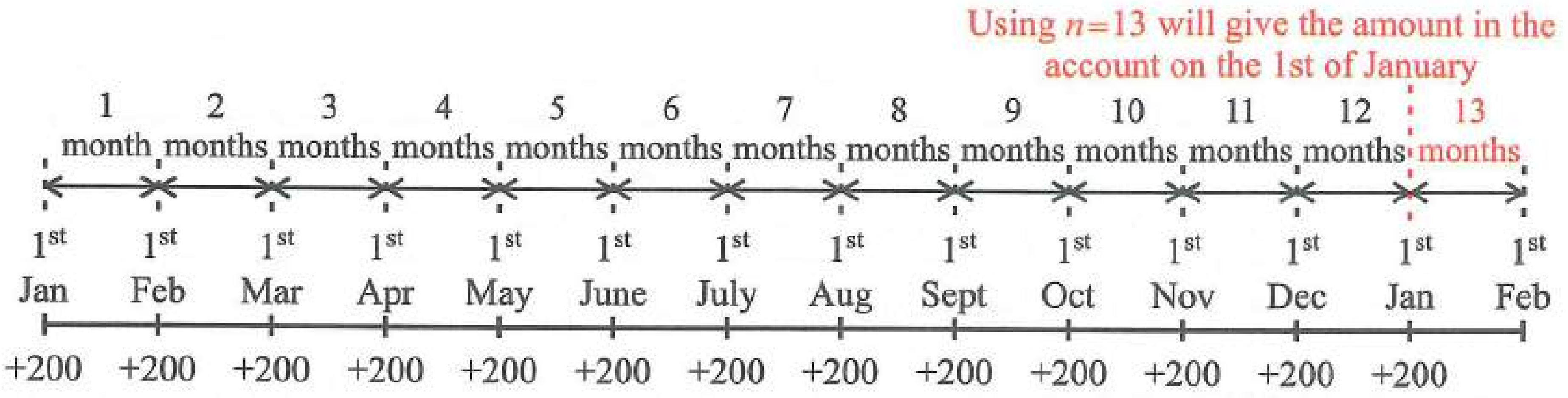
$\therefore A=\text{R}6335,14$

This can be combined into one step as shown below:

$$F=\frac{500\left[\left(1+\frac{0,1}{12}\right)^{12}-1\right]}{\frac{0,1}{12}}\times\left(1+\frac{0,1}{12}\right)=\text{R}6335,14$$

In the future value formula, n is the number of payments that are made. Interest is earned for 1 period less than the number of payments. This is because in the formula, it is assumed that no interest is earned on the last payment.

Example 2: Arshad's birthday is on the 1st of January. On the day he turns 20 he starts to save for his 21st birthday party by placing R200 into a savings account every month with his last payment being made on his 21st birthday. How much money will Arshad have for his party, if the account promises an interest rate of 4,5% per annum compounded monthly and his party is to be held on the 1st of February?


Using $n=13$ will give the amount in the account on the 1st of January

1 month
2 months
3 months
4 months
5 months
6 months
7 months
8 months
9 months
10 months
11 months
12 months
13 months
1st Jan
1st Feb
1st Mar
1st Apr
1st May
1st June
1st July
1st Aug
1st Sept
1st Oct
1st Nov
1st Dec
1st Jan
1st Feb
+200
+200
+200
+200
+200
+200
+200
+200
+200
+200
+200
+200
+200

The 1st deposit is made on the 1st of January and the last deposit is made on the 1st of January the next year. This means 13 payments are made. However, the money was left in the account for 1 extra month which has to be added to the amount in the account on 1st of January.

$$\text{Investment}: F=\frac{x\left[(1+i)^n-1\right]}{i}$$

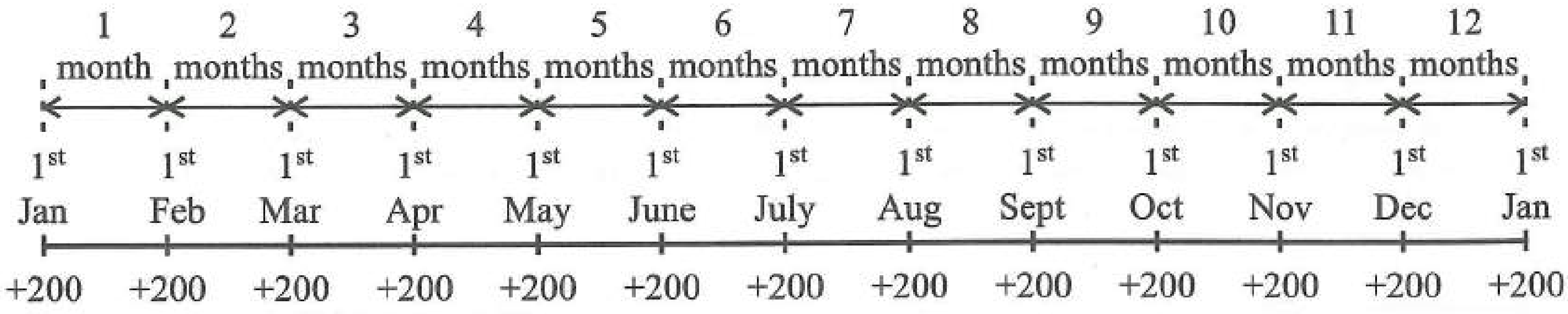
$$F=\frac{x\left[(1+i)^n-1\right]}{i}\times(1+i)^1$$

$$\therefore F=\frac{200\left[\left(1+\frac{0{,}045}{12}\right)^{13}-1\right]}{\frac{0{,}045}{12}}\times\left(1+\frac{0{,}045}{12}\right)^1$$

$$\therefore F=\text{R}\,2\,669{,}28$$

By using $n=13$ you are calculating the amount that would be in the account on the 1st of January (13 payments 12 months interest). As you are looking for the amount that would be in the account on the 1st of February you have to add on 1 month's interest. This is achieved by multiplying the final amount by $(1+i)^1$.

Example 3: **Sheryl's birthday is on the 1st of January. On the day she turns 20 she starts to save for her 21st birthday party by placing R200 into a savings account every month with her last payment being made on her 21st birthday. How much money will Sheryl have for her party on her birthday, if the account promises an interest rate of 4,5% per annum compounded monthly?**


1 month
2 months
3 months
4 months
5 months
6 months
7 months
8 months
9 months
10 months
11 months
12 months
1st Jan
1st Feb
1st Mar
1st Apr
1st May
1st June
1st July
1st Aug
1st Sept
1st Oct
1st Nov
1st Dec
1st Jan
+200
+200
+200
+200
+200
+200
+200
+200
+200
+200
+200
+200
+200

In this case, 13 payments are made and interest is earned for 12 months. This is a normal future value calculation for an investment of 13 months.

$$\text{Investment}: F = \frac{x\left[(1+i)^n - 1\right]}{i}$$

$$\therefore F = \frac{200\left[\left(1+\frac{0,045}{12}\right)^{13} - 1\right]}{\frac{0,045}{12}}$$

$$\therefore F = \text{R}\,2\,659,31$$

Complete Exercise 2 on pages 85 and 86.

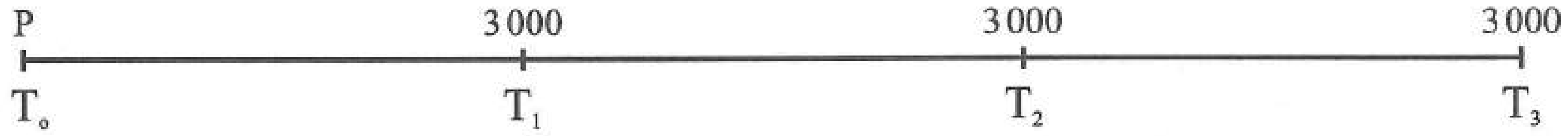
2. PRESENT VALUE CALCULATIONS

Present value calculations are used when payments are made on a regular basis in order to pay back a loan. In these calculations the money is first received and then paid back. Therefore, present value calculations are used for:

- **Loans**
- **Bond repayments**

DERIVATION OF THE PRESENT VALUE FORMULA

Illustrative Example: **A loan is paid off by 3 equal payments of R3 000 over a period of 3 years. Determine the original amount of the loan if interest is charged at a rate of 10% per annum compounded annually.**



Consider the payment made at T_1:

- A portion of this payment is used to pay off the original loan and a portion of this payment is used to pay off the interest on the loan for 1 year.
- In order to determine the amount of the original loan, you have to determine the amount that each payment represents without interest. In other words you have to calculate the value of each payment at T_0. The original value of the loan will be the sum of the payments without interest.

For the payment at T_1: $A = P(1+i)^n$

$$\therefore 3000 = P(1+0{,}1)^1$$

$$\therefore P = \frac{3000}{(1+0{,}1)^1}$$

$\therefore P = 3000(1+0{,}1)^{-1} \rightarrow$ this is the value of the first payment without interest

For the payment at T_2: $A = P(1+i)^n$

$$\therefore 3000 = P(1+0{,}1)^2$$

$$\therefore P = \frac{3000}{(1+0{,}1)^2}$$

$\therefore P = 3000(1+0{,}1)^{-2} \rightarrow$ this is the value of the second payment without interest

For the payment at T_3: $A = P(1+i)^n$

$$\therefore 3000 = P(1+0{,}1)^3$$

$$\therefore P = \frac{3000}{(1+0{,}1)^3}$$

$\therefore P = 3000(1+0{,}1)^{-3} \rightarrow$ this is the value of the third payment without interest

$\therefore$ The total value of the loan is:

$$P = 3000(1+0{,}1)^{-1} + 3000(1+0{,}1)^{-2} + 3000(1+0{,}1)^{-3}$$

$$\therefore P = \text{R}7\,460{,}56$$

In order to derive a general formula that can be used for present value annuities, we will apply the same principle that was used in the illustrative example on the previous page. Assume a loan of P is taken out at an interest rate of i. The loan is paid off by n equal payments of x Rand.

The derivation of the present value formula is not examinable.

From the example on the previous page it can be seen that:

$$P = x(1+i)^{-1} + x(1+i)^{-2} + x(1+i)^{-3} + \ldots + x(1+i)^{-n}$$

The right hand side of the equation forms a geometric series with:

$a = x(1+i)^{-1}$ **and** $r = (1+i)^{-1}$

$$S_n = \frac{a\left(r^n - 1\right)}{r-1}$$

$$\therefore P = \frac{x(1+i)^{-1}\left[(1+i)^{-n}-1\right]}{(1+i)^{-1}-1} \rightarrow x(1+i)^{-1} = \frac{x}{(1+i)^1}$$

$$\therefore P = \frac{x\left[(1+i)^{-n}-1\right]}{(1+i)^1\left[(1+i)^{-1}-1\right]}$$

$$\therefore P = \frac{x\left[(1+i)^{-n}-1\right]}{(1+i)^1(1+i)^{-1}-1(1+i)^1} \rightarrow (1+i)^1(1+i)^{-1} = \frac{1+i}{1+i} = 1$$

$$\therefore P = \frac{x\left[(1+i)^{-n}-1\right]}{1-1-i}$$

$$\therefore P = \frac{x\left[(1+i)^{-n}-1\right]}{-i}$$

$$\therefore P = \frac{x\left[1-(1+i)^{-n}\right]}{i}$$

Formula:

$$P = \frac{x\left[1-(1+i)^{-n}\right]}{i}$$

Where: $P \rightarrow$ **Present value \ Outstanding Balance**

$x \rightarrow$ **Value of the instalment (usually paid monthly)**

$i \rightarrow$ **Interest rate (not in a percentage)**

$n \rightarrow$ **Number of payments (deposits) that still have to be made**

When using the present value formula take note that:

- **P represents the outstanding balance on a loan with n payments still to go.**
- **n is always the number of payments left on the loan.**
- **Payments always start 1 period after a loan is granted.**

Example 1: **Simon wishes to purchase a car for R80 000. If he borrows the money at an interest rate of 8,5% per annum compounded monthly, calculate:**

a) The monthly instalments if the loan is to be paid back over 5 years.
b) The outstanding balance on the loan after 3 years.
c) The amount that would be saved by settling the loan after 3 years.

a) Loan Repayment: $P=\dfrac{x\left[1-(1+i)^{-n}\right]}{i}$

$$\therefore 80\,000=\frac{x\left[1-\left(1+\frac{0{,}085}{12}\right)^{-60}\right]}{\frac{0{,}085}{12}}$$

$\rightarrow$ interest is compounded monthly

60 payments are made in 5 years

$$\therefore 80\,000\left(\frac{0{,}085}{12}\right)=x\left[1-\left(1+\frac{0{,}085}{12}\right)^{-60}\right]$$

$\rightarrow$ **do not** round off

$$\therefore x=\frac{80\,000\left(\frac{0{,}085}{12}\right)}{1-\left(1+\frac{0{,}085}{12}\right)^{-60}}$$

$\therefore x=\text{R}1\,641{,}32$

$\therefore$ The monthly instalments would be R1641,32

b) Outstanding Balance on a loan: $P=\dfrac{x\left[1-(1+i)^{-n}\right]}{i}$

$$P=\frac{1641{,}32\left[1-\left(1+\frac{0{,}085}{12}\right)^{-24}\right]}{\frac{0{,}085}{12}}$$

$\rightarrow$ answer from (a),

there are 24 payments to be made after 3 years

$\therefore P=\text{R}36\,108{,}14$

$\therefore$ The outstanding balance after 3 years is R36 108,14

c) **If the loan was not settled after 3 years there would still be 24 payments of R1 641,32 still to be made. Therefore, the total paid in the final 2 years is:**

Total paid in the final 2 years = 1641,32 × 24 = R39 391,68

By settling the loan after 3 years Simon would only pay R36 108,14.

Saving = the amount that would have been paid − outstanding balance

$\therefore \text{Saving}=39\,391{,}68-36\,108{,}14$

$\therefore \text{Saving}=\text{R}3\,283{,}54$

The amount saved is the interest on the loan in the final 2 years.

Example 2: **A loan of R50 000 is to be paid off by equal quarterly payments over 10 years. If interest is charged at a rate of 10% per annum compounded quarterly, determine:**

a) The value of the repayments.

b) The outstanding balance of the loan after 7 years.

a) Loan Repayment: $P=\dfrac{x\left[1-(1+i)^{-n}\right]}{i}$

$$\therefore 50\,000=\frac{x\left[1-\left(1+\frac{0,1}{4}\right)^{-40}\right]}{\frac{0,1}{4}} \rightarrow \text{interest is compounded quarterly, 40 payments are made in 10 years}$$

$$\therefore 50\,000\left(\frac{0,1}{4}\right)=x\left[1-\left(1+\frac{0,1}{4}\right)^{-40}\right] \rightarrow \textbf{do not} \text{ round off}$$

$$\therefore x=\frac{50\,000\left(\frac{0,1}{4}\right)}{1-\left(1+\frac{0,1}{4}\right)^{-40}}$$

$\therefore x=\text{R}1\,991,81$

$\therefore$ The quarterly payments are R1991,81

b) Outstanding Balance on a loan: $P=\dfrac{x\left[1-(1+i)^{-n}\right]}{i}$

$$P=\frac{1991,81\left[1-\left(1+\frac{0,1}{4}\right)^{-12}\right]}{\frac{0,1}{4}} \rightarrow \text{answer from (a), there are 12 payments to be made after 7 years}$$

$\therefore P=\text{R}20\,431,52$

$\therefore$ The outstanding balance after 7 years is R20 431,52

Example 3: **Thabo wishes to purchase a fridge for R6 000. The fridge can either be paid off over a period of 6 months at an interest rate of 12% p.a. compounded monthly, or he can pay cash and receive a 12,5% discount. Determine:**

a) The cash price.
b) The monthly instalments on the loan.
c) The saving if cash is paid.

a) $\text{Cash price} = 6\,000 \times 0,875$ or $\text{Cash price} = 6\,000 - (6\,000 \times 0,125)$

$= \text{R}5\,250$ $\qquad$ $= 6000 - 750$

$= \text{R}5\,250$

b) Loan Repayment: $P = \dfrac{x\left[1-(1+i)^{-n}\right]}{i}$

$$\therefore 6000 = \frac{x\left[1-\left(1+\frac{0,12}{12}\right)^{-6}\right]}{\frac{0,12}{12}} \rightarrow \text{interest is compounded monthly, 6 payments are made in 6 months}$$

$$\therefore 6\,000\left(\frac{0,12}{12}\right) = x\left[1-\left(1+\frac{0,12}{12}\right)^{-6}\right] \rightarrow \textbf{do not} \text{ round off}$$

$$\therefore x = \frac{6\,000\left(\frac{0,12}{12}\right)}{1-\left(1+\frac{0,12}{12}\right)^{-6}}$$

$\therefore x = \text{R}1\,035,29$

$\therefore$ The monthly instalments are R1 035, 29

c) Saving = the amount paid with interest – cash price

$\therefore \text{Saving} = (1035,29 \times 6) - 5250$

$\therefore \text{Saving} = \text{R}961,74$

ALTERNATIVE METHOD OF CALCULATING THE BALANCE OUTSTANDING ON A LOAN

Up until now you have only calculated the balance outstanding using the present value formula. It is also possible to calculate the balance outstanding on a loan using the future value formula. This method works by calculating the value of the loan (including interest) up to the point which you are looking for the outstanding balance, less the total payments that would have been made up until this point (including interest).

Outstanding Balance on a Loan using the Future Value Formula:

$$\text{Outstanding Balance on a Loan} = P(1+i)^n - \frac{x\left[(1+i)^n - 1\right]}{i}$$

Example 1: A loan of R60 000 is paid off over a period of 5 years by equal monthly payments at an interest rate of 9,5% per annum compounded monthly. Determine the balance outstanding on the loan after 3 years.

You have to determine monthly repayments before calculating the outstanding balance on the loan:

$$P = \frac{x\left[1-(1+i)^{-n}\right]}{i}$$

$$\therefore 60000 = \frac{x\left[1-\left(1+\frac{0,095}{12}\right)^{-60}\right]}{\frac{0,095}{12}}$$

$$\therefore x = \text{R}1260,111679$$

You can now use the future value formula to determine the balance outstanding on the loan:

$$\text{Outstanding Balance on a Loan} = P(1+i)^n - \frac{x\left[(1+i)^n - 1\right]}{i}$$

$$= 60000\left(1+\frac{0,095}{12}\right)^{36} - \frac{1260,111679\left[\left(1+\frac{0,095}{12}\right)^{36} - 1\right]}{\frac{0,095}{12}}$$

$$= \text{R}27\,444,75$$

Example 2: **A loan of R120 000 is paid off over a period of 8 years by equal monthly payments at an interest rate of 8% per annum compounded monthly. Determine the balance outstanding on the loan after 6 years.**

You have to determine monthly repayments before calculating the outstanding balance on the loan:

$$P=\frac{x\left[1-(1+i)^{-n}\right]}{i}$$

$$\therefore 120000=\frac{x\left[1-\left(1+\frac{0,08}{12}\right)^{-96}\right]}{\frac{0,08}{12}}$$

$$\therefore x=\text{R}1696,401511$$

You can now use the future value formula to determine the balance outstanding on the loan:

$$\text{Outstanding Balance on a Loan}=P(1+i)^n-\frac{x\left[(1+i)^n-1\right]}{i}$$

$$=120000\left(1+\frac{0,08}{12}\right)^{72}-\frac{1696,401511\left[\left(1+\frac{0,08}{12}\right)^{72}-1\right]}{\frac{0,08}{12}}$$

$$=\text{R}37508,36$$

It is important to note that when using the future value formula to calculate the balance outstanding on a loan, the value of *n* is the number of payments which have been made. When using the present value formula as you have done previously, the value of *n* is the number of payments that still have to be made.

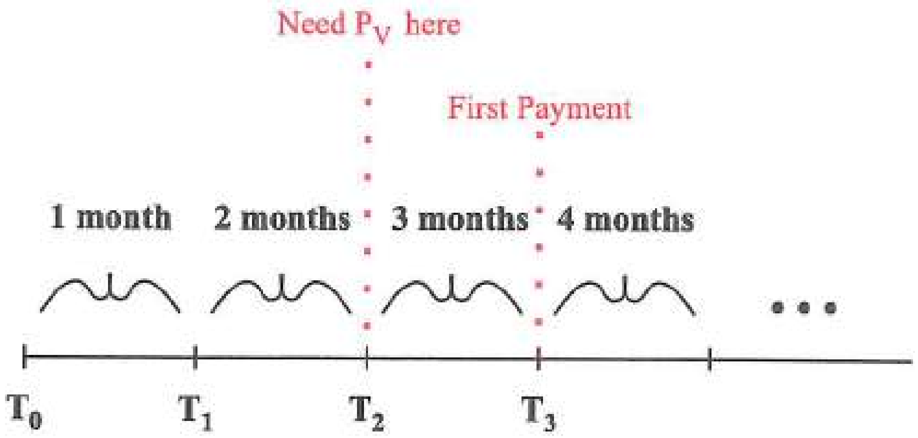
You may use whichever method you prefer to determine the balance outstanding on a loan. In the examples which follow, the present value formula will be used where necessary in order to calculate the balance outstanding on a loan. As an exercise, redo example 1 through to 3 on pages 58 to 60 using the future value formula.

Complete Exercise 3 on page 86.

LOANS WITH DELAYED PAYMENTS (DEFERRED PAYMENTS)

Example 1:

Lerato takes out a loan of R100 000 to buy a car. The loan agreement stipulates that it has to be repaid by means of 238 equal monthly payments at an interest rate of 18% per annum compounded monthly starting 3 months after the loan is granted. Determine Lerato's monthly payments.



When dealing with loans and the present value formula, the 1st payment is usually made **1 month after** the loan is granted. In other words, if a loan is taken out at T_0 the first payment is due at T_1.

In this question the **first payment is made at T_3**, therefore the amount owing on the **loan at T_2** is required to use the present value formula and calculate the monthly payments.

$T_0 - T_2$:

$$A = P(1+i)^n$$

$$\therefore A = 100000\left(1+\frac{0,18}{12}\right)^2$$

$$\therefore A = \text{R}103022,50$$

$T_2 - T_{240}$: → there are 238 payments made

$$P = \frac{x\left[1-(1+i)^{-n}\right]}{i}$$

$$\therefore 103022,5 = \frac{x\left[1-\left(1+\frac{0,18}{12}\right)^{-238}\right]}{\frac{0,18}{12}}$$

$$\therefore x = 103022,5 \div \frac{\left[1-\left(1+\frac{0,18}{12}\right)^{-238}\right]}{\frac{0,18}{12}}$$

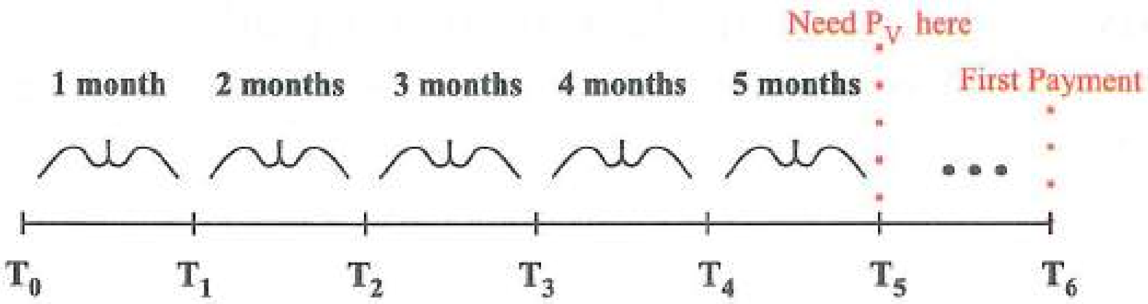
$$\therefore x = \text{R}1591,35$$

$\therefore$ The monthly payments would be R1591,35

Always calculate the value of the loan 1 period before the 1st payment is made.

Example 2:

A loan of R50 000 is repaid over a period of 5 years, from the time that the loan is taken, by equal monthly payments at an interest rate of 5,8% per annum compounded monthly. If payments start 6 months after the loan is granted, what monthly payments are required to repay the loan?


Need P_V here
1 month
2 months
3 months
4 months
5 months
First Payment
T_0
T_1
T_2
T_3
T_4
T_5
T_6

$T_0 - T_5$:

$$A = P(1+i)^n$$

$$\therefore A = 50000\left(1+\frac{0,058}{12}\right)^5$$

$$\therefore A = \text{R}51220{,}07048$$

$T_5 - T_{60}$: → there are 60 months in 5 years and 5 have already elapsed

$$P = \frac{x\left[1-(1+i)^{-n}\right]}{i}$$

$$\therefore 51220{,}07048 = \frac{x\left[1-\left(1+\frac{0,058}{12}\right)^{-55}\right]}{\frac{0,058}{12}}$$ → there are 55 payments to be made

$$\therefore x = 51220{,}07048 \div \frac{1-\left(1+\frac{0,058}{12}\right)^{-55}}{\frac{0,058}{12}}$$

$$\therefore x = \text{R}1062{,}77$$

∴ The monthly payments would be R1062,77

Complete Exercise 4 on page 86.

FINAL PAYMENTS WITH REDUCED VALUES

Example 1:

Candice borrows R100 000 at an interest rate of 10% p.a. compounded monthly. She repays the loan by means of equal monthly payments of R5 500 and a final payment which is less than R5 500.

a) How many months will it take for the loan to be repaid?

b) What will the outstanding balance on the loan be after the 19th payment?

c) What is the final payment?

a) $P=\dfrac{x\left[1-(1+i)^{-n}\right]}{i}$

$$\therefore 100000=\frac{5500\left[1-\left(1+\frac{0,1}{12}\right)^{-n}\right]}{\frac{0,1}{12}} \rightarrow \text{multiply both sides by } \frac{0,1}{12} \text{ and divide by } 5500$$

$$\therefore \frac{5}{33}=1-\left(1+\frac{0,1}{12}\right)^{-n}$$

$$\therefore \left(1+\frac{0,1}{12}\right)^{-n}=1-\frac{5}{33}$$

$$\therefore -n=\log_{\left(1+\frac{0,1}{12}\right)}\left(1-\frac{5}{33}\right)$$

$\therefore n=19,79840405 \rightarrow$ do not round off as you need this decimal for the next question

This means that there will be 19 payments of R5500 and the last payment will be less than R5500. Therefore there will be 20 payments in total.

$\therefore$ It will take 20 months to pay back the loan.

b) $P=\dfrac{x\left[1-(1+i)^{-n}\right]}{i}$

$$P=\frac{5500\left[1-\left(1+\frac{0,1}{12}\right)^{-0,79840405}\right]}{\frac{0,1}{12}} \rightarrow \text{0,79840405 payments are left after the 19}^{\text{th}}\text{ payment}$$

$P=\text{R}4358,57$

$\therefore$ There is R4358,57 outstanding after the 19th payment

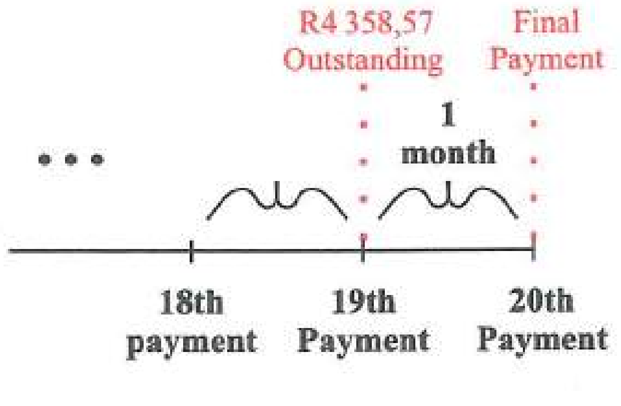
c) After the 19th payment there is R4 358,57 outstanding. The final payment will be made one month later and interest for that month has to be added onto the outstanding balance.

$\therefore P=A(1+i)^{n}$

$$\therefore P=4358,57\left(1+\frac{0,1}{12}\right)^{1} \rightarrow \text{1 month's interest}$$

$\therefore P=\text{R}4394,89$

$\therefore$ The final payment will be R4394,89



Example 2:

Thabo borrows R5 000 from a micro lender at an interest rate of 28% p.a. compounded monthly. He repays the loan by means of equal monthly payments of R800 and a final payment of less than R800.

a) Determine the number of payments at R800?

b) What will the value of the final payment be?

a) Loan Repayment: $P=\dfrac{x\left[1-(1+i)^{-n}\right]}{i}$

$$\therefore 5000=\frac{800\left[1-\left(1+\frac{0,28}{12}\right)^{-n}\right]}{\frac{0,28}{12}} \rightarrow \text{multiply both sides by } \frac{0,28}{12} \text{ and divide by } 800$$

$$\therefore \frac{7}{48}=1-\left(1+\frac{0,28}{12}\right)^{-n}$$

$$\therefore \left(1+\frac{0,28}{12}\right)^{-n}=1-\frac{7}{48}$$

$$\therefore -n=\log_{\left(1+\frac{0,28}{12}\right)}\left(1-\frac{7}{48}\right)$$

$\therefore n=6,834037675 \rightarrow$ do not round off as you need this decimal for the next question

$\therefore n\approx 7$

$\therefore$ There will be **6 payments at R800** as the final payment is less than R800.

b) You cannot calculate the value of the final payment straight away. You first have to calculate the amount outstanding after the 6th payment and then add 1 month's interest as you did in Example 1 on the previous page.

$$P=\frac{x\left[1-(1+i)^{-n}\right]}{i}$$

$$\therefore P=\frac{800\left[1-\left(1+\frac{0,28}{12}\right)^{-0,834037675}\right]}{\frac{0,28}{12}} \rightarrow$$ 0,834037675 payments are left after the 6th payment

$P=\text{R}653,2611593 \rightarrow$ do not round off during a financial maths calculation

$\therefore$ There is R653,2611593 outstanding after the 6th payment

Final Payment: $A=P(1+i)^n$

$$\therefore A=653,2611593\left(1+\frac{0,28}{12}\right)^1 \rightarrow \text{1 month's interest}$$

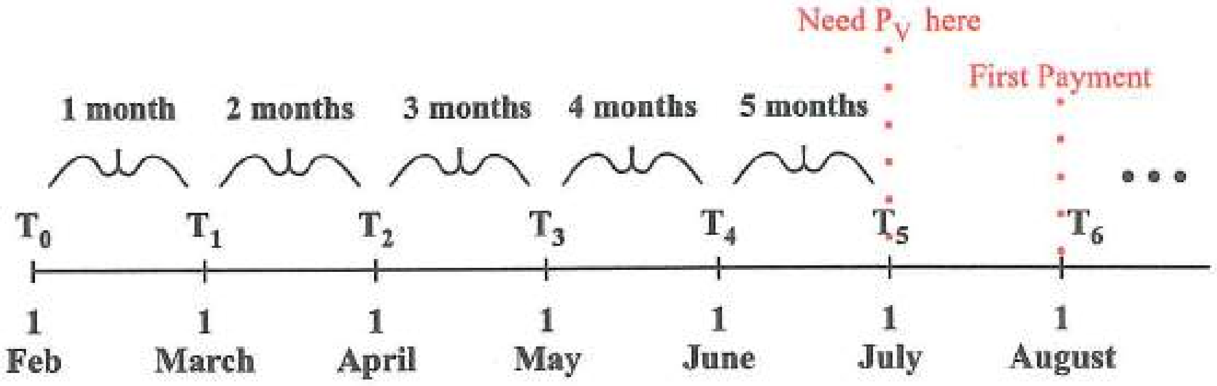
$\therefore A=\text{R}668,50$

$\therefore$ The final payment will be R668,50

Example 3:

Siphiwe borrows R10 000 on the 1st of February 2011 at an interest rate of 9,5% p.a. compounded monthly. He pays the loan back via monthly instalments of R450. The loan agreement allows Siphiwe to make his first payment on the 1st of August 2011.

a) Determine how many months it will take Siphiwe to pay back the loan once payments start.
b) What is the balance of the loan immediately after the 25th payment?
c) What is the value of the final payment?



a) $T_0 - T_5$:

$$A = P(1+i)^n$$

$$\therefore A = 10000\left(1+\frac{0,095}{12}\right)^5$$

$$\therefore A = \text{R}10402,15051 \rightarrow \text{amount owing on the loan at } T_5$$

$T_5 -$ end:

$$P = \frac{x\left[1-(1+i)^{-n}\right]}{i}$$

$$\therefore 10402,15051 = \frac{450\left[1-\left(1+\frac{0,095}{12}\right)^{-n}\right]}{\frac{0,095}{12}} \rightarrow \text{only round off at the end of each question}$$

$$\therefore \frac{10402,15051\left(\frac{0,095}{12}\right)}{450} = 1-\left(1+\frac{0,095}{12}\right)^{-n}$$

$$\therefore \left(1+\frac{0,095}{12}\right)^{-n} = 1-\frac{10402,15051\left(\frac{0,095}{12}\right)}{450}$$

$$\therefore -n = \log_{\left(1+\frac{0,095}{12}\right)}\left[1-\frac{10402,15051\left(\frac{0,095}{12}\right)}{450}\right]$$

$$\therefore -n = -25,6315142 \rightarrow \text{don't round off as you need this decimal for the next question}$$

$$\therefore n = 26$$

$\therefore$ The loan will be paid 26 months after payments start.

b) After the 25th payment, there are 0,6315142 payments still to be made

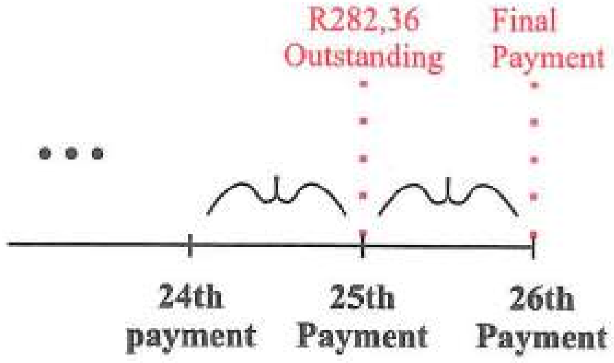
$$P=\frac{x\left[1-(1+i)^{-n}\right]}{i}$$

$$=\frac{450\left[1-\left(1+\frac{0,095}{12}\right)^{-0,6315142}\right]}{\frac{0,095}{12}}\rightarrow 0,6315142\text{ payments left after the }25^{\text{th}}\text{ payment}$$

$$=\text{R}282,36$$

$\therefore$ There is R282,36 outstanding after the 25th payment.

c)


R282,36 Outstanding
Final Payment
• • •
24th payment
25th Payment
26th Payment

After the 25th payment there is R282,36 still outstanding. The 26th payment is made one month later and will accrue interest for that month.

$$A=P(1+i)^{n}$$

$$\therefore A=282,36\left(1+\frac{0,095}{12}\right)^{1}\rightarrow\text{interest is accrued for 1 month}$$

$$=\text{R}284,60$$

$\therefore$ The final payment will be R284,60

Example 4: **Mr Mzinyati takes out a home loan for 1,2 million rand to be paid back over 30 years at an interest rate of 10% p.a compounded monthly.**

a) Determine Mr Mzinyati's monthly repayments.

b) Determine the balance outstanding on the loan after the 20th payment.

c) Mr Mzinyati has financial difficulties and is unable to make the 21st, 22nd and 23rd payment. The bank agrees to restructure the loan so that it is paid off in the same amount of time. Determine the new monthly repayments.

a) $P=\dfrac{x\left[1-(1+i)^{-n}\right]}{i}$

$$1200000=\frac{x\left[1-\left(1+\dfrac{0,1}{12}\right)^{-360}\right]}{\dfrac{0.1}{12}}\rightarrow 30\times 12$$

$$\therefore x=1200000\left(\frac{0,1}{12}\right)\div\left[1-\left(1+\frac{0,1}{12}\right)^{-360}\right]$$

$\therefore x=10530,86$

$\therefore$ The monthly repayments R10530,86 per month

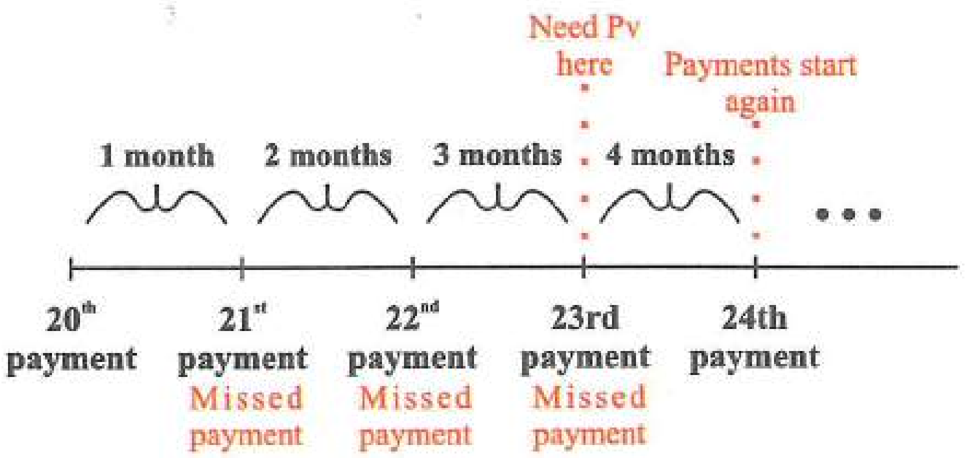
b) $P=\dfrac{x\left[1-(1+i)^{-n}\right]}{i}$

$$P=\frac{10530,86\left[1-\left(1+\dfrac{0,1}{12}\right)^{-340}\right]}{\dfrac{0,1}{12}}\rightarrow \text{340 payments are left after the } 20^{\text{th}} \text{ payment}$$

$\therefore P=1188498,87$

$\therefore$ There is R1188498,87 outstanding after the 20th payment

c) This a two part question. You first have to calculate the amount owing on the loan after the last missed payment.


Need Pv here
Payments start again
1 month
2 months
3 months
4 months
20th payment
21st payment
Missed payment
22nd payment
Missed payment
23rd payment
Missed payment
24th payment

$A = P(1+i)^n$ → compound interest formula as no payments are being made

$= 1188498,87\left(1+\frac{0,1}{12}\right)^3$ → payments were missed for 3 months

$= R1218459,633$ → do not round off within questions

You now have to calculate the new monthly payments

$$P = \frac{x\left[1-(1+i)^{-n}\right]}{i}$$

$\therefore 1218459,633 = \dfrac{x\left[1-\left(1+\frac{0,1}{12}\right)^{-337}\right]}{\frac{0,1}{12}}$ → there are still 337 payments to be made

$$\therefore 1218459,633\left(\frac{0,1}{12}\right) = x\left[1-\left(1+\frac{0,1}{12}\right)^{-337}\right]$$

$$\therefore x = \frac{1218459,633\left(\frac{0,1}{12}\right)}{1-\left(1+\frac{0,1}{12}\right)^{-337}}$$

$$\therefore x = R10813,58$$

∴ the new monthly payments will be R10813,58 per month

Complete Exercise 5 on page 87.

C. COMPARING INVESTMENT AND LOAN OPTIONS AND APPLICATIONS OF FINANCE

COMPARING INVESTMENT OPTIONS

There are various factors that have to be taken into account when comparing different investment options:

- **The return on the investment (how much would be made on the investment).**
- **The investment terms (the length of time before the investment is paid out).**
- **The cost of the investment (any fees, commissions, taxes, etc.).**
- **The type of interest rate (fixed or variable).**

Example: **Mr Brown plans to save R2400 per annum in order to pay for an overseas holiday for his family in 5 years' time. He has approached three different banks, which have offered him different savings options. These options are shown below. Which option should Mr Brown choose?**

Option	Payment Terms	Interest
1	R200 per month	9% p.a. compounded monthly
2	R1 200 every six months	9,5% p.a. compounded six monthly
3	R2 400 yearly	9,8% p.a. compounded yearly

In order to determine the best option for Mr Brown, you have to calculate the amount that would be paid to him after 5 years for each option.

Option 1:

$$\text{Investment}: F=\frac{x\left[(1+i)^n-1\right]}{i}$$

$$\therefore F=\frac{200\left[\left(1+\frac{0,09}{12}\right)^{60}-1\right]}{\frac{0,09}{12}}$$

$$\therefore F=\text{R}\,15084,83$$

Option 2:

$$\text{Investment}: F=\frac{x\left[(1+i)^n-1\right]}{i}$$

$$\therefore F=\frac{1200\left[\left(1+\frac{0,095}{2}\right)^{10}-1\right]}{\frac{0,095}{2}}$$

$$\therefore F=\text{R}\,14918,51$$

Option 3:

$$\text{Investment}: F=\frac{x\left[(1+i)^n-1\right]}{i}$$

$$\therefore F=\frac{2400\left[(1+0,098)^5-1\right]}{0,098}$$

$$\therefore F=\text{R}\,14594,01$$

Mr Brown should choose Option 1 as this would provide the greatest return on his investment.

The highest interest rate does not always offer the greatest return on an investment. You have to take into account all of the variables.

COMPARING LOAN OPTIONS

There are various factors that have to be taken into account when comparing different loan options:

- The total amount of the loan with interest (how much will actually be paid).
- The loan terms (the length of time before the loan is paid off).
- The cost of the loan (any fees, commissions, taxes, etc.).
- The type of interest rate (fixed or variable).

Example: **Mr Dasoo wants to take out a loan for a house over 20 years. He has approached two banks and has been offered two different options. The two options are shown in the table below. Which option should Mr Dasoo choose?**

Variables	Option 1	Option 2
Loan amount	R800 000	R800 000
Interest rate (compounded monthly)	12%	11,8%
Repayments	Monthly	Monthly
Bank charges	R0	R200 per month
Commissions	R6 000	R0

In order to determine the best option for Mr Dasoo, you have to calculate the total amount that he will have to pay over 10 years for each option. In order to do this, you need to calculate his monthly repayments.

Option 1:

$$\text{Loan}: P=\frac{x\left[1-(1+i)^{-n}\right]}{i}$$

$$\therefore 800\,000=\frac{x\left[1-\left(1+\frac{0{,}12}{12}\right)^{-240}\right]}{\frac{0{,}12}{12}}$$

$$\therefore x=\frac{800\,000\left(\frac{0{,}12}{12}\right)}{1-\left(1+\frac{0{,}12}{12}\right)^{-240}}$$

$$\therefore x=\text{R}\,8808{,}70$$

To calculate the total amount paid on the loan you have to add in all bank charges and commissions.

$$\therefore \text{Total paid}=(8\,808{,}70\times240)+6000$$

$$\therefore \text{Total paid}=\text{R}2\ 120\ 088$$

Option 2:

$$\text{Loan}: P=\frac{x\left[1-(1+i)^{-n}\right]}{i}$$

$$\therefore 800\,000=\frac{x\left[1-\left(1+\frac{0{,}118}{12}\right)^{-240}\right]}{\frac{0{,}118}{12}}$$

$$\therefore x=\frac{800\,000\left(\frac{0{,}118}{12}\right)}{1-\left(1+\frac{0{,}118}{12}\right)^{-240}}$$

$$\therefore x=\text{R}\,8697{,}40$$

To calculate the total amount paid on the loan you have to add in all bank charges and commissions.

$$\therefore \text{Total paid}=(8\,697{,}40\times240)+(200\times240)$$

$$\therefore \text{Total paid}=\text{R}2\ 135\ 376$$

Mr Dasoo should choose Option 1 as it has the lowest total repayment.

The lowest interest rate on a loan does not always guarantee the lowest total payment.

PYRAMID SCHEMES

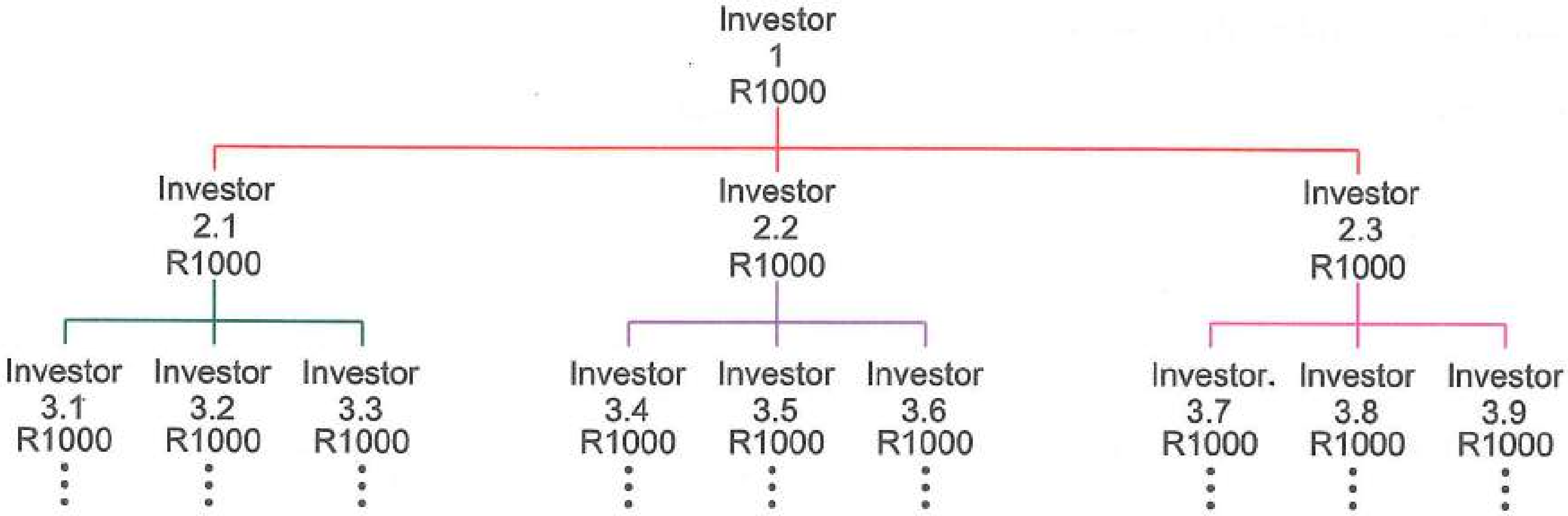
A pyramid scheme is an illegal multilevel marketing scheme. It relies on a participant recruiting new members into the scheme with the promise of very high returns on an investment. As each investor joins, a new level of the pyramid is created and the investors in each level above gain from the new investor.

In reality the pyramid will always collapse because the scheme relies on an infinite number of investors joining the scheme for those above them to make money. The only investors that actually make money are those in the top few levels of the pyramid while those below lose their investments.

Pyramid schemes come in many forms, ultimately any scheme which requires you to recruit investors and promises very high returns due to investors below, is a pyramid scheme and should be avoided.

Illustrative Example:

The example below shows how a pyramid scheme works in theory:
An investor invests R1 000 into the scheme either by purchasing goods or making a cash investment. He then recruits 3 new investors who each invest R1 000. This means a further R3 000 is invested into the scheme. The recruiter gets his R1 000 initial investment returned to him and the remaining R2 000 is shared among the investors above the recruiter. This illustrated in the diagram below.



a) How much money will each investor in levels 1 and 2 make when all the investors in level 3 invest?

b) How many new investors will be added to the scheme in the 4th level?

c) How much money will each investor in levels 1,2 and 3 make when all the investors in level 4 invest.

a) The total amount invested in level 3 will be R9 000. Each of the 3 investors in level 2 will get their initial investment back leaving R6 000 to be split among the 4 members above level 3. Therefore, each investor will receive R1 500 profit.

b) Each recruiter in level 3 is required to recruit three new members. This means the total number of new members will be 27.

c) The total amount invested in level 4 will be R27 000. Each of the 9 investors in level 3 will get their initial investment back leaving R18 000 to be split among the 13 members above level 4. Therefore, each investor will receive R1 384,62 profit.

As the pyramid grows the profit that each investor makes will decrease until the pyramid collapses and all the investors in the last level lose their money.

APPLICATIONS OF FINANCE TO GROWTH AND DECAY

Example 1: A population of Impala on a game reserve increased from 120 000 to 214 000 in 5 years. At what annual compound rate did the population grow?

Compound growth: $A = P(1+i)^n$

$\therefore 214000 = 120000(1+i)^5 \rightarrow$ divide both sides by 120000

$\therefore \dfrac{214000}{120000} = (1+i)^5 \rightarrow$ take the $\sqrt[5]{\ }$ of both sides

$\therefore \sqrt[5]{\dfrac{214000}{120000}} = 1+i$

$\therefore i = \sqrt[5]{\dfrac{107}{60}} - 1$

$\therefore i = 0{,}1227$

$\therefore$ The population grows at a rate of 12,27% per annum

Example 2: After a rainstorm, a puddle begins to evaporate. If the puddle loses 10% of it volume every minute, how long will take for the puddle to halve in volume? Give your answer to the nearest minute.

Compound (reducing balance) decay: $A = P(1-i)^n$

$\therefore \dfrac{1}{2}x = x(1-0{,}1)^n \rightarrow$ divide both sides by x

$\therefore \dfrac{1}{2} = (1-0{,}1)^n \rightarrow$ you are solving for the exponent, write in logarithmic form

$\therefore n = \log_{(1-0{,}1)}\left(\dfrac{1}{2}\right)$

$\therefore n = 6{,}58$

$\therefore$ To the nearest minute it will take 7 minutes for half the puddle to evaporate.

Always use logarithms to solve for the exponent in compound growth and decay problems.

Complete the Mixed Exercise on pages 87 and 88.

Chapter 3 Financial Mathematics (Exercises)

Future Value Calculations

Exercise 1: (answers on page 347)

1) **A factory purchases new equipment which costs R5,2 million. It depreciates at a rate of 12,5% per annum on a reducing balance. The replacement cost of the equipment escalates at a rate of 7,5% per annum. A sinking fund is set up to pay for the new machinery in 6 years' time. If the sinking fund earns 9% p.a. compounded monthly, determine:**
   a) The book value of the old equipment in 6 years' time.
   b) The cost of the new equipment in 6 years' time.
   c) The value the sinking fund must attain, if the book value of the current equipment is used towards the cost of the new equipment.
   d) The monthly payments that need to be made into the sinking fund if payments commence one month after the current equipment is purchased and will continue until the new equipment is purchased.

2) **Lesedi purchases a new car for R189 000. He expects to replace the car in 5 years' time. The replacement cost of the car increases at a rate of 12% p.a. compounded annually and the rate of depreciation on his current car is 18,5% p.a. on a reducing balance. Lesedi sets up a sinking fund to pay for a new car in 5 years. Determine:**
   a) The trade-in value of Lesedi's car in 5 years' time.
   b) The cost of a new car in 5 years' time.
   c) The value of the sinking fund in 5 years' time if Lesedi trades in his old car.
   d) The quarterly payments into the sinking fund, if it earns an interest rate of 6,5% p.a. compounded quarterly and payments start 3 months after the car is initially purchased.

3) **Bridgette started saving for a car in Grade 8. She deposited R1500 into a saving account at the end of each term until the end of Matric. If there are 4 terms per year:**
   a) How much will Bridgette have saved for her car, if she earns 4,3% interest p.a. compounded quarterly?
   b) Bridgette's parents would like her to have R95 000 to spend on a car at the end of Matric. To help her achieve this, they make equal monthly deposits into an account that earns 3,8% interest p.a. compounded monthly. How much do Bridgette's parents need to deposit each month?

4) **At the end of January 2012 Brian deposits R2 500 into a savings account that promises an interest rate of 4,5% compounded quarterly. How many months will Brian have to wait before he has R372 533,37 in his account, assuming that he continues making the same payment every 3 months?**

5) **Siphiwe wants to save R100 000 for a deposit on a house. He can afford to save R4 000 per month and invests in unit trusts that provide an average return of 8,5% p.a. compounded monthly. How long will Siphiwe have to wait to buy his house? Give your answer to the nearest year.**

Future Value Calculations when payments are made at the beginning of the period

Exercise 2: (answers on page 347)

1) **Thembi starts saving for her pension on her 24th birthday by depositing R650 into an account every month. Interest is calculated at 9,6% p.a. compounded monthly. She plans to retire at the age of 60.**
   a) How much did she pay into the account if the last payment is made one month before her 60th birthday?
   b) How much money will she have in her account on her 60th birthday?
   c) If the first payment was made the month after her 24th birthday and the last on her 60th birthday, how much will she have in her account on her 60th birthday?
   d) Explain why there is a difference between your answers in b) and c).

2) **On the day Charlie is born his father immediately starts saving for university by placing R1000 every month into a savings account that earns 6% interest p.a. compounded monthly. His father makes the last payment the month before Charlie's 18th birthday and presents his son with his savings on his 18th birthday. How much money will Charlie have for university? Give your answer to the nearest rand.**

3) **Gina starts saving on her 15th birthday by depositing R8000 into a bank account with an interest rate of 8% p.a. compounded quarterly. She continues to make quarterly payments until her last payment, 3 months before her 24th birthday. How much will Gina have in her account on her 24th birthday?**

Present Value Calculations

Exercise 3: (answers on page 347)

1) **Donna wishes to purchase a car that costs R192 000. She takes out a 5 year loan at an interest rate of 12% p.a. compounded monthly.**
   a) What are the monthly instalments that Donna will have to pay on her loan?
   b) What is the balance outstanding on the loan after Donna makes her 45th payment?

2) **Louisa takes out a loan to purchase a house that costs R700 000. However, the bank will not grant the entire loan and she has to put down a deposit of R50 000 and take out a loan for the remaining balance at 18% per annum compounded monthly.**
   a) Calculate the monthly payments that Louisa needs to make in order to pay back the loan in 15 years.
   b) If Louisa wins the lottery after 10 years and wishes to pay off the loan in full, how much will she have to pay?

3) **Ted applies for a loan at a bank but it is denied due to a poor credit rating. He takes out a short term loan from a micro lender at in interest rate of 16% per annum compounded monthly. He pays the loan back by means of 72 equal monthly payments of R2 169,18 each.**
   a) What was the original amount of the loan? Give your answer to the nearest rand.
   b) What was the interest that Ted paid to the micro lender?

Loans with Delayed Payments (Deferred Payments)

Exercise 4: (answers on page 347)

1) **A loan of R150 000 is taken out at an interest rate of 7% p.a. compounded monthly. Payments start 6 months after the loan is taken out and is repaid by means of 60 equal monthly payments.**
   a) Calculate the outstanding balance 5 months after the loan is taken out.
   b) Determine the monthly repayments required to pay back the loan.

2) **Jessica takes out a loan of R180 000 which is to be repaid over a 5 year period. The loan agreement allows for payments to start 3 months after the granting of the loan at an interest rate of 10,5% per annum compounded monthly.**
   a) Determine the outstanding balance of the loan 2 months after the loan is granted.
   b) What are the monthly repayments that Jessica has to make in order to pay the loan off in 5 years?

3) **A loan of R350 000 is taken out at an interest rate of 5,8% p.a. compounded quarterly. The loan is taken out on the 1st March 2011. The first payment is made on the 1st of December 2011 and is repaid by 72 equal quarterly payments.**
   a) What is the outstanding balance of the loan on the 1st of September 2011?
   b) Determine the monthly repayments required to pay back the loan.

Exercise 5: (answers on page 347)

1) **A loan of R500 000 is taken out at an interest rate of 9% p.a. compounded monthly. It is repaid by equal monthly payments of R6000 and a final payment less than R6 000.**
   - a) How many payments of R6 000 are needed to pay off the loan?
   - b) What is the outstanding balance on the loan after the last payment of R6 000.
   - c) What is the value of the final payment?

2) **A loan of R750 000 is taken out at an interest rate of 12,5% p.a. compounded biannually. It is repaid by equal biannual payments of R55 000 and a final payment less than R55 000.**
   - a) How many payments are required to settle the loan?
   - b) What is the outstanding balance on the loan after the final payment of R55 000?
   - c) What is the value of the final payment?

Mixed Exercise: (answers on page 348)

1) **If R1 820 is invested at 6,8% p.a. compounded annually, how many years will it take for the money to grow to R2 700?**

2) **R5 000 was invested into a fund paying $i\%$ interest compounded monthly. After 26 months the value of the fund was R5 942,89. Determine the value of i, the interest rate.**

3) **Moses purchases a car for R800 000. In 5 years' time Moses will trade in his current car for R200 000 and use this towards the purchase of a new car.**
   - a) If inflation on a new car is 8% per annum, how much will Moses have to pay for a new car in 5 years?
   - b) One month after purchasing his car, Moses sets up a sinking fund in order to pay for his new car in 5 years. How much must Moses deposit into the fund every month if the fund earns an interest rate of 12% p.a. compounded monthly?
   - c) If Moses withdraws R5 000 at the end of each year for maintenance of his current car (making 5 withdrawals in total), what will his new monthly deposit have to be in order to still have the correct amount in the fund to purchase a new car in 5 years' time?

4) **Jason deposits R50 000 into a savings account with an interest rate of 8,2% p.a. compounded monthly. He withdraws R5 500 per month for his monthly expenses.**
   - a) How many full months will Jason be able to use his investment to finance his expenses?
   - b) How much will Jason have in his account after his 9th withdrawal?
   - c) How much will Jason have for his final withdrawal?

5) **Tali wants to retire at the age of 40 and decides to start saving for her retirement on her 21st birthday. She does so by paying R6000 per month into a pension fund starting immediately. The fund pays 8,3% p.a. compounded monthly. How much money will she have one month after she turns 40, if she makes the last payment on her 40th birthday?**

6) **A loan of R500 000 is taken out at an interest rate of 9,07% p.a. compounded quarterly. The loan is repaid over a period of 25 years from the day the loan is drawn by equal monthly payments, starting 6 months after the loan is drawn. Calculate the monthly payments.**

7) **A loan of R130 000 is repaid by means of equal monthly payments over a period of 5 years.**
   - a) Determine the monthly repayments if the interest on the loan is 7% p.a. compounded monthly.
   - b) Due to financial difficulties, the lender is unable to make the 18th, 19th, 20th, and 21st payments. Determine the outstanding balance of the loan at the end of the 17th month.
   - c) The payments are adjusted so that the loan is amortised over the same period with the same interest rate. What will the new monthly payments be after the defaulted payments?

8) **A mortgage bond of R350 000 is taken out. The bond is repaid in equal monthly instalments at the end of each month, over a period of 20 years at an interest rate of 12,5 % p.a. compounded monthly.**

   a) Determine the monthly instalments required to pay back the loan in 20 years.

   b) The loan is actually paid off by paying R4 000 per month for the first 4 years and R5 000 per month thereafter until the loan is paid off, with the final payment being less than R5 000. Determine the outstanding balance after 4 years.

   c) Determine the number of full R5000 payments that have to be made.

   d) What is the value of the final payment?

9) **Abdul purchases a car and starts a sinking fund to replace it in 5 years' time. He expects the new car to cost R500 000 after he trades in his current car. What monthly payments must Abdul make into the sinking fund if payments start 3 months after the purchase of his current car and he receives an interest rate of 7,5 % per annum compounded monthly?**

10) **Bongani and Stephen are two friends that have babies on the same day. Bongani decides to invest R3000 into a savings account at the end of every month in order to save for his child's tertiary education. After 9 years Bongani stops investing money into the account and lets what he has saved grow in the account. The account offers an interest rate of 12% per annum compounded monthly.**

   a) How much would Bongani have in his account at the end of 9 years?

   b) At the end of 9 years, Stephen realises that it is wise to save for his child's education and starts the same savings plan and begins to invest R3000 into a saving account that also offers an interest rate of 12% per annum compounded monthly. How long will it take for Stephen to have saved as much as Bongani?

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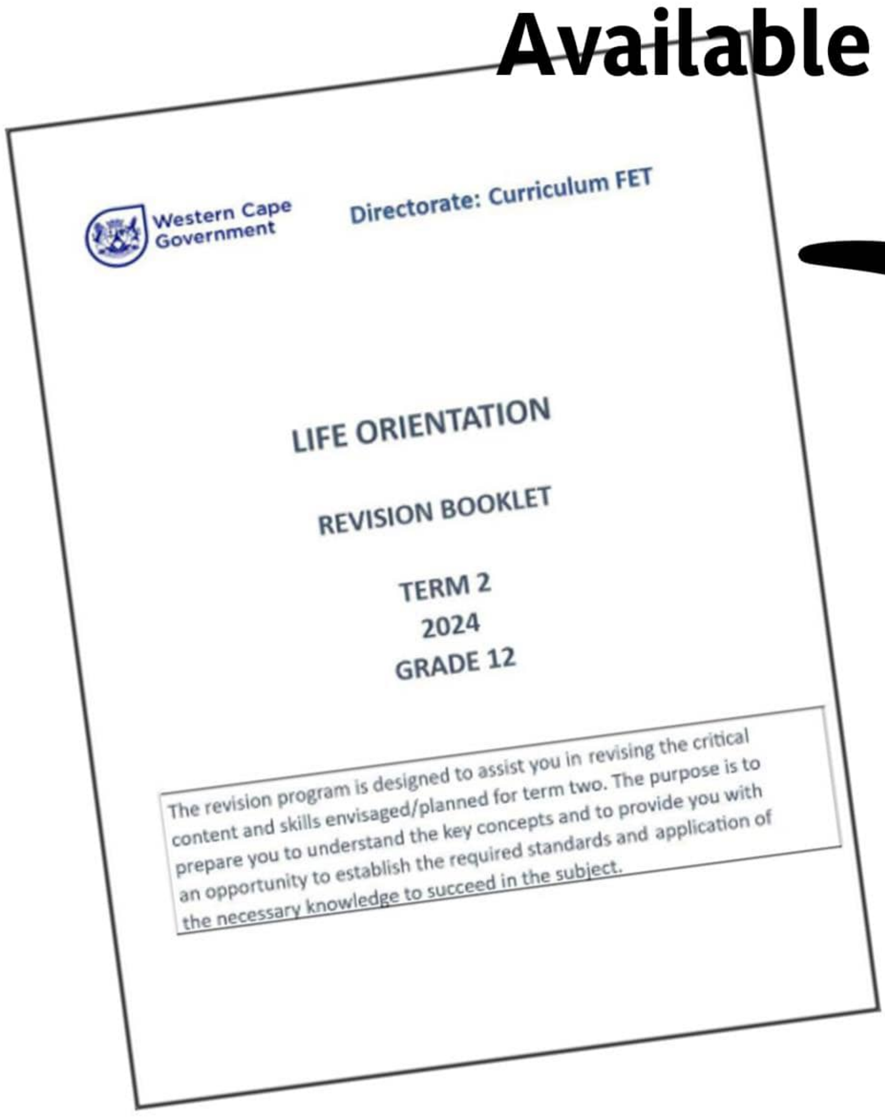

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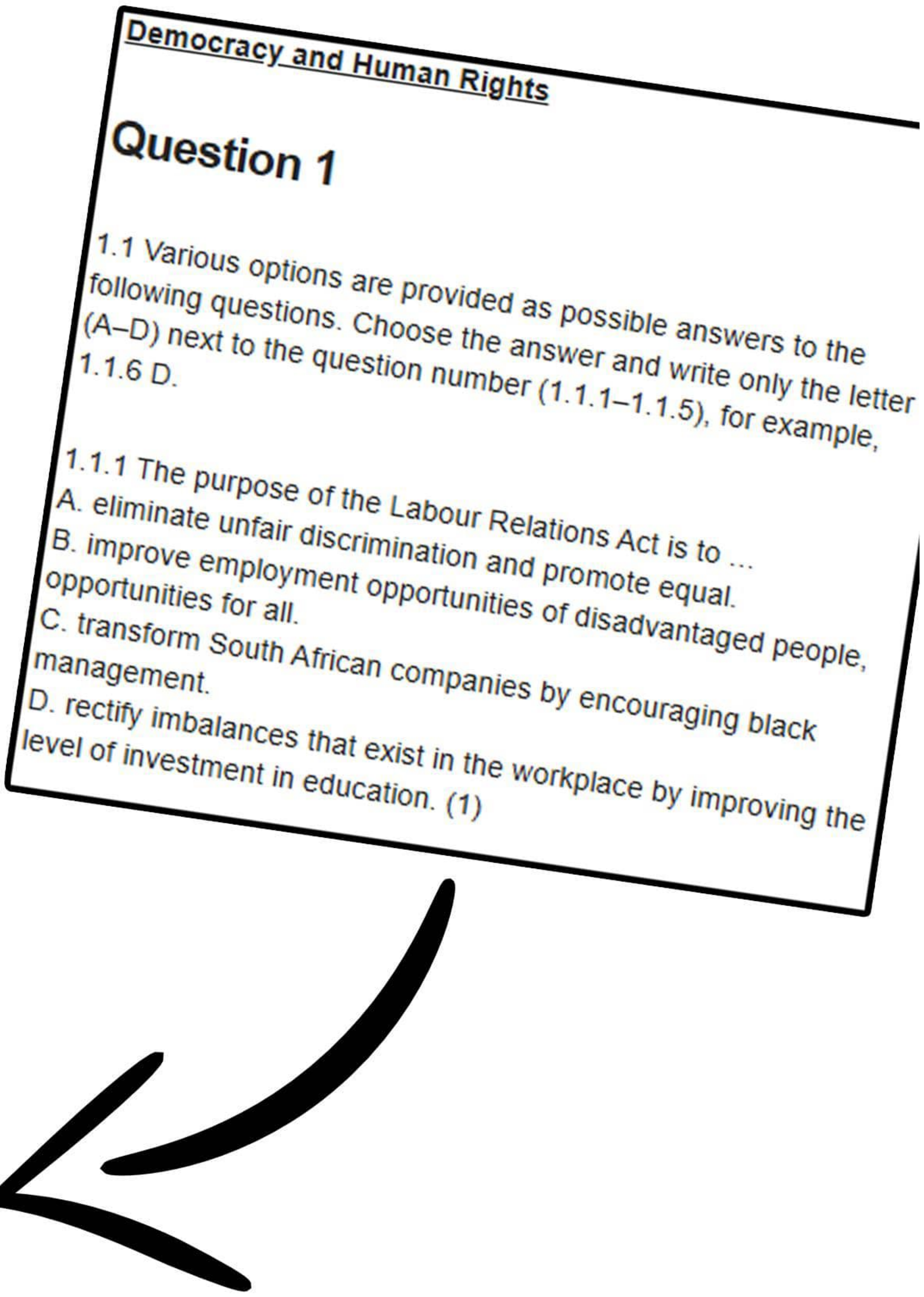
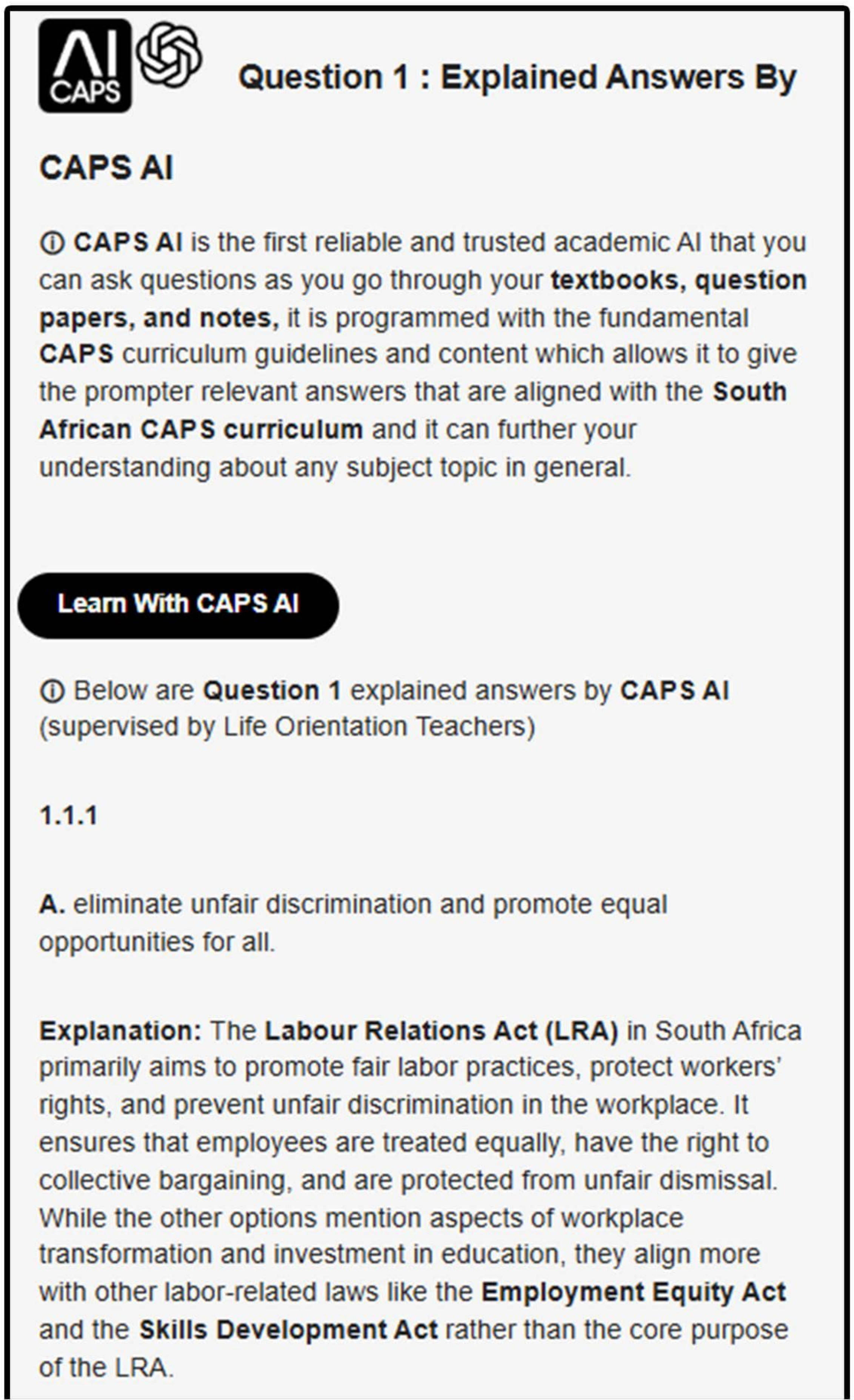


Democracy and Human Rights

Question 1

1.1 Various options are provided as possible answers to the following questions. Choose the answer and write only the letter (A–D) next to the question number (1.1.1–1.1.5), for example, 1.1.6 D.

1.1.1 The purpose of the Labour Relations Act is to …
A. eliminate unfair discrimination and promote equal opportunities for all.
B. improve employment opportunities of disadvantaged people.
C. transform South African companies by encouraging black management.
D. rectify imbalances that exist in the workplace by improving the level of investment in education. (1)



Question 1 : Explained Answers By CAPS AI

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ⓘ Below are **Question 1** explained answers by **CAPS AI** (supervised by Life Orientation Teachers)

1.1.1

A. eliminate unfair discrimination and promote equal opportunities for all.

Explanation: The **Labour Relations Act (LRA)** in South Africa primarily aims to promote fair labor practices, protect workers' rights, and prevent unfair discrimination in the workplace. It ensures that employees are treated equally, have the right to collective bargaining, and are protected from unfair dismissal. While the other options mention aspects of workplace transformation and investment in education, they align more with other labor-related laws like the **Employment Equity Act** and the **Skills Development Act** rather than the core purpose of the LRA.

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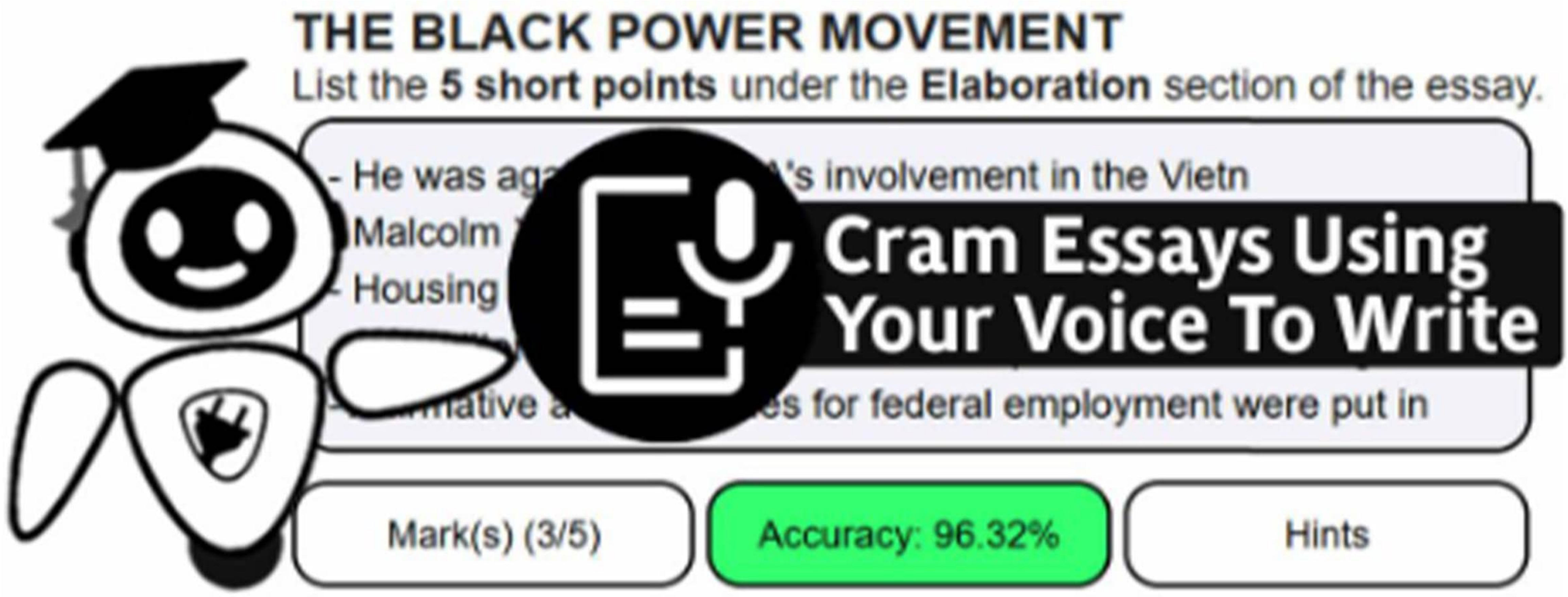
THE BLACK POWER MOVEMENT
List the **5 short points** under the **Elaboration** section of the essay.

- He was ag 's involvement in the Vietn
Malcolm
- Housing
- mative a es for federal employment were put in

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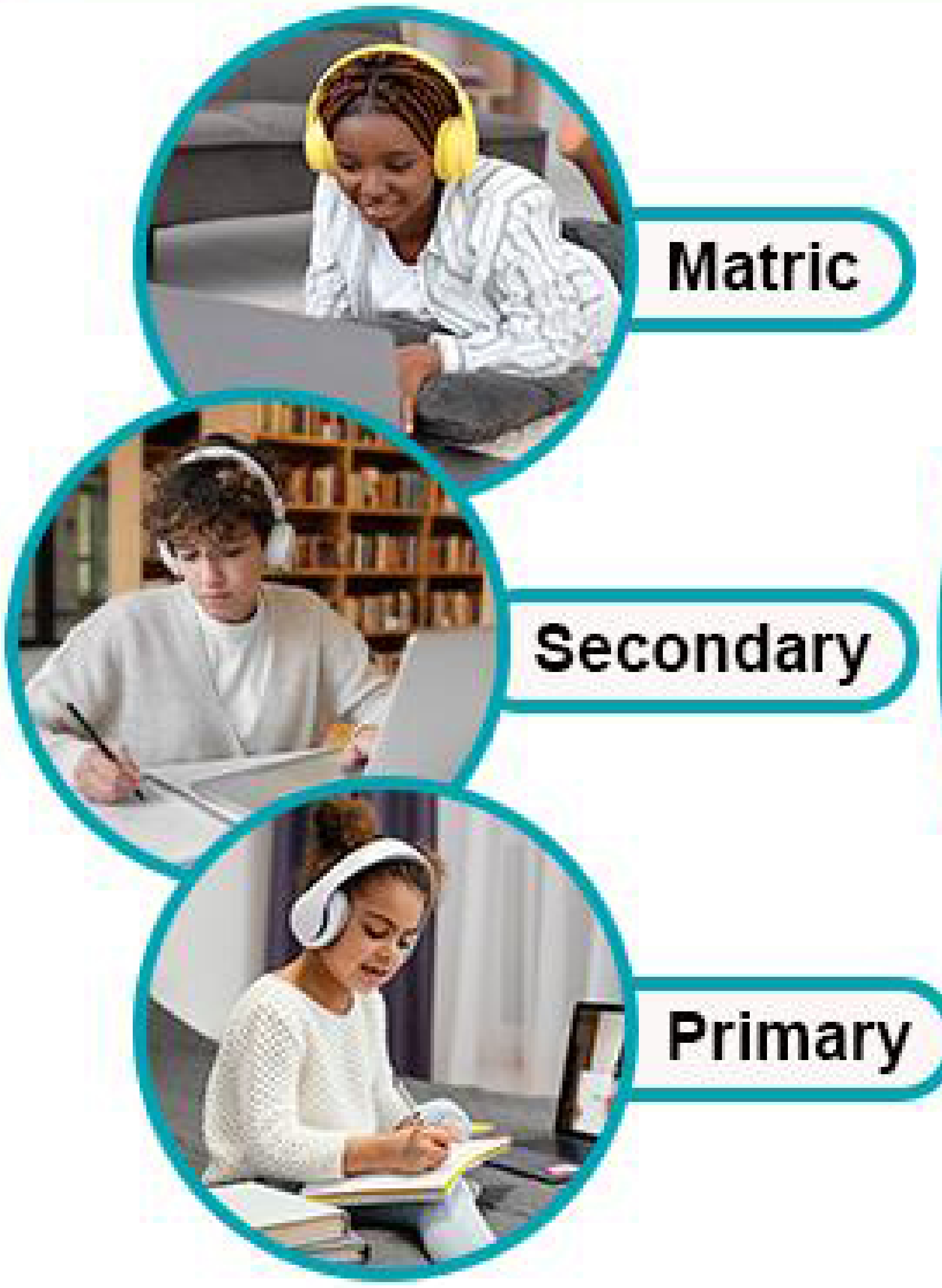

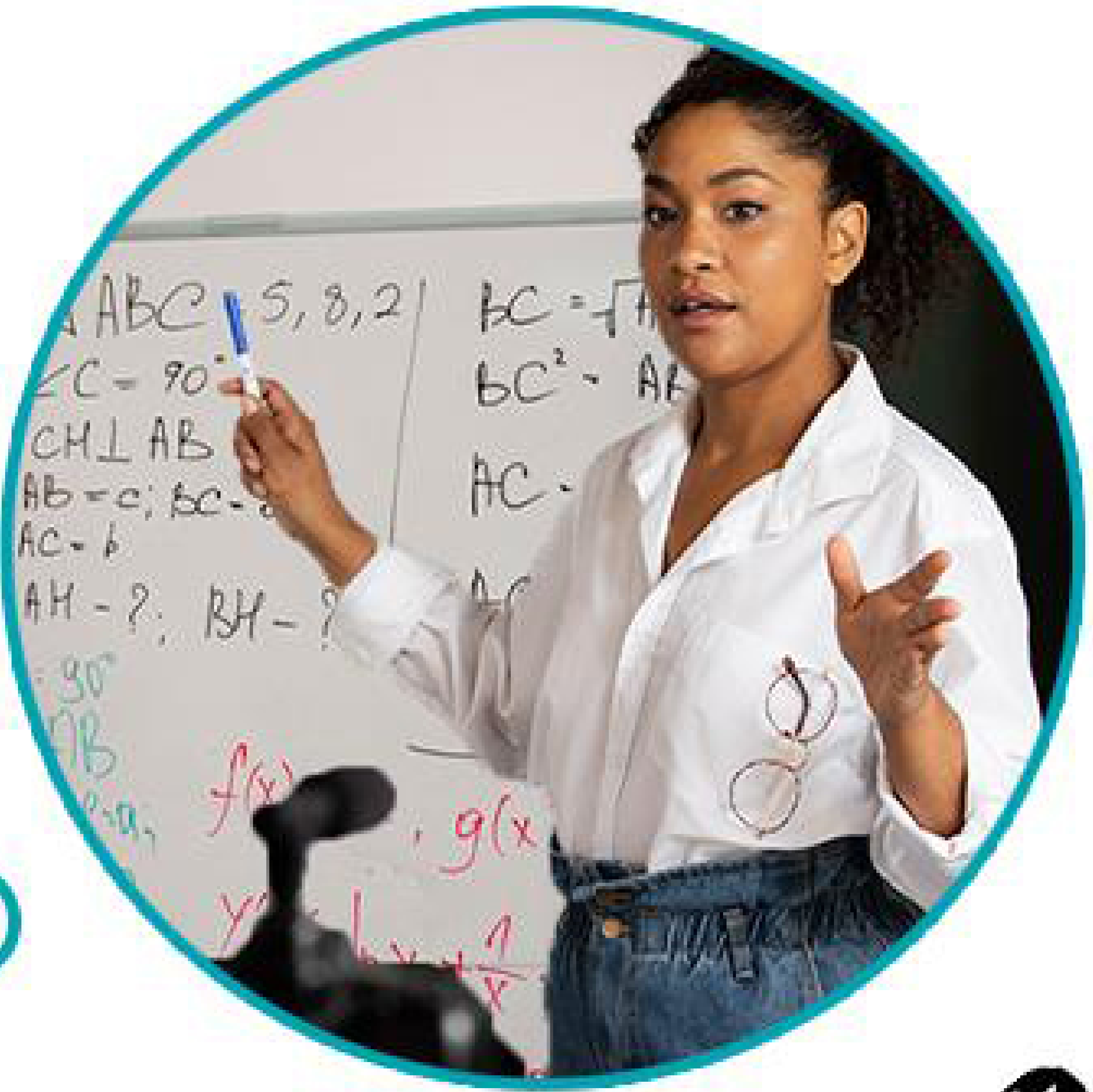

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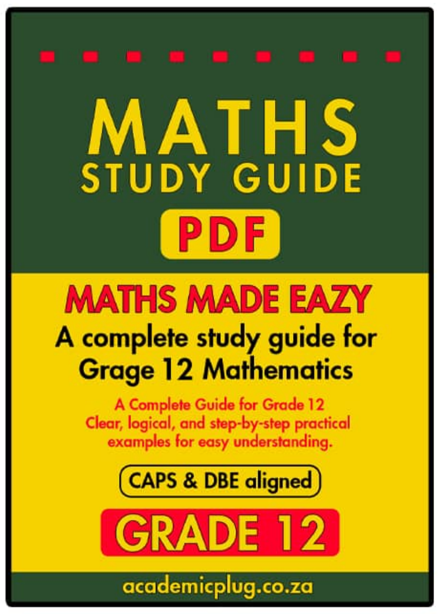
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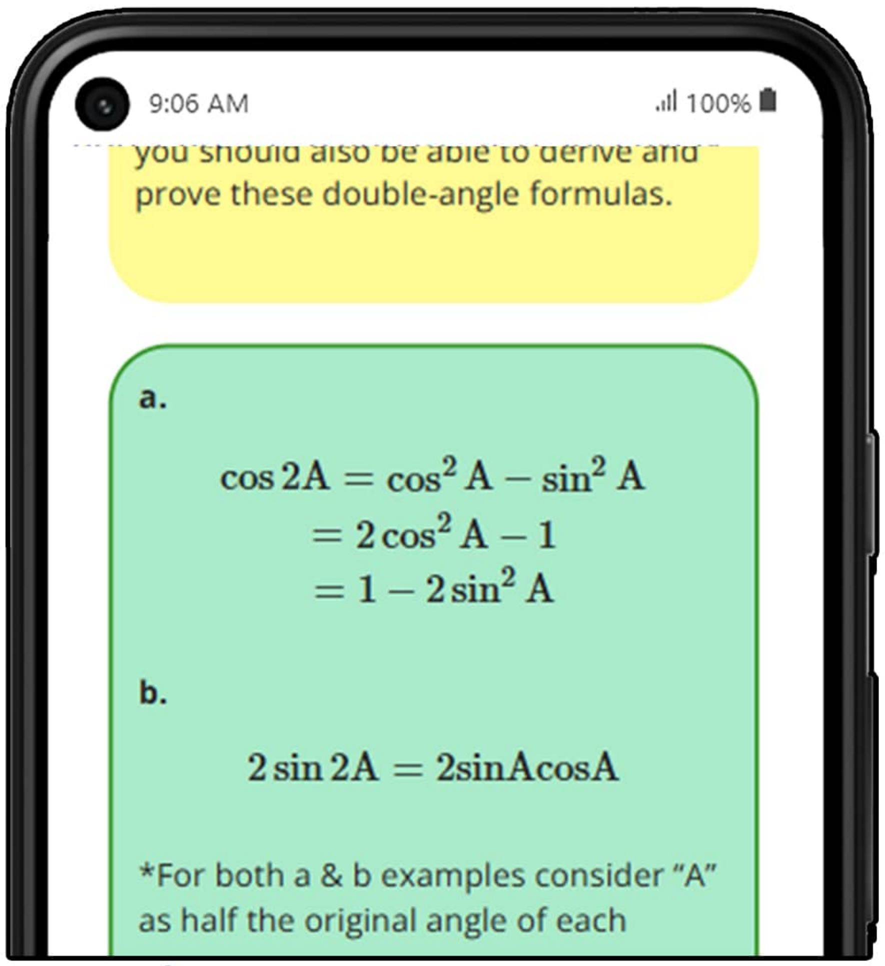
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you should also be able to derive and prove these double-angle formulas.

a.

$$\cos 2A = \cos^2 A - \sin^2 A$$
$$= 2\cos^2 A - 1$$
$$= 1 - 2\sin^2 A$$

b.

$$2\sin 2A = 2\sin A\cos A$$

*For both a & b examples consider "A" as half the original angle of each

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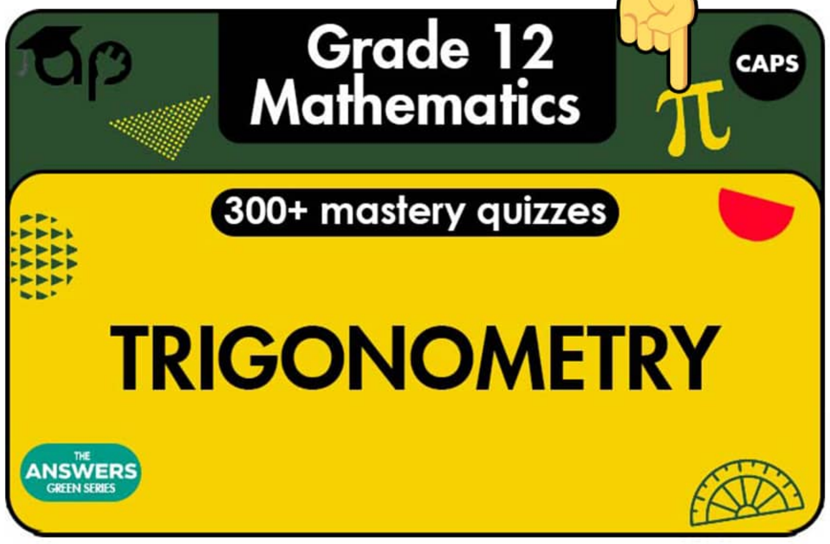
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CHAPTER 4
TRIGONOMETERY

WHAT YOU WILL LEARN IN THIS CHAPTER

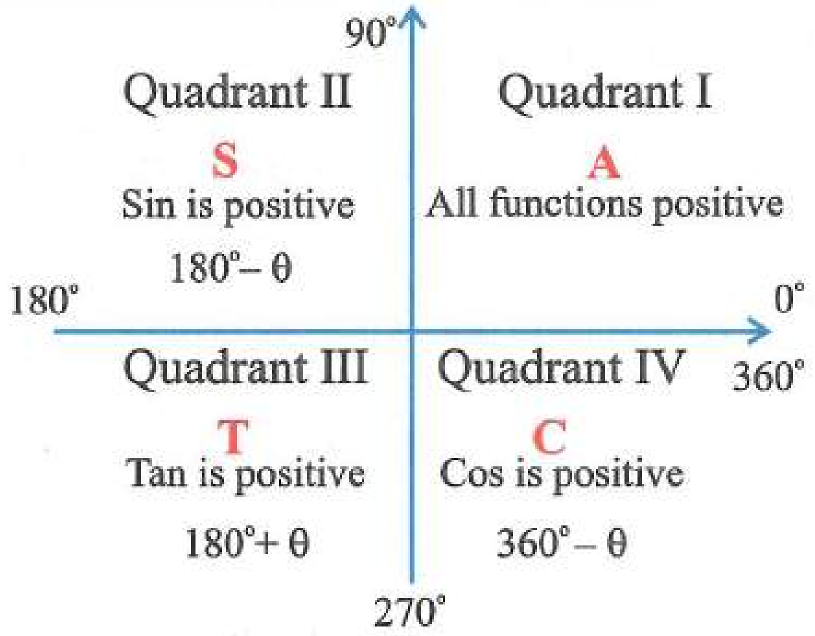
PAGE	CONTENT	CAPS	ASSESSMENTS YOU WILL BE ABLE TO:
90	**A. Grade 11 Trigonometry Revision**	8	• Prove and compound angle and double angle identities.
90	• Reduction formula		
90	• Negative Angles		• Work with general trigonometry problems and trigonometric equations.
91	• Trigonometric identities		
91	• Special Angles		
94	**B. Compound Angles**		
94	• Compound Angle Formulas		
95	• Exceptions to using Compound Angles		
96	• Compound Angle Formula Proofs		
99	**C. Double Angles**		
99	• Double Angle Formula Proofs		
100	• Mixed Examples with Double and Compound Angles		
103	**D. Trigonometric Identities Using Double Angles**		
106	**E. Trigonometric Equations**		
106	• Grade 11 Revision		
108	• Trigonometric Equations using Double and Compound Angles		
110	• Restrictions on Identities and Trigonometric Equations		

A. GRADE 11 TRIGONOMETRY REVISION

REDUCTION FORMULA

Reduction formulas are used to simplify trigonometric expressions.
There are two mnemonics that may be used to remember which functions are positive in the different quadrants.

1. Moving **anti-clockwise** from the **first quadrant:** All Students Take Care
2. Moving **anti-clockwise** from the **fourth quadrant:** CAST

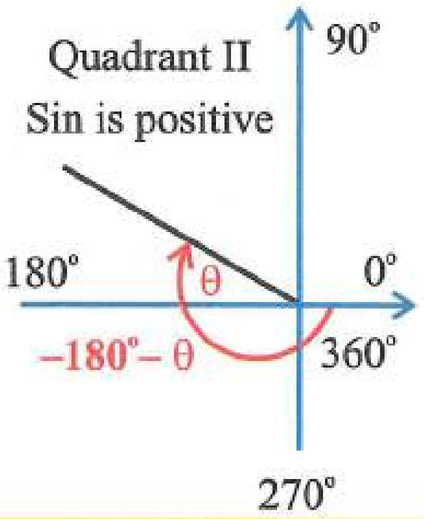
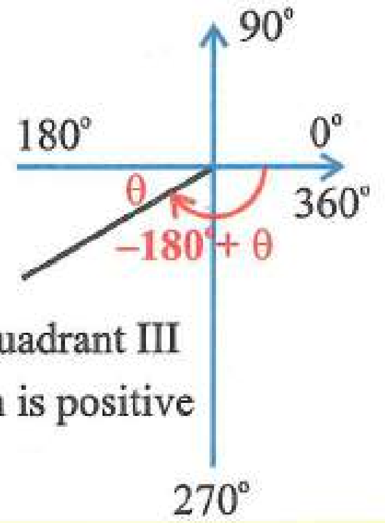
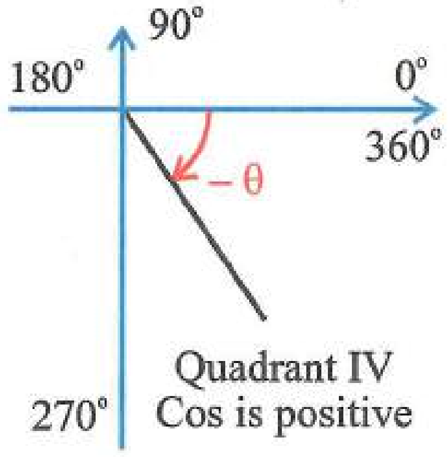

90°
Quadrant II
S
Sin is positive
180°– θ
Quadrant I
A
All functions positive
180°
0°
360°
Quadrant III
T
Tan is positive
180°+ θ
Quadrant IV
C
Cos is positive
360° – θ
270°

QUADRANT II	QUADRANT III	QUADRANT IV
$\sin(180° - \theta) = +\sin\theta$	$\tan(180° + \theta) = +\tan\theta$	$\cos(360° - \theta) = +\cos\theta$
$\cos(180° - \theta) = -\cos\theta$	$\sin(180° + \theta) = -\sin\theta$	$\tan(360° - \theta) = -\tan\theta$
$\tan(180° - \theta) = -\tan\theta$	$\cos(180° + \theta) = -\cos\theta$	$\sin(360° - \theta) = -\sin\theta$

Do not use reduction formulas on angles less than 90° (angles in Quadrant I).

NEGATIVE ANGLES

The same CAST diagram that is used for reduction formulas can be used for negative angles. In the Cartesian plane, angles measured clockwise are negative. For instance, if given the ratio $\sin(-180° - \theta)$ this is equivalent to moving 180° clockwise and then moving another $\theta°$ clockwise. This would take you into the second quadrant where $\sin\theta$ is positive and $\cos\theta$ and $\tan\theta$ are negative.

$-180° - \theta$	$-180° + \theta$	$-\theta$
 Quadrant II Sin is positive 90° 180° θ 0° –180°– θ 360° 270°	 90° 180° 0° θ 360° –180°+ θ Quadrant III Tan is positive 270°	 180° 90° 0° 360° – θ Quadrant IV Cos is positive 270°
$\sin(-180° - \theta) = +\sin\theta$ $\cos(-180° - \theta) = -\cos\theta$ $\tan(-180° - \theta) = -\tan\theta$	$\tan(-180° + \theta) = +\tan\theta$ $\sin(-180° + \theta) = -\sin\theta$ $\cos(-180° + \theta) = -\cos\theta$	$\cos(-\theta) = +\cos\theta$ $\sin(-\theta) = -\sin\theta$ $\tan(-\theta) = -\tan\theta$

In all of the above expressions, θ has to be an acute angle.

TRIGONOMETRIC IDENTITIES

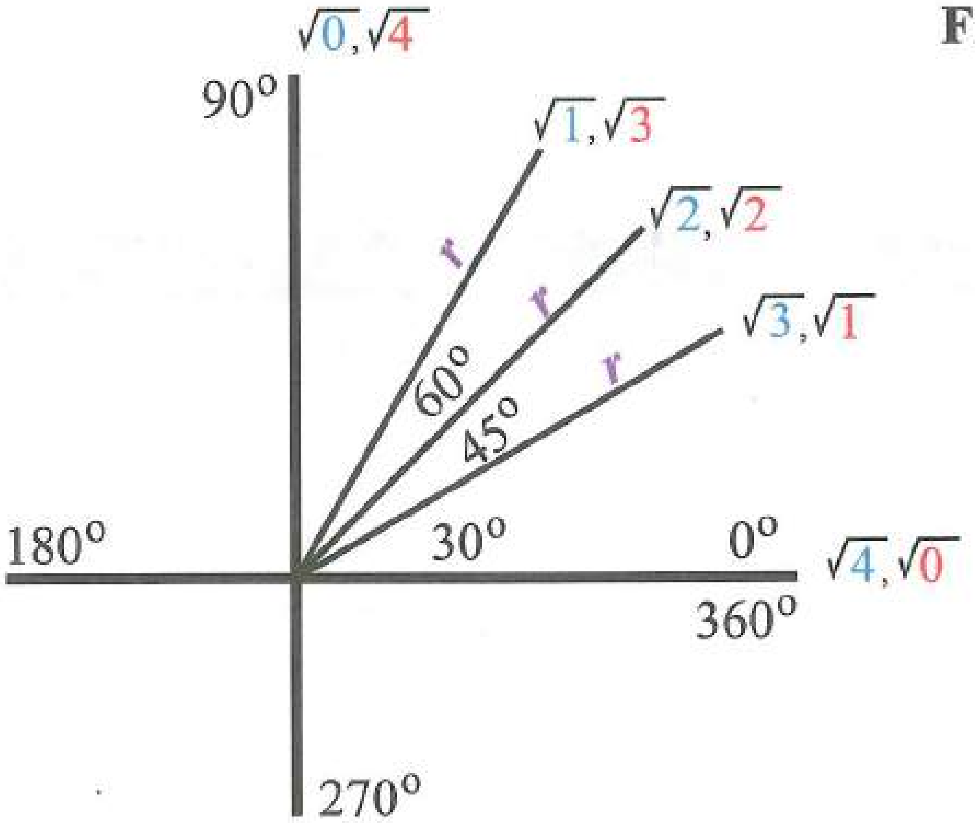
The identities from Grade 11 which you are required to know are shown below:

1. $\sin^2\theta+\cos^2\theta=1$ **or** $\sin^2\theta=1-\cos^2\theta$ **or** $\cos^2\theta=1-\sin^2\theta$

2. $\tan\theta=\dfrac{\sin\theta}{\cos\theta}$

SPECIAL ANGLES

The Fan diagram below is useful in remembering special angles. The fan is constructed by drawing a terminal arm ***r*** for each of the special angles 30°, 45° and 60°. The *x*-coordinates are then labelled 0, 1, 2, 3 and 4 starting at 90° moving clockwise to 0°. The *y*-coordinates are then labelled 0, 1, 2, 3 and 4 starting at 0° moving anti-clockwise to 90°. Once all of the co-ordinates are in place, **square root** each coordinate and use your diagram to solve for all the special angles as shown below. Each terminal arm has a length ***r*** which is always equal to **2**.



From the diagram it can be seen that when:

$\theta=0^\circ \rightarrow (x;y)=(2;0)$

$\theta=30^\circ \rightarrow (x;y)=(\sqrt{3};1)$

$\theta=45^\circ \rightarrow (x;y)=(\sqrt{2};\sqrt{2})$

$\theta=60^\circ \rightarrow (x;y)=(1;\sqrt{3})$

$\theta=90^\circ \rightarrow (x;y)=(0;2)$

When dealing with special angles *r* is always equal to 2.

It is important to remember that the trigonometric ratios are defined as follows:

$\sin\theta=\dfrac{\text{Opposite}}{\text{Hypotenuse}}\rightarrow$ Oh Heaven $\qquad \sin\theta=\dfrac{y}{r}$

$\cos\theta=\dfrac{\text{Adjacent}}{\text{Hypotenuse}}\rightarrow$ Another Hour $\qquad \cos\theta=\dfrac{x}{r}$

$\tan\theta=\dfrac{\text{Opposite}}{\text{Adjacent}}\rightarrow$ Of Algebra $\qquad \tan\theta=\dfrac{y}{x}$

In order to remember the trigonometric ratios use the mnemonic:
Oh Heaven Another Hour Of Algebra

It is important to note that adding or subtracting 360° does not change the value of a trigonometric ratio.

$\sin(\theta\pm 360^\circ)=+\sin\theta \qquad \cos(\theta\pm 360^\circ)=+\cos\theta \qquad \tan(\theta\pm 360^\circ)=+\tan\theta$

Example 1: **Simplify** $\dfrac{\cos(360^\circ - x)\sin(360^\circ - x)\cos(90^\circ - x)}{\sin^2(90^\circ - x)\sin(180^\circ + x)}$.

$$\frac{\cos(360^\circ - x)\sin(360^\circ - x)\cos(90^\circ - x)}{\sin^2(90^\circ - x)\sin(180^\circ + x)} \rightarrow \text{reduction formula and co-ratios}$$

$$= \frac{(\cos x)(-\sin x)(\sin x)}{(\cos^2 x)(-\sin x)}$$

$$= \frac{\sin x}{\cos x} \rightarrow \text{trig identity}$$

$$= \tan x$$

Example 2: **Prove that** $\dfrac{1}{1-\sin\theta} + \dfrac{1}{1+\sin\theta} = \dfrac{2}{\cos^2\theta}$ **ignoring any restrictions.**

$$\text{LHS} = \frac{1}{1-\sin\theta} + \frac{1}{1+\sin\theta} \rightarrow \text{LCD} = (1-\sin\theta)(1+\sin\theta)$$

$$= \frac{1(1+\sin\theta) + 1(1-\sin\theta)}{(1-\sin\theta)(1+\sin\theta)}$$

$$= \frac{1+\sin\theta+1-\sin\theta}{1-\sin^2\theta} \rightarrow 1-\sin^2\theta = \cos^2\theta$$

$$= \frac{2}{\cos^2\theta} = \text{RHS}$$

Example 3: **Prove that** $\dfrac{\sin\theta(1+\cos\theta)}{1-\cos^2\theta} + \tan\theta = \dfrac{1+\cos\theta}{\sin\theta\cos\theta}$ **ignoring any restrictions.**

$$\text{LHS} = \frac{\sin\theta(1+\cos\theta)}{1-\cos^2\theta} + \tan\theta \rightarrow \tan\theta = \frac{\sin\theta}{\cos\theta} \text{ and } 1-\cos^2\theta = \sin^2\theta$$

$$= \frac{\sin\theta(1+\cos\theta)}{\sin^2\theta} + \frac{\sin\theta}{\cos\theta}$$

$$= \frac{1+\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta} \rightarrow \text{LCD} = \sin\theta\cos\theta$$

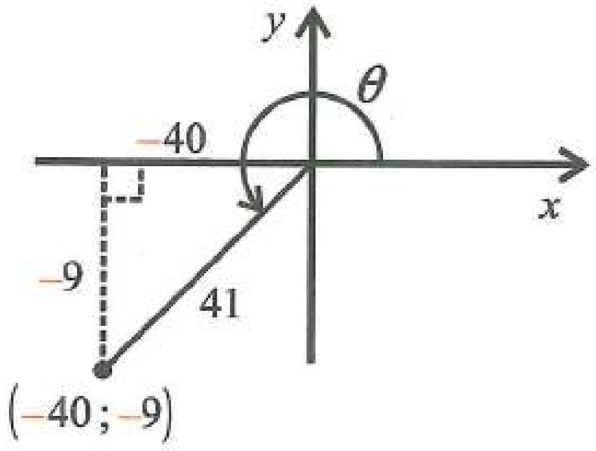
$$= \frac{\cos\theta(1+\cos\theta)+\sin^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{\cos\theta+\cos^2\theta+\sin^2\theta}{\sin\theta\cos\theta} \rightarrow \cos^2\theta+\sin^2\theta = 1$$

$$= \frac{\cos\theta+1}{\sin\theta\cos\theta} = \text{RHS} \rightarrow \text{remember that } \cos\theta+1 \text{ is the same as } 1+\cos\theta$$

Example 4: **Given that $\sin\theta = -\dfrac{9}{41}$ and $\cos\theta < 0$, without a calculator and with aid of a diagram, determine the value of $\cos\theta \div \sin\theta$.**

When drawing the diagram, you have to determine which quadrant the angle lies in. $\sin\theta < 0$ and $\cos\theta < 0$ this occurs in Quadrant III. Therefore, draw the terminal arm in the 3rd quadrant.



$\sin\theta = -\dfrac{9}{41} \rightarrow \dfrac{y}{r}$

Calculate the value of x using Pythagoras:

$x^2 = r^2 - y^2$

$\therefore x = \sqrt{(41)^2 - (-9)^2} \rightarrow r$ is always positive

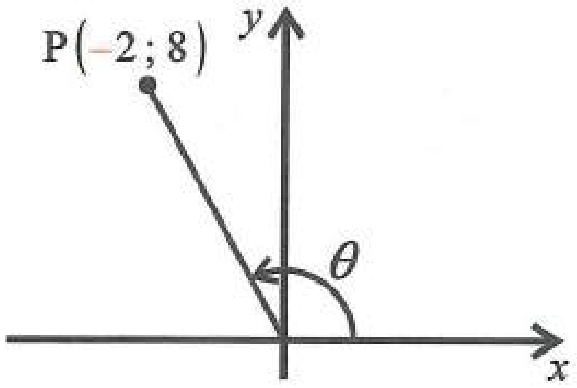
$\therefore x = -40 \rightarrow x$ is negative in the 3rd quadrant

$$\therefore \cos\theta \div \sin\theta = \left(-\frac{40}{41}\right) \div \left(-\frac{9}{41}\right)$$

$$= \left(-\frac{40}{41}\right) \times \left(-\frac{41}{9}\right)$$

$$= \frac{40}{9}$$

Example 5: **Using the diagram below and without the use of a calculator, determine the value of $\cos\theta\cos 60°$.**



Calculate the value of r using Pythagoras:

$r^2 = x^2 + y^2$

$\therefore r = \sqrt{(-2)^2 + (8)^2} = 2\sqrt{17}$

$$\therefore \cos\theta\cos 60° = \left(\frac{-2}{2\sqrt{17}}\right)\left(\frac{1}{2}\right) \rightarrow \frac{x}{r}$$

$$= -\frac{1}{2\sqrt{17}} \rightarrow \text{this can also be written as } -\frac{\sqrt{17}}{34}$$

Complete Exercise 1 on page 111.

B. COMPOUND ANGLES

When two angles are added or subtracted they are known as compound angles. When dealing with trigonometric ratios it is important to note that:

$$\cos(A \pm B) \neq \cos(A) \pm \cos(B) \text{ and } \sin(A \pm B) \neq \sin(A) \pm \sin(B)$$

When dealing with compound angles, the formulas below must be used in order to expand compound angles.

COMPOUND ANGLE FORMULAS

$$\cos(A - B) = \cos A.\cos B + \sin A.\sin B$$

$$\cos(A + B) = \cos A.\cos B - \sin A.\sin B$$

Note: With $\cos(A \pm B)$ there is a sign change.

$$\sin(A + B) = \sin A.\cos B + \cos A.\sin B$$

$$\sin(A - B) = \sin A.\cos B - \cos A.\sin B$$

Note: With $\sin(A \pm B)$ there is no sign change.

Example 1: Simplify $\cos(A + 60^\circ) + \cos(A - 60^\circ)$

$\cos(A + 60^\circ) + \cos(A - 60^\circ) \rightarrow$ use compound angles

$= \cos A \cos 60^\circ - \sin A \sin 60^\circ + \cos A \cos 60^\circ + \sin A \sin 60^\circ$

$= 2\cos A \cos 60^\circ$

$= 2\cos A\left(\frac{1}{2}\right)$

$= \cos A$

Example 2: Prove that $\sin(90^\circ - \theta) = \cos\theta$

$\text{LHS} = \sin(90^\circ - \theta)$

$= \sin 90^\circ \cos\theta - \cos 90^\circ \sin\theta$

$= (1)\cos\theta - (0)\sin\theta$

$= \cos\theta = \text{RHS}$

Example 3: Simplify $\sin(169^\circ)\sin(41^\circ)+\sin(79^\circ)\sin(131^\circ)$.

$$\sin(169^\circ)\sin(41^\circ)+\sin(79^\circ)\sin(131^\circ)$$

$$=\sin(180^\circ-11^\circ)\sin(41^\circ)+\sin(79^\circ)\sin(180^\circ-49^\circ)\rightarrow\text{if possible always reduce first}$$

$$=\sin(11^\circ)\sin(41^\circ)+\sin(79^\circ)\sin(49^\circ)\rightarrow\text{put into the form of }\cos A\cos B+\sin A\sin B$$

$$=\sin(90^\circ-79^\circ)\sin(90^\circ-49)+\sin(79^\circ)\sin(49^\circ)\rightarrow\text{co-functions}$$

$$=\cos(79^\circ)\cos(49^\circ)+\sin(79^\circ)\sin(49^\circ)$$

$$=\cos(79^\circ-49^\circ)$$

$$=\cos(30^\circ)\rightarrow\text{special angles}$$

$$=\frac{\sqrt{3}}{2}$$

Example 4: Without using a calculator, find the value of $\sin(75^\circ)$.

$$\sin(75^\circ)\rightarrow\text{break apart into two special angles}$$

$$=\sin(30^\circ+45^\circ)\rightarrow\text{use compound angle formula}$$

$$=\sin(30^\circ)\cos(45^\circ)+\cos(30^\circ)\sin(45^\circ)$$

$$=\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)+\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$

$$=\frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4}$$

$$=\frac{\sqrt{2}+\sqrt{6}}{4}$$

EXCEPTIONS TO USING COMPOUND ANGLES

Do not use compound angle formulas if you can use reduction formulas or co-functions.

Examples:

1. $\sin(90^\circ-\theta)=\cos\theta$

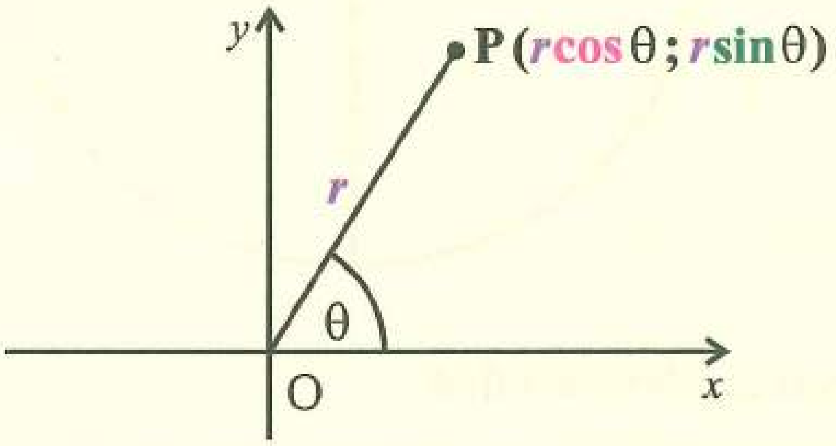
2. $\sin(180^\circ+\theta)=-\sin\theta$

3. $\cos(\theta-180^\circ)=\cos(-180^\circ+\theta)$

$$=-\cos\theta$$

COMPOUND ANGLE FORMULA PROOFS

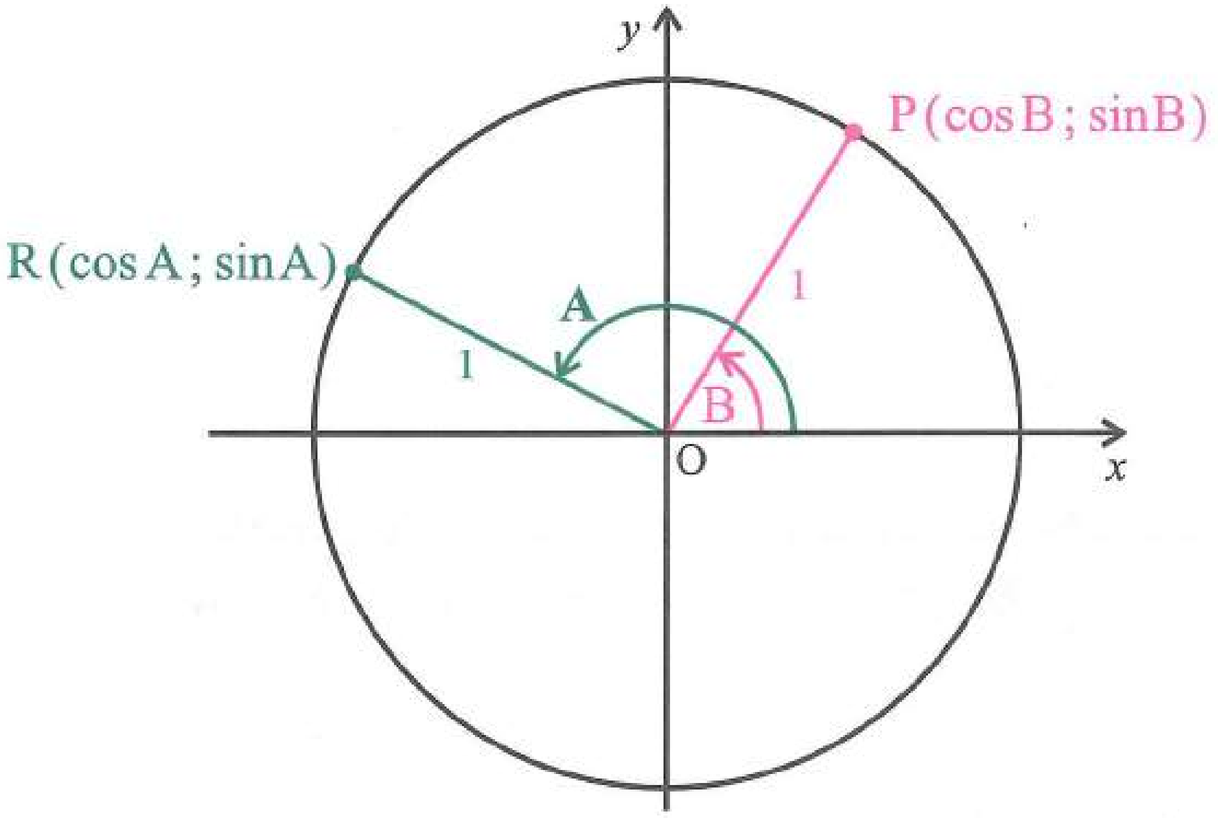
Before proving the formula $\cos(A-B)=\cos A.\cos B+\sin A.\sin B$ it is important to note that any x and y coordinate in the Cartesian plane can be written in terms of trigonometric ratios. Any point $P(x;y)$ can be written as $P(r\cos\theta;r\sin\theta)$, where r is the distance from the origin to P and θ is the angle measured anticlockwise from the x-axis to line OP.



You will also have to be familiar with the distance formula (pg. 207) and Cosine rule (pg. 119). These are all concepts from Grade 11 which you should be familiar with.

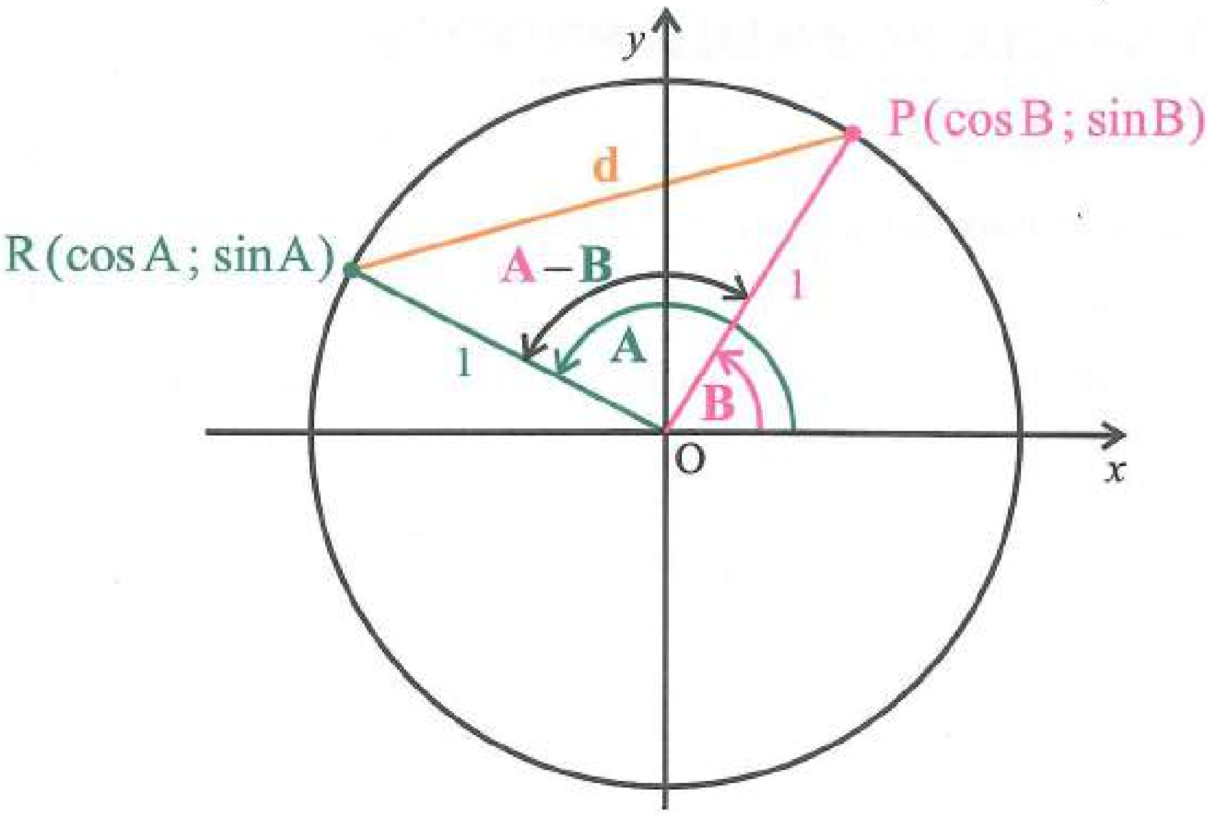
Proof 1: $\cos(A-B)=\cos A.\cos B+\sin A.\sin B$.

In order to prove this formula you have to construct a circle with centre O and a radius of 1 (this is known as the unit circle). On the unit circle, mark point $P(\cos B;\sin B)$ and point $R(\cos A;\sin A)$ where $B<A$. This is shown on the diagram below:



We will now join line **PR** and let the length of $PR=d$. It is also important to note that the angle formed between line OP and OR is $(A-B)$. This is shown over the page.

Note that $\cos B$ is the x-coordinate of point P and $\sin B$ is the y-coordinate of point P. Similarly $\cos A$ is the x-coordinate of point R and $\sin A$ is the y-coordinate of point R.



From the distance formula it can be seen that:

$d^2 = (\sin A - \sin B)^2 + (\cos A - \cos B)^2 \rightarrow$ distance formula: $d^2 = (y_2 - y_1)^2 + (x_2 - x_1)^2$

$= \sin^2 A - 2\sin A \sin B + \sin^2 B + \cos^2 A - 2\cos A \cos B + \cos^2 B \rightarrow \sin^2\theta + \cos^2\theta = 1$

$= 1 + 1 - 2\sin A \sin B - 2\cos A \cos B$

$= 2 - 2\sin A \sin B - 2\cos A \cos B \rightarrow$ take out a HCF of -2

$= 2 - 2(\sin A \sin B + \cos A \cos B) \rightarrow \boxed{1}$

Now use the Cosine rule in order to solve for d^2:

$d^2 = OP^2 + OR^2 - 2(OP)(OR)\cos(A - B) \rightarrow$ Cosine Rule

$= 1^2 + 1^2 - 2(1)(1)\cos(A - B) \rightarrow$ Cosine Rule

$= 2 - 2\cos(A - B) \rightarrow \boxed{2}$

$\boxed{2} = \boxed{1}$:

$\therefore 2 - 2\cos(A - B) = 2 - 2(\sin A \sin B + \cos A \cos B) \rightarrow$ subtract 2 from both sides

$\therefore -2\cos(A - B) = -2(\sin A \sin B + \cos A \cos B) \rightarrow$ divide both sides by -2

$\therefore \cos(A - B) = \sin A \sin B + \cos A \cos B$

In order to prove the remaining compound angle formulas, we will accept that $\cos(A - B) = \cos A.\cos B + \sin A.\sin B$ is true and use it to prove the remaining formulas.

Proof 2: Using $\cos(A - B) = \cos A.\cos B + \sin A.\sin B$ prove that:

$\cos(A + B) = \cos A \cos B - \sin A \sin B$

LHS $= \cos[A - (-B)] \rightarrow \cos(A + B)$ a minus has been 'taken out'

$= \cos A.\cos(-B) + \sin A.\sin(-B)$

$= \cos A.\cos B - \sin A.\sin B$

$=$ RHS

Proof 3: **Using $\cos(A-B)=\cos A\cos B+\sin A\sin B$ prove that:**

$$\sin(A+B)=\sin A\cos B+\cos A\sin B$$

LHS $= \cos\left[90^\circ-(A+B)\right]$ → $\sin(A+B)=\cos\left[90^\circ-(A+B)\right]$ → co-ratios

$= \cos\left[(90^\circ-A)-B\right]$ → the '$-$' doesn't belong to the angle

$= \cos(90^\circ-A)\cos B+\sin(90^\circ-A)\sin B$

$= \sin A\cos B+\cos A\sin B$

$=$ RHS

Proof 4: **Using $\cos(A-B)=\cos A\cos B+\sin A\sin B$ prove that:**

$$\sin(A-B)=\sin A\cos B-\cos A\sin B$$

LHS $= \cos\left[90^\circ-(A-B)\right]$ → $\sin(A-B)=\cos\left[90^\circ-(A-B)\right]$ → co-ratios

$= \cos\left[(90^\circ-A)+B\right]$ → the '$+$' doesn't belong to the angle

$= \cos(90^\circ-A)\cos B-\sin(90^\circ-A)\sin B$

$= \sin A\cos B-\cos A\sin B$

$=$ RHS

Complete Exercise 2 on pages 111 and 112.

C. DOUBLE ANGLES

In Grade 12 you are expected to use double angle formulas in order to simplify trigonometric expressions and solve equations.
In addition, you are expected to be able to prove the double angle formulas. In order to do this, you use compound angle formulas (see pg. 94).

1. $\cos 2A = \cos^2 A - \sin^2 A \rightarrow$ **A is half the original angle**

   $= 2\cos^2 A - 1$

   $= 1 - 2\sin^2 A$

2. $\sin 2A = 2\sin A\cos A \rightarrow$ **A is half the original angle**

The formula $\sin 2A = 2\sin A\cos A$ can be manipulated to give:

$$\sin A\cos A = \frac{1}{2}\sin 2A$$

DOUBLE ANGLE FORMULA PROOFS

Proof 1: Show that $\cos 2A = \cos^2 A - \sin^2 A$

$$\begin{aligned}\cos 2A &= \cos(A + A)\\ &= \cos A\cos A - \sin A\sin A\\ &= \cos^2 A - \sin^2 A\end{aligned}$$

Proof 2: Express cos2A in terms of cosA and sinA only

$$\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A\\ &= \cos^2 A - (1 - \cos^2 A)\\ &= 2\cos^2 A - 1\end{aligned}$$

or

$$\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A\\ &= (1 - \sin^2 A) - \sin^2 A\\ &= 1 - 2\sin^2 A\end{aligned}$$

Proof 3: Show that $\sin 2A = 2\sin A\cos A$

$$\begin{aligned}\sin 2A &= \sin(A + A)\\ &= \sin A\cos A + \sin A\cos A\\ &= 2\sin A\cos A\end{aligned}$$

MIXED EXAMPLES WITH DOUBLE AND COMPOUND ANGLES

Example 1: Simplify the following without the use of a calculator:

a) $\cos^2 75^\circ + \sin^2 75^\circ$ b) $\cos^2 75^\circ - \sin^2 75^\circ$ c) $1 - 2\sin^2 75^\circ$

d) $\cos 75^\circ \sin 75^\circ$ e) $\dfrac{1-\sin^2 30^\circ}{2\cos^2 15^\circ - 1}$ f) $\cos 22{,}5^\circ$

a) $\cos^2 75^\circ + \sin^2 75^\circ = 1 \rightarrow \cos^2 A + \sin^2 A = 1$

b) $\cos^2 75^\circ - \sin^2 75^\circ \rightarrow \cos^2 A - \sin^2 A = \cos 2A$

$$= \cos\left(2 \times 75^\circ\right)$$
$$= \cos(150^\circ)$$
$$= \cos(180^\circ - 30^\circ)$$
$$= -\cos(30^\circ)$$
$$= -\frac{\sqrt{3}}{2}$$

c) $1 - 2\sin^2 75^\circ \rightarrow 1 - 2\sin^2 A = \cos 2A$

$$= \cos\left(2 \times 75^\circ\right)$$
$$= \cos(150^\circ)$$
$$= -\frac{\sqrt{3}}{2} \rightarrow \text{using the above answer from b)}$$

d) $\cos 75^\circ \sin 75^\circ \rightarrow \sin A \cos A = \frac{1}{2}\sin 2A$

$$= \frac{1}{2}\sin 150^\circ = \frac{1}{2}\sin\left(180^\circ - 30^\circ\right) = \frac{1}{2}\sin 30^\circ = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4}$$

e) $\dfrac{1-\sin^2 30^\circ}{2\cos^2 15^\circ - 1} = \dfrac{\cos^2 30^\circ}{\cos 30^\circ} = \dfrac{\cos 30^\circ}{1} = \dfrac{\sqrt{3}}{2}$

f) Using $\cos 2A = 2\cos^2 A - 1$

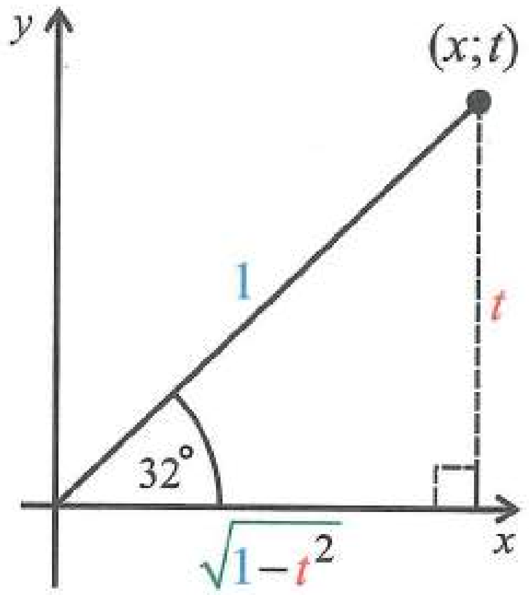
$$\cos 45^\circ = 2\cos^2\left(22{,}5^\circ\right) - 1 \rightarrow \text{solve for } \cos\left(22{,}5^\circ\right)$$
$$\therefore \frac{\sqrt{2}}{2} + 1 = 2\cos^2\left(22{,}5^\circ\right)$$
$$\therefore \frac{\sqrt{2}+2}{2} = 2\cos^2\left(22{,}5^\circ\right)$$
$$\therefore \frac{\sqrt{2}+2}{4} = \cos^2\left(22{,}5^\circ\right)$$
$$\therefore \cos\left(22{,}5^\circ\right) = \pm\sqrt{\frac{\sqrt{2}+2}{4}} \rightarrow 22{,}5^\circ \text{ is in the } 1^{\text{st}} \text{ quadrant, } \therefore \text{ the '}-\text{' is not applicable}$$
$$\therefore \cos\left(22{,}5^\circ\right) = +\frac{\sqrt{\sqrt{2}+2}}{2}$$

Example 2: If $\sin 32^\circ = t$, express each of the following in terms of t:

a) $\sin 212^\circ$ b) $\cos 58^\circ$ c) $\cos 64^\circ$

d) $\cos 62^\circ$ e) $\sin 2^\circ$ f) $\sin 16^\circ$

Draw a diagram when expressing trigonometric ratios in terms of a variable.



$\sin 32^\circ = \frac{t}{1} \rightarrow \sin\theta = \frac{y}{r}$

$\therefore$ The hypotenuse $(r) = 1$ and the y-value $= t$

Use the theorem of Pythagoras to solve for the x-value.

$x^2 = r^2 - y^2$

$\therefore x^2 = 1^2 - t^2$

$\therefore x = \sqrt{1-t^2} \rightarrow x$ is positive as 32° is in the 1st quadrant

When answering the questions try and get the angles in terms of 32° so you can use the diagram.

a) $\sin 212^\circ \rightarrow$ use reduction formulas

$= \sin(180^\circ + 32^\circ)$

$= -\sin 32^\circ$

$= -t$

b) $\cos 58^\circ \rightarrow 90^\circ - 32^\circ = 58^\circ$, use co-functions

$= \cos(90^\circ - 32^\circ) \rightarrow$ co-functions

$= \sin 32^\circ$

$= t$

c) $\cos 64^\circ \rightarrow 64 = 2(32)$, use double angles

$= 1 - 2\sin^2 32^\circ \rightarrow 2\cos^2 32^\circ - 1$ or $\cos^2 32^\circ - \sin^2 32^\circ$ could also have been used

$= 1 - 2t^2$

d) $\cos 62^\circ \rightarrow$ split 62° into 32° and a special angle and then use compound angles

$= \cos(32^\circ + 30^\circ) \rightarrow$ compound angles

$= \cos 32^\circ \cos 30^\circ - \sin 32^\circ \sin 30^\circ$

$= \sqrt{1-t^2}\left(\frac{\sqrt{3}}{2}\right) - t\left(\frac{1}{2}\right)$

$= \frac{\sqrt{3-3t^2} - t}{2}$

e) $\sin 2^\circ \rightarrow$ split 2° into 32° and a special angle and then use compound angles

$$= \sin\left(32^\circ - 30^\circ\right)$$

$$= \sin 32^\circ \cos 30^\circ - \cos 32^\circ \sin 30^\circ$$

$$= t\left(\frac{\sqrt{3}}{2}\right) - \sqrt{1-t^2}\left(\frac{1}{2}\right)$$

$$= \frac{t\sqrt{3} - \sqrt{1-t^2}}{2}$$

f) $\sin 16^\circ \rightarrow 16^\circ$ is half of 32°, use $\cos 2A = 1 - 2\sin^2 A$

$$\therefore \cos 32^\circ = 1 - 2\sin^2 16^\circ$$

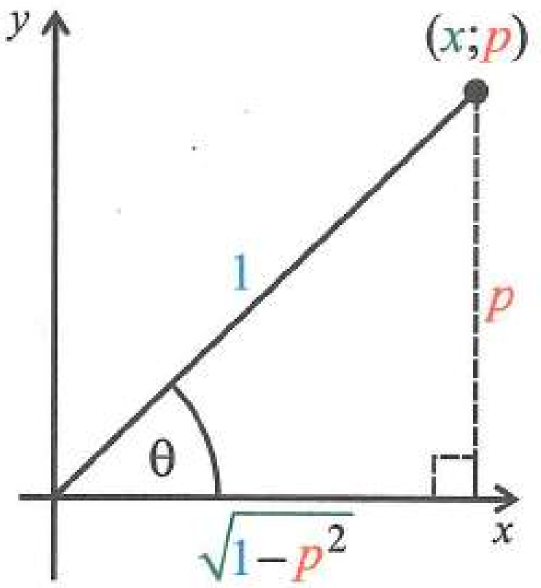
$$\therefore \cos 32^\circ - 1 = -2\sin^2 16^\circ$$

$$\therefore \sqrt{1-t^2} - 1 = -2\sin^2 16^\circ$$

$$\therefore \sin^2 16^\circ = \frac{1-\sqrt{1-t^2}}{2}$$

$$\therefore \sin 16^\circ = +\sqrt{\frac{1-\sqrt{1-t^2}}{2}} \rightarrow \text{positive as } 16^\circ \text{ is in the } 1^{\text{st}} \text{ quadrant}$$

Example 3: If $\sin\theta = p$ and $\theta \in \left[0^\circ; 90^\circ\right]$, with aid of a diagram, determine the value of $\sin\left(\frac{\theta}{2} + 45^\circ\right)\cos\left(\frac{\theta}{2} + 45^\circ\right)$.



$\sin\theta = \frac{p}{1} \rightarrow \sin\theta = \frac{y}{r}$, θ is in quadrant I as $\theta \in \left[0^\circ; 90^\circ\right]$

$\therefore$ The hypotenuse $(r) = 1$ and the y-value $= p$

Use the theorem of Pythagoras to solve for the x-value.

$x^2 = r^2 - y^2$

$\therefore x = \sqrt{1-p^2} \rightarrow x$ is positive as θ is in the 1^{st} quadrant

$$\sin\left(\frac{\theta}{2} + 45^\circ\right)\cos\left(\frac{\theta}{2} + 45^\circ\right) \rightarrow \text{double angle } \sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$$

$$= \frac{1}{2}\sin\left[2\left(\frac{\theta}{2} + 45^\circ\right)\right]$$

$$= \frac{1}{2}\sin\left(\theta + 90^\circ\right) \rightarrow \text{co-functions}$$

$$= \frac{1}{2}\cos\theta$$

$$= \frac{1}{2}\sqrt{1-p^2}$$

Complete Exercise 3 on page 112.

D. TRIGONOMETRIC IDENTITIES USING DOUBLE ANGLES

If the identity that you are attempting to simplify contains a $\cos 2A$ it is often useful to choose the option that eliminates the constant. For example:

	or
$1+\cos 2A$	$\cos 2A - 1$
$=1+\left(2\cos^2 A-1\right)$	$=\left(1-2\sin^2 A\right)-1$
$=2\cos^2 A$	$=-2\sin^2 A$

Only use the identity $\cos^2 A+\sin^2 A=1$ as a last resort.

Example 1: **Prove that $\dfrac{1+\cos x+\cos 2x}{\sin 2x+\sin x}=\dfrac{\cos x}{\sin x}$.**

$$\text{LHS}=\frac{1+\cos x+\cos 2x}{\sin 2x+\sin x}\rightarrow \text{choose the option that eliminates the 1, } \cos 2x=2\cos^2 x-1$$

$$=\frac{1+\cos x+\left(2\cos^2 x-1\right)}{2\sin x\cos x+\sin x}\rightarrow \text{take out a HCF of } \sin x$$

$$=\frac{2\cos^2 x+\cos x}{\sin x\left(2\cos x+1\right)}\rightarrow \text{take out a HCF of } \cos x$$

$$=\frac{\cos x\left(2\cos x+1\right)}{\sin x\left(2\cos x+1\right)}$$

$$=\frac{\cos x}{\sin x}=\text{RHS}$$

Example 2: **Prove that $\dfrac{1-\cos 2\theta}{\sin 2\theta}=\tan\theta$.**

$$\text{LHS}=\frac{1-\cos 2\theta}{\sin 2\theta}\rightarrow \text{choose the option that eliminates the } 1, \cos 2\theta=1-2\sin^2\theta$$

$$=\frac{1-\left(1-2\sin^2\theta\right)}{2\sin\theta\cos\theta}$$

$$=\frac{1-1+2\sin^2\theta}{2\sin\theta\cos\theta}$$

$$=\frac{2\sin^2\theta}{2\sin\theta\cos\theta}$$

$$=\frac{\sin\theta}{\cos\theta}$$

$$=\tan\theta=\text{RHS}$$

Example 3: **Prove that** $\dfrac{\sin 2x - \sin x}{\cos 2x + \cos x} = \dfrac{\sin x}{\cos x + 1}$.

$$\text{LHS} = \frac{\sin 2x - \sin x}{\cos 2x + \cos x} \rightarrow \text{the RHS only has cos in the denominator, } \cos 2x = 2\cos^2 x - 1$$

$$= \frac{2\sin x\cos x - \sin x}{2\cos^2 x - 1 + \cos x}$$

$$= \frac{\sin x(2\cos x - 1)}{2\cos^2 x + \cos x - 1} \rightarrow \text{always look for a trinomial that can factorise}$$

$$= \frac{\sin x(2\cos x - 1)}{(2\cos x - 1)(\cos x + 1)}$$

$$= \frac{\sin x}{\cos x + 1} = \text{RHS}$$

Example 4: **Prove that** $\dfrac{1+\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)} = \dfrac{\cos\left(\frac{\theta}{4}\right)+\sin\left(\frac{\theta}{4}\right)}{\cos\left(\frac{\theta}{4}\right)-\sin\left(\frac{\theta}{4}\right)}$.

Let $\frac{\theta}{4} = \text{A}$

$\therefore \ \frac{\theta}{2} = 2\text{A}$

$\therefore$ RTP: $\dfrac{1+\sin 2\text{A}}{\cos 2\text{A}} = \dfrac{\cos \text{A} + \sin \text{A}}{\cos \text{A} - \sin \text{A}}$ → RTP stands for Required To Prove

$$\text{LHS} = \frac{1+\sin 2\text{A}}{\cos 2\text{A}} \rightarrow \text{there is a cosA and a sinA in the denominaotor of the RHS,}$$

$\therefore$ use $\cos 2\text{A} = \cos^2 \text{A} - \sin^2 \text{A}$

$$= \frac{1 + 2\sin \text{A}\cos \text{A}}{\cos^2 \text{A} - \sin^2 \text{A}} \rightarrow \text{difference of squares}$$

$$= \frac{1 + 2\sin \text{A}\cos \text{A}}{(\cos \text{A} - \sin \text{A})(\cos \text{A} + \sin \text{A})} \rightarrow \text{nothing else can be done } \therefore \text{ use } 1 = \sin^2 \text{A} + \cos^2 \text{A}$$

$$= \frac{\sin^2 \text{A} + \cos^2 \text{A} + 2\sin \text{A}\cos \text{A}}{(\cos \text{A} - \sin \text{A})(\cos \text{A} + \sin \text{A})} \rightarrow \cos^2 \text{A} + 2\sin \text{A}\cos \text{A} + \sin^2 \text{A} = (\cos \text{A} + \sin \text{A})^2$$

$$= \frac{(\cos \text{A} + \sin \text{A})^2}{(\cos \text{A} - \sin \text{A})(\cos \text{A} + \sin \text{A})}$$

$$= \frac{\cos \text{A} + \sin \text{A}}{\cos \text{A} - \sin \text{A}} = \text{RHS}$$

Example 5: Prove that $\dfrac{\sin 8A - \sin 4A}{\cos 8A + \cos 4A} = \tan 2A$ **.**

$$\text{LHS} = \frac{\sin 8A - \sin 4A}{\cos 8A + \cos 4A} \rightarrow \text{use double angles to get all angles in terms of } 4A$$

$$= \frac{2\sin 4A\cos 4A - \sin 4A}{2\cos^2 4A - 1 + \cos 4A} \rightarrow \text{take out a highest common factor of } \sin 4A$$

$$= \frac{\sin 4A(2\cos 4A - 1)}{2\cos^2 4A + \cos 4A - 1} \rightarrow \text{always look for a trinomial that can factorise}$$

$$= \frac{\sin 4A(2\cos 4A - 1)}{(2\cos 4A - 1)(\cos 4A + 1)}$$

$$= \frac{\sin 4A}{\cos 4A + 1} \rightarrow \text{use double angles to get all angles in terms of } 2A$$

$$= \frac{2\sin 2A\cos 2A}{2\cos^2 2A - 1 + 1} \rightarrow \text{the option of } \cos 4A = 2\cos^2 2A \text{ was chosen to eliminate the 1}$$

$$= \frac{2\sin 2A\cos 2A}{2\cos^2 2A}$$

$$= \frac{\sin 2A}{\cos 2A} = \tan 2A = \text{RHS}$$

Example 6: Without the use of a calculator prove that:

a) $\dfrac{\sin 2x}{1 + \cos 2x} = \tan x$ **.** **b) Hence determine the value of** $\tan 15^\circ$**.**

a) $$\text{LHS} = \frac{\sin 2x}{1 + \cos 2x} \rightarrow \text{choose the option that eliminates the constant}$$

$$= \frac{2\sin x\cos x}{1 + 2\cos^2 x - 1}$$

$$= \frac{2\sin x\cos x}{2\cos^2 x} = \frac{\sin x}{\cos x} = \tan x = \text{RHS}$$

b) Using the identity above $\tan 15^\circ = \dfrac{\sin 30^\circ}{1 + \cos 30^\circ}$

$$\therefore \tan 15^\circ = \frac{1}{2} \div \left(1 + \frac{\sqrt{3}}{2}\right) \rightarrow \text{LCD} = 2$$

$$\therefore \tan 15^\circ = \frac{1}{2} \div \left(\frac{2 + \sqrt{3}}{2}\right)$$

$$\therefore \tan 15^\circ = \frac{1}{2} \times \left(\frac{2}{2 + \sqrt{3}}\right) = \frac{1}{2 + \sqrt{3}}$$

Complete Exercise 4 on page 112.

GRADE 11 REVISION

Recap:

When solving trig equations you have to find a reference angle (RA) and remember that the $\sin\theta$ and $\cos\theta$ functions have a period of 360º, while $\tan\theta$ has a period of 180°.

If $\sin\theta = \text{m}$ and $-1 \le \text{m} \le 1$ then:

$\theta = \text{RA} + 360^\circ k$ **or** $\theta = 180^\circ - \text{RA} + 360^\circ k$; $k \in \mathbb{Z}$

If $\cos\theta = \text{m}$ and $-1 \le \text{m} \le 1$ then:

$\theta = \pm\text{RA} + 360^\circ k$; $k \in \mathbb{Z}$

If $\tan\theta = \text{m}$ and $\text{m} \in \mathbb{R}$ then:

$\theta = \text{RA} + 180^\circ k$; $k \in \mathbb{Z}$

In order to find the reference angle:

$\sin\theta / \tan\theta / \cos\theta = \text{m}$

The RA = $\boxed{2nd\ \text{F}}$ or $\boxed{\text{shift}}$ $\boxed{\sin\theta}/\boxed{\tan\theta}/\boxed{\cos\theta}$ $\boxed{\text{m}}$, on the calculator.

Example 1: Determine the general solution of $2\cos^2\theta + \cos\theta - 1 = 0$.

$2\cos^2\theta + \cos\theta - 1 = 0 \rightarrow 2x^2 + x - 1$, factorise

$\therefore (2\cos\theta - 1)(\cos\theta + 1) = 0$

$\therefore \cos\theta = \dfrac{1}{2}$ or $\cos\theta = -1$

$\therefore \text{RA} = 60^\circ$ or $\text{RA} = 180^\circ$

$\therefore \theta = \pm 60^\circ + 360^\circ k$ *or* $\theta = \pm 180^\circ + 360^\circ k$; $k \in \mathbb{Z} \rightarrow$ always put in $k \in \mathbb{Z}$

Example 2: Solve for x if $2\sin x\cos x = \sin x$.

$2\sin x\cos x = \sin x \rightarrow$ never divide both sides by $\sin x$ as it could be equal to 0

$\therefore 2\sin x\cos x - \sin x = 0 \rightarrow$ factorise, take out a HCF of $\sin x$

$\therefore \sin x(2\cos x - 1) = 0$

$\therefore \sin x = 0$ or $\cos x = \dfrac{1}{2}$

$\therefore \text{RA} = 0^\circ$ or $\text{RA} = 60^\circ$

$\therefore x = 0^\circ + 360^\circ k$ or $x = 180^\circ - 0^\circ + 360^\circ k$ or $x = \pm 60^\circ + 360^\circ k$; $k \in \mathbb{Z}$

Example 3: Solve for x if $\sin x = 3\cos x$ and $x \in \left[-180^\circ ; 360^\circ\right]$.

$\sin x = 3\cos x \rightarrow$ you can only divide by $\cos x$ to form a $\tan x$

$\therefore \tan x = 3$

$\therefore \text{RA} = 71{,}57^\circ$

$\therefore x = 71{,}57^\circ + 180^\circ k;\ k \in \mathbb{Z}$

But $x \in \left[-180^\circ ; 360^\circ\right]$, therefore you have to draw a table and substitute different values for k, to see which values of x fall inside this interval.

k	-2	-1	0	1	2
x	$-288{,}43^\circ$	$-108{,}43^\circ$	$71{,}57^\circ$	$251{,}57^\circ$	$431{,}57^\circ$
	N/A	✓	✓	✓	**N/A**

$\therefore x = -108{,}43^\circ$ or $x = 71{,}57^\circ$ or $x = 251{,}57^\circ$

The table function on your calculator can be used to draw the above table. No marks are awarded for the table, only the answers for x.

Example 4: Determine the general solution of $\sin 2x = \cos\left(x - 10^\circ\right)$.

$\sin 2x = \cos\left(x - 10^\circ\right) \rightarrow$ use co-ratios

$\therefore \cos\left(90^\circ - 2x\right) = \cos\left(x - 10^\circ\right)$

$\therefore \text{RA} = x - 10^\circ \rightarrow 90^\circ - 2x$ could also have been used as the RA

$\therefore 90^\circ - 2x = x - 10^\circ + 360^\circ k$ or $90^\circ - 2x = -\left(x - 10^\circ\right) + 360^\circ k;\ k \in \mathbb{Z}.$

$\therefore -3x = -100^\circ + 360^\circ k$ or $-2x = -x + 10^\circ - 90^\circ + 360^\circ k$

$\therefore x = 33{,}33^\circ - 120^\circ k$ or $-x = -80^\circ + 360^\circ k$

$x = 80^\circ - 360^\circ k$

Example 5: Determine the general solution of $3\cos^2 x - 5\sin x - 1 = 0$.

$3\cos^2 x - 5\sin x - 1 = 0 \rightarrow$ get the equation in terms of $\sin x$ using $\cos^2 x = 1 - \sin^2 x$

$\therefore 3\left(1 - \sin^2 x\right) - 5\sin x - 1 = 0$

$\therefore 3 - 3\sin^2 x - 5\sin x - 1 = 0$

$\therefore 3\sin^2 x + 5\sin x - 2 = 0$

$\therefore \left(3\sin x - 1\right)\left(\sin x + 2\right) = 0$

$\therefore \sin x = \frac{1}{3}$ or $\sin x = -2 \rightarrow$ N/A, $-1 \le \sin x \le 1$

$\therefore \text{RA} = 19{,}47^\circ$

$\therefore x = 19{,}47^\circ + 360^\circ k$ or $x = 180^\circ - 19{,}47^\circ + 360^\circ k;\ k \in \mathbb{Z}$

$= 160^\circ{,}53 + 360^\circ k$

Complete Exercise 5 on page 113.

TRIGONOMETRIC EQUATIONS USING DOUBLE AND COMPOUND ANGLES

Example 1: Determine the general solution of $\sin\theta\cos 60^\circ - \sin 60^\circ\cos\theta = 0{,}5$.

$\sin\theta\cos 60^\circ - \sin 60^\circ\cos\theta = 0{,}5 \rightarrow$ compound angles, $\sin(A-B) = \sin A\cos B - \sin B\cos A$

$\therefore \sin(\theta - 60^\circ) = 0{,}5$

$\therefore \text{RA} = 30^\circ$

$\therefore \theta - 60^\circ = 30^\circ + 360^\circ k$ or $\theta - 60^\circ = 180^\circ - 30^\circ + 360^\circ k$; $k \in \mathbb{Z}$

$\therefore \theta = 90^\circ + 360^\circ k$ or $\theta = 210^\circ + 360^\circ k$

Example 2: Solve for x if $\cos 2x + \cos x + 1 = 0$.

$\cos 2x + \cos x + 1 = 0$

$2\cos^2 x - 1 + \cos x + 1 = 0 \rightarrow \cos 2x = 2\cos^2 x - 1$

$\therefore 2\cos^2 x + \cos x = 0$

$\therefore \cos x(2\cos x + 1) = 0$

$\therefore \cos x = 0$ or $\cos x = -\frac{1}{2}$

$\therefore \text{RA} = 90^\circ$ or $\text{RA} = 120^\circ$

$\therefore x = \pm 90^\circ + 360^\circ k$ or $x = \pm 120^\circ + 360^\circ k$; $k \in \mathbb{Z}$

Example 3: Solve for θ if $\cos 2\theta - 3\cos\theta - 1 = 0$.

$\cos 2\theta - 3\cos\theta - 1 = 0 \rightarrow$ write in terms of $\cos\theta$ using $\cos 2\theta = 2\cos^2\theta - 1$

$\therefore 2\cos^2\theta - 1 - 3\cos\theta - 1 = 0$

$\therefore 2\cos^2\theta - 3\cos\theta - 2 = 0 \rightarrow$ factorise the trinomial

$\therefore (2\cos\theta + 1)(\cos\theta - 2)$

$\therefore \cos\theta = -\frac{1}{2}$ or $\cos\theta = 2 \rightarrow$ N/A, $-1 \le \cos\theta \le 1$

$\therefore \text{RA} = 120^\circ$

$\therefore \theta = \pm 120^\circ + 360^\circ k$; $k \in \mathbb{Z}$

Example 4: Solve for θ if $4\cos^2\theta + 2\sin 2\theta + \sin^2\theta = 0$.

$4\cos^2\theta + 2\sin 2\theta + \sin^2\theta = 0 \rightarrow \sin 2\theta = 2\sin\theta\cos\theta$

$\therefore 4\cos^2\theta + 2(2\sin\theta\cos\theta) + \sin^2\theta = 0$

$\therefore 4\cos^2\theta + 4\sin\theta\cos\theta + \sin^2\theta = 0 \rightarrow$ factorise

$\therefore (2\cos\theta + \sin\theta)^2 = 0$

$\therefore 2\cos\theta + \sin\theta = 0$

$\therefore \sin\theta = -2\cos\theta \rightarrow$ divide both sides by $\cos\theta$

$\therefore \tan\theta = -2$

$\therefore \text{RA} = -63{,}43^\circ$

$\therefore \theta = -63{,}43^\circ + 180^\circ k$; $k \in \mathbb{Z}$

Example 5: **Solve for x if $\cos^2 x + \sin 2x - 1 = 0$ and $\sin x \neq 0$.**

$\cos^2 x + \sin 2x - 1 = 0 \rightarrow \sin 2x = 2\sin x \cos x$

$\therefore \cos^2 x + 2\sin x \cos x - 1 = 0 \rightarrow \cos^2 x + \sin^2 x = 1$

$\therefore \cos^2 x + 2\sin x \cos x - \left(\cos^2 x + \sin^2 x\right) = 0$

$\therefore \cos^2 x + 2\sin x \cos x - \cos^2 x - \sin^2 x = 0$

$\therefore 2\sin x \cos x - \sin^2 x = 0 \rightarrow$ factorise, take out a HCF of $\sin x$

$\therefore \sin x(2\cos x - \sin x) = 0$

$\therefore 2\cos x - \sin x = 0$ or $\sin x = 0 \rightarrow$ N/A as $\sin x \neq 0$

$\therefore 2\cos x = \sin x$

$\therefore \tan x = 2$

$\therefore \text{RA} = 63.43^\circ$

$\therefore x = 63.43^\circ + 180^\circ k;\ k \in \mathbb{Z}$

Example 6: **Solve for θ if $\cos\theta - \sqrt{3}\sin\theta = \sqrt{3}$.**

$\cos\theta - \sqrt{3}\sin\theta = \sqrt{3}$

$\therefore \cos\theta - \tan 60^\circ \sin\theta = \tan 60^\circ \rightarrow \tan 60^\circ = \sqrt{3}$, look for special angles

$$\therefore \cos\theta - \frac{\sin 60^\circ}{\cos 60^\circ}\sin\theta = \frac{\sin 60^\circ}{\cos 60^\circ}$$

$\therefore \cos\theta\cos 60^\circ - \sin 60^\circ \sin\theta = \sin 60^\circ \rightarrow \cos(\text{A}+\text{B}) = \cos\text{A}\cos\text{B} - \sin\text{A}\sin\text{B}$

$\therefore \cos\left(\theta + 60^\circ\right) = \sin 60^\circ$

$\therefore \cos\left(\theta + 60^\circ\right) = \cos 30^\circ \rightarrow \sin(60^\circ) = \sin(90^\circ - 30^\circ) = \cos 30^\circ$

$\therefore \text{RA} = 30^\circ$

$\therefore \theta + 60^\circ = \pm 30^\circ + 360^\circ k \rightarrow$ this represents two answers

$\therefore \theta = -30^\circ + 360^\circ k$ or $\theta = -90^\circ + 360^\circ k;\ k \in \mathbb{Z}$

RESTRICTIONS ON IDENTITIES AND TRIGONOMETRIC EQUATIONS

It is important to realise that the denominator of a fraction can never be equal to 0.

From the identities learnt in Grade 11 it is clear that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.

$\therefore \tan\theta$ is undefined when $\cos\theta = 0$

In other words when: $\theta = \pm 90^\circ + 360^\circ k\,;\, k \in \mathbb{Z}$

This is the reason for the asymptotes on the tan graph.

Example 1: **Given the identity $\dfrac{2\sin^3 x + (\sin 2x)(\cos x)}{2\cos x} = \tan x$.**

a) Prove the identity above.
b) For what values of x is the above identity undefined?

a) $\text{LHS} = \dfrac{2\sin^3 x + (\sin 2x)(\cos x)}{2\cos x} \rightarrow$ double angle, $\sin 2x = 2\sin x\cos x$

$= \dfrac{2\sin^3 x + 2\sin x\cos x(\cos x)}{2\cos x}$

$= \dfrac{2\sin^3 x + 2\sin x\cos^2 x}{2\cos x} \rightarrow$ take out a HCF of $2\sin x$

$= \dfrac{2\sin x(\sin^2 x + \cos^2 x)}{2\cos x} \rightarrow \sin^2 x + \cos^2 x = 1$

$= \tan x = \text{RHS}$

b) For the identity to be undefined $2\cos x = 0$

$\therefore \cos x = 0$

$\therefore \text{RA} = 90^\circ$

$\therefore x = \pm 90^\circ + 360^\circ k\,;\, k \in \mathbb{Z}$

Example 2: **Determine the general solution of $\tan\theta\sin\theta = \tan\theta$.**

$\tan\theta\sin\theta = \tan\theta \rightarrow$ never divide by $\tan\theta$ as it could be equal to 0

$\therefore \tan\theta\sin\theta - \tan\theta = 0 \rightarrow$ factorise, take out a HCF of $\tan\theta$

$\therefore \tan\theta(\sin\theta - 1) = 0$

$\therefore \tan\theta = 0$ or $\sin\theta = 1$

$\therefore \text{RA} = 0^\circ$ or $\text{RA} = 90^\circ$

$\therefore \theta = 180^\circ k$ or $\therefore \theta = 90^\circ + 360^\circ k \rightarrow \text{N/A}$

$\tan\theta$ is undefined when $\cos\theta = 0$ $\left(\tan\theta = \dfrac{\sin\theta}{\cos\theta}\right)$,

$\cos\theta = 0$ when $\theta = \pm 90^\circ + 360^\circ k\,;\; k \in \mathbb{Z}$

Complete Exercise 6 on page 113 and the Mixed Exercise on pages 113 and 114.

Chapter 4 Trigonometry (Exercises)

Grade 11 Revision

Exercise 1: (answers on page 349)

1) **Simplify the following expressions:**

a) $\sin^2 108^\circ + \sin^2 18^\circ$

b) $\dfrac{\sin\left(360^\circ - \theta\right)\cos\left(180^\circ + \theta\right)}{\sin\left(270^\circ\right)\cos\left(360^\circ - \theta\right)\sin\left(90 + \theta\right)}$

c) $\dfrac{\sin\left(180^\circ - \theta\right)\tan\left(180^\circ + \theta\right)\cos\left(\theta + 360^\circ\right)}{\cos\left(90^\circ - \theta\right)\cos\left(-\theta\right)\tan\left(360^\circ - \theta\right)}$

d) $\dfrac{\sin\left(315^\circ\right)\cos\left(-315^\circ\right)\sin\left(210^\circ\right)}{\tan\left(225^\circ\right)}$

2) **Prove the following trigonometric identities. You may ignore all restrictions.**

a) $\sin x \tan x + \cos x = \dfrac{1}{\cos x}$

b) $\dfrac{\cos x}{1 - \sin x} - \dfrac{\cos x}{1 + \sin x} = 2\tan x$

c) $\dfrac{2\sin^2 x - 1}{\cos x \sin x} = \tan x - \dfrac{\cos x}{\sin x}$

d) $\dfrac{\sin^3 x + \sin x \cos^2 x}{\cos x} = \tan x$

3) **Determine the value of $\sin\theta\cos\theta$ if $\tan\theta = \dfrac{3}{4}$ and $\theta \in \left[180^\circ; 360^\circ\right]$**

Compound Angles

Exercise 2: (answers on page 349)

1) **Expand each of the following using a compound angle formula:**

a) $\sin\left(\theta + 50^\circ\right)$

b) $\sin\left(\theta - 25^\circ\right)$

c) $\cos\left(3x - 40^\circ\right)$

d) $\cos\left(2x + y\right)$

2) **Use the compound angle formulas to write each of the following as a single term:**

a) $\cos 4A \cos 2A + \sin 2A \sin 4A$

b) $\sin 3\theta \cos\theta - \cos 3\theta \sin\theta$

c) $\sin x \sin y - \cos x \cos y$

d) $\cos 70^\circ \cos 10^\circ - \cos 20^\circ \cos 80^\circ$

3) **Evaluate each of the following without the use of a calculator:**

a) $\cos 10^\circ \cos 20^\circ - \sin 10^\circ \sin 20^\circ$

b) $\cos 230^\circ \cos 160^\circ + \sin 310^\circ \sin 200^\circ$

c) $\cos 62^\circ \sin 28^\circ - \cos 28^\circ \sin 242^\circ$

d) $\sin 20^\circ \sin 170^\circ - \cos 350^\circ \sin 70^\circ$

4) **Prove the following identities using compound angle formulas:**

a) $\sin\left(270^\circ + \theta\right) = -\cos\theta$

b) $\cos\left(180^\circ + \theta\right) = -\cos\theta$

5) **Prove the following identities without the use of a calculator:**

a) $\cos\left(\theta + 60^\circ\right) - \sqrt{3}\cos\left(\theta + 30^\circ\right) = -\cos\theta$

b) $\cos\left(\alpha + \beta\right) + \cos\left(\alpha - \beta\right) = 2\cos\alpha\cos\beta$

c) $\sin 2\theta = 2\sin\theta\cos\theta$

d) $\cos 2\theta = 1 - 2\sin^2\theta$

e) $\sin\left(x + 30^\circ\right) - \sin\left(x - 30^\circ\right) = \cos x$

f) $\cos 7\mathrm{A} + \cos \mathrm{A} = 2\cos 4\mathrm{A} \cos 3\mathrm{A}$

6) **Evaluate the following without the use of a calculator:**

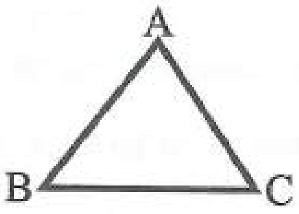
a) $\sin 15^\circ$

b) $\cos 105^\circ$

7) Given triangle ABC alongside:

a) Determine an expression for Angle A in terms of B and C.

b) Hence, prove that $\cos A = \sin B \sin C - \cos B \cos C$.



Double and Compound Angles

Exercise 3: (answers on pages 349 and 350)

1) Evaluate the following without the use of a calculator:

a) $2\sin 15^\circ \cos 15^\circ$

b) $\cos^2 15^\circ - \sin^2 15^\circ$

c) $\cos^2 15^\circ + \sin^2 15^\circ$

d) $1 - 2\sin^2 22{,}5^\circ$

e) $-\sin 165^\circ \cos 165^\circ$

f) $2\cos^2 75^\circ - 1$

g) $4\sin 22{,}5^\circ \cos 22{,}5^\circ$

h) $(\sin 15^\circ + \cos 15^\circ)^2$

2) Given that $\sin 35^\circ = p$, express each of the following in terms of p:

a) $\cos 325^\circ$

b) $\sin 55^\circ$

c) $\cos 70^\circ$

d) $\cos 95^\circ$

3) Simplify the following expressions:

a) $\dfrac{\sin 140^\circ}{\sin 20^\circ \cos 20^\circ}$

b) $\sin 67{,}5^\circ \cos 67{,}5^\circ$

c) $\dfrac{\tan(-60^\circ)\cos(-145^\circ)\cos(235^\circ)}{\sin(470^\circ)}$

d) $\dfrac{8\cos^2 35^\circ - 8\sin^2 35^\circ}{2\sin 20^\circ}$

Trigonometric Identities

Exercise 4: (answers on page 350)

1) Prove the following identities:

a) $(\cos\theta - \sin\theta)^2 = 1 - \sin 2\theta$

b) $\dfrac{\sin x - \cos x}{\cos 2x} = -\dfrac{1}{\cos x + \sin x}$

c) $\cos^4 x - \sin^4 x = 2\cos^2 x - 1$

d) $\tan\theta + \dfrac{1}{\tan\theta} = \dfrac{2}{\sin 2\theta}$

e) $\sin 4\theta = 4\sin\theta\cos\theta\cos 2\theta$

f) $\cos 4x = 1 - 8\sin^2 x + 8\sin^4 x$

g) $\dfrac{\tan\theta}{\tan 2\theta} = \dfrac{\cos^2\theta - \sin^2\theta}{1 + \cos 2\theta}$

h) $\dfrac{\cos 2\theta + \cos\theta + 1}{\sin\theta + \sin 2\theta} = \dfrac{1}{\tan\theta}$

i) $\dfrac{\sin 2x - \sin x}{\cos 2x + \cos x} = \dfrac{\sin x}{\cos x + 1}$

j) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

2) Given that $\cos(45^\circ + x)\cos(45^\circ - x) = \frac{1}{2}\cos 2x$.

a) Prove the above identity.

b) Hence, determine the value of $\cos 22{,}5^\circ \cos 67{,}5^\circ$ without the use of a calculator.

3) Given that $\dfrac{1+\tan\theta}{1-\tan\theta} = \dfrac{1+\sin 2\theta}{\cos 2\theta}$.

a) Prove the above identity.

b) Hence, determine the value of $\dfrac{1+\tan 15^\circ}{1-\tan 15^\circ}$ without the use of a calculator.

4) Given that $\tan\theta = \dfrac{\sin 2\theta}{1+\cos 2\theta}$.

a) Prove the above identity.

b) Hence, determine the value of $\tan 22{,}5^\circ$ without the use of a calculator.

Grade 11 Trigonometric Equations Revision

Exercise 5: (answers on page 350)

1) **Solve for θ in the following equations, correct to 1 decimal place where applicable:**

   a) $2\sin\theta\tan\theta - \tan\theta = 0$ if $\theta \in \left[0^\circ;360^\circ\right]$

   b) $4\cos\theta = \dfrac{1}{\cos\theta}$

   c) $3\sin^2\theta - 5\sin\theta = 2$

   d) $2\tan^2\theta + \tan\theta - 1 = 0$

Trigonometric Equations Using Double and Compound Angles

Exercise 6: (answers on page 350)

1) **Solve for θ in the following equations:**

   a) $\cos 2\theta + \cos^2\theta - 5\cos\theta - 1 = 0$

   b) $\cos^2\theta + 2\sin 2\theta + 3\sin^2\theta = 0$

   c) $2\cos\theta - \cos 2\theta - 1 = 0$

   d) $2\cos^2\theta + 2\sin 2\theta + 1 = 0$

   e) $\cos 2\theta\cos 30^\circ - \sin 2\theta\sin 30^\circ = 0,5$

   f) $\cos 2\theta\sin 10^\circ - \cos 10^\circ\sin 2\theta = \sin 20^\circ$

2) **Given the identity** $\dfrac{\sin 2x + \sin x}{\cos 2x + \cos x + \sin 90^\circ} = \tan x$ **:**

   a) Prove the above identity.

   b) Hence, determine the value/s of x for which the identity is undefined.

3) **Given the identity:** $\dfrac{\sin 2x}{\cos 2x} + \dfrac{1}{\cos 2x} = \dfrac{\sin x + \cos x}{\cos x - \sin x}$ **where** $x \in \left[-180^\circ;180^\circ\right]$

   a) Prove the above identity.

   b) Hence, determine the value/s of x for which the identity is undefined.

Mixed Exercise: (answers on pages 350 and 351)

1) **Simplify the following:**

   a) $\dfrac{\cos\left(90^\circ + x\right)\cos\left(x - 180^\circ\right)\tan\left(180^\circ + x\right)}{\sin(-x)\cos(x - 360)}$

   b) $\cos 72^\circ\sin 48^\circ - \sin 108^\circ\sin\left(-42^\circ\right)$

   c) $\dfrac{2\cos^2 40^\circ - 2\sin^2 40^\circ}{8\sin 5^\circ\cos 5^\circ}$

   d) $\dfrac{\sin 99^\circ}{\sin 33^\circ} - \dfrac{\cos 99^\circ}{\cos 33^\circ}$

2) **Given that** $\dfrac{\sin^4\theta + \sin^2\theta\cos^2\theta}{1 + \cos\theta} = 1 - \cos\theta$ **.**

   a) Prove the above identity.

   b) For which values of θ is the above identity undefined?

3) **Given that** $\dfrac{\sin 2x + \sin x}{\cos 2x + \cos x + 1} = \tan x$ **.**

   a) Prove the above identity.

   b) For which values of x is the above identity undefined?

4) **Given that** $\cos(A + B) + \cos(A - B) = 2\cos A\cos B$ **.**

   a) Prove the above identity.

   b) Hence, determine the value of $\cos 80^\circ + \cos 40^\circ$ if $\cos 20^\circ = 0,94$

5) **If** $\sin 50^\circ = k$, **determine the value of** $\dfrac{\sin 25^\circ\cos 25^\circ}{4 - 8\cos^2 25^\circ}$ **in terms of** k**.**

6) **If** $\cos 4^\circ = m$, **determine the value of:**

   a) $\cos 38^\circ\cos 34^\circ + \sin 34^\circ\sin 38^\circ$

   b) $\cos^2 178^\circ - \cos^2 272^\circ$

7) Determine the general solution of $\sin 2x - \sin x = 2\cos^2 x - \cos x$.

8) Prove the following identities:

a) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

b) $2\sin^2\left(\theta - 45^\circ\right) + \sin 2\theta = 1$

9) Determine the value of $\frac{1}{2}\left(\cos 15 + \sqrt{3}\sin 15\right)$.

CHAPTER 5
ASSESSMENT ON TERM 1

A. TERM 1 ASSESSMENT: EXEMPLAR TEST

Marks: 50
Time: 60 minutes

INSTRUCTIONS AND INFORMATION

Read the following instructions carefully before answering the questions:

1. This question paper consists of 5 questions.
2. Answer ALL the questions.
3. Clearly show ALL calculations, diagrams, graphs, et cetera that you have used in determining the answers.
4. Answers only will not necessarily be awarded full marks.
5. An approved scientific calculator (non-programmable and non-graphical) may be used, unless stated otherwise.
6. Round your answers off to TWO decimal places if necessary, unless stated otherwise.
7. Diagrams are NOT necessarily drawn to scale.
8. Number the answers correctly according to the numbering system used in this question paper.
9. Write legibly and present your work neatly.

Question 1

The table below shows two number sequences:

- One of these sequences is geometric
- The other is arithmetic

	Sequence
1	$-1;4;-16;64;...$
2	$10;\ 8,75;\ 7,5;\ 6,25\ ;...$

1.1 From the sequences given above, identify and write down the geometric sequence. (1)

1.2 Explain why the sum to infinity of this sequence cannot be found. (1)

1.3 Determine the sum to 11 terms of the geometric sequence. (3)

1.4 Determine which term of the arithmetic sequence is equal to –35 (3)

[8]

Question 2

In a clinical trial it is found that the half-life of a new drug is 1 hour. This means that the amount of drugs present in a person's blood stream will be half of the original amount after 1 hour.

2.1 Write down the first 3 terms of the geometric sequence that would model the amount of drugs in a person's bloodstream at the beginning of each hour if 5 000 mg of the drug is initially administered. (1)

2.2 Determine the common ratio of this sequence. (1)

2.3 How long will it take for there to be less than 0,05mg of the drug remaining in a person's blood stream? (6)

[8]

Question 3

A printing company has just bought a new printing press for R1 200 000. Due to improvements in technology, the company expects to have to replace the press in 8 years' time. The replacement cost of a new printing press is expected to increase by 7% per annum.

3.1 In 8 years' time, the current press will be sold for R300 000. If the proceeds of the sale are used to purchase a new printing press, how much extra will need to be paid. (3)

3.2 One month after purchasing the present press, the company deposited x Rand into an account that pays interest at a rate of 11,5% p.a. compounded monthly. The same amount was then deposited at the end of each month for 8 years. At the end of 8 years, the company has exactly the amount that is needed to purchase a new press. Determine the value of x, if the proceeds of selling the current press is used to contribute to the purchase of the new printing press. (6)

3.3 Suppose that 12 months after the purchase of the present press and every 12 months thereafter, R6 500 is withdrawn from the account, to pay for maintenance and running expenses of the press. If 8 such withdrawals are made, what will the new monthly deposit be? (4)

[13]

Question 4

4.1

4.1.1 Prove that $\sin^2 2\theta\left(\dfrac{1}{\tan^2\theta}-\tan^2\theta\right)=4\cos 2\theta$. (5)

4.1.2 Hence, determine the general solution for:

$$\left(\frac{\sin^2 2\theta}{4}\right)\left(\frac{1}{\tan^2\theta}-\tan^2\theta\right)=\frac{\sqrt{3}}{2}$$ (5)

4.2 If $12\tan p-5=0$ and $\cos p<0$, using a diagram and without the use of a calculator, determine the value of:

4.2.1 $\cos 2p$ (4)

4.2.2 $\sin\left(p-45^\circ\right)$ (3)

[17]

Question 5

The identity $\sin\left(30^\circ+\theta\right)+\sin\left(30^\circ-\theta\right)=\cos\theta$ is given. Using this identity it can be shown that $\sin 43^\circ+\sin 17^\circ=\cos 13^\circ$ without a calculator:

5.1 Use the above identity to determine the value of $\sin 40^\circ+\sin 20^\circ$. You may leave your answer in terms of a trigonometric ratio. (1)

5.2 Use the above identity to determine the exact value of $\sin 35^\circ+\sin 25^\circ-\sin 85^\circ$. (3)

[4]

Total: [50]

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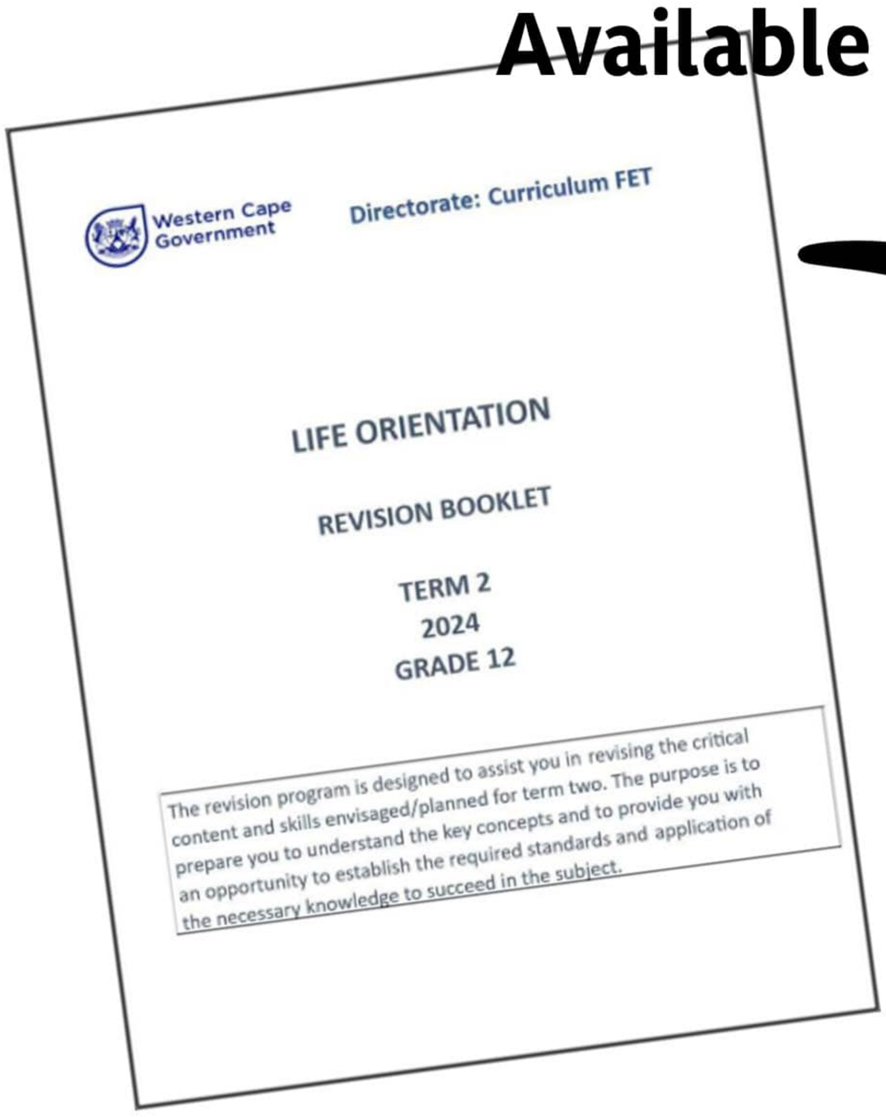

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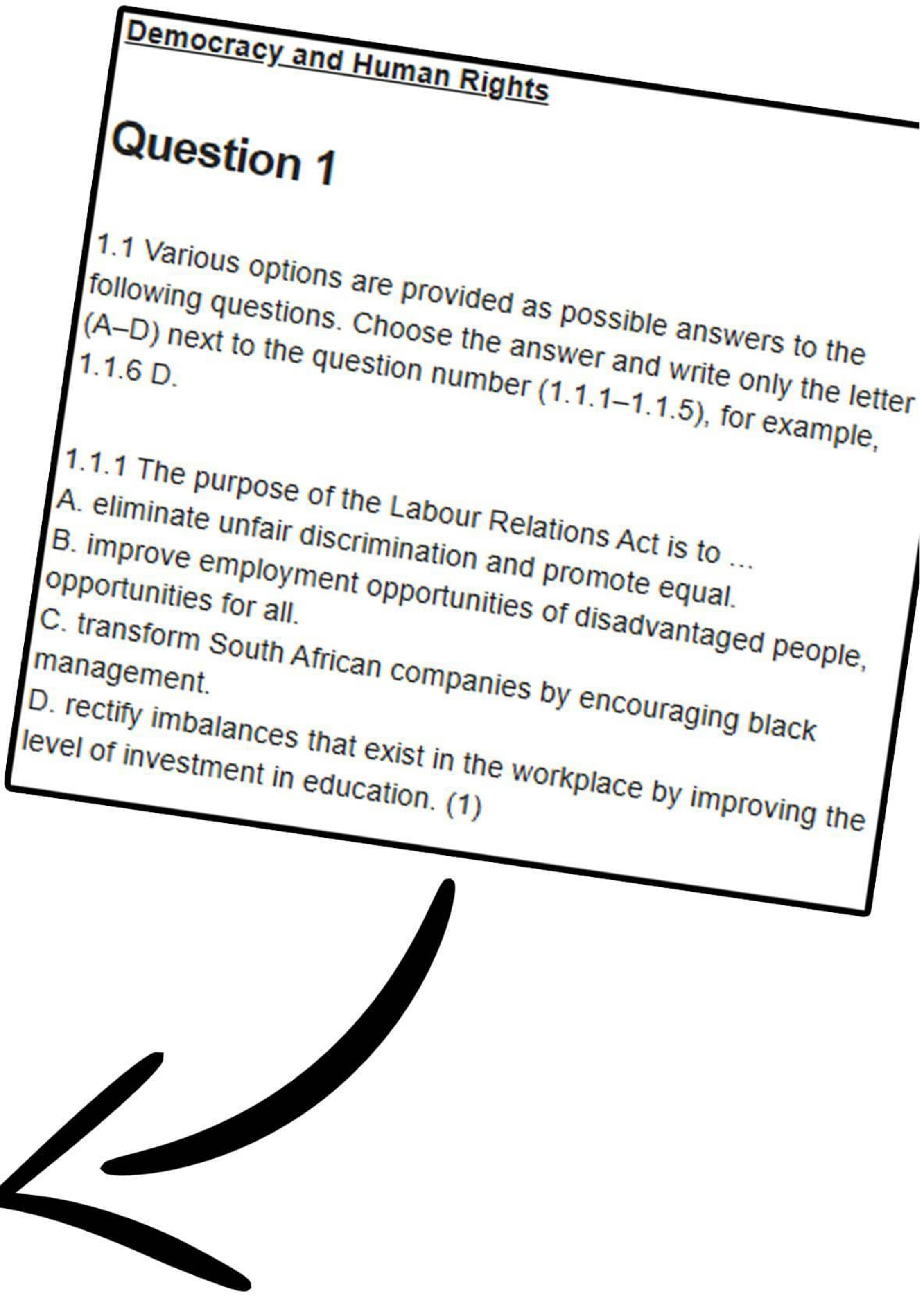
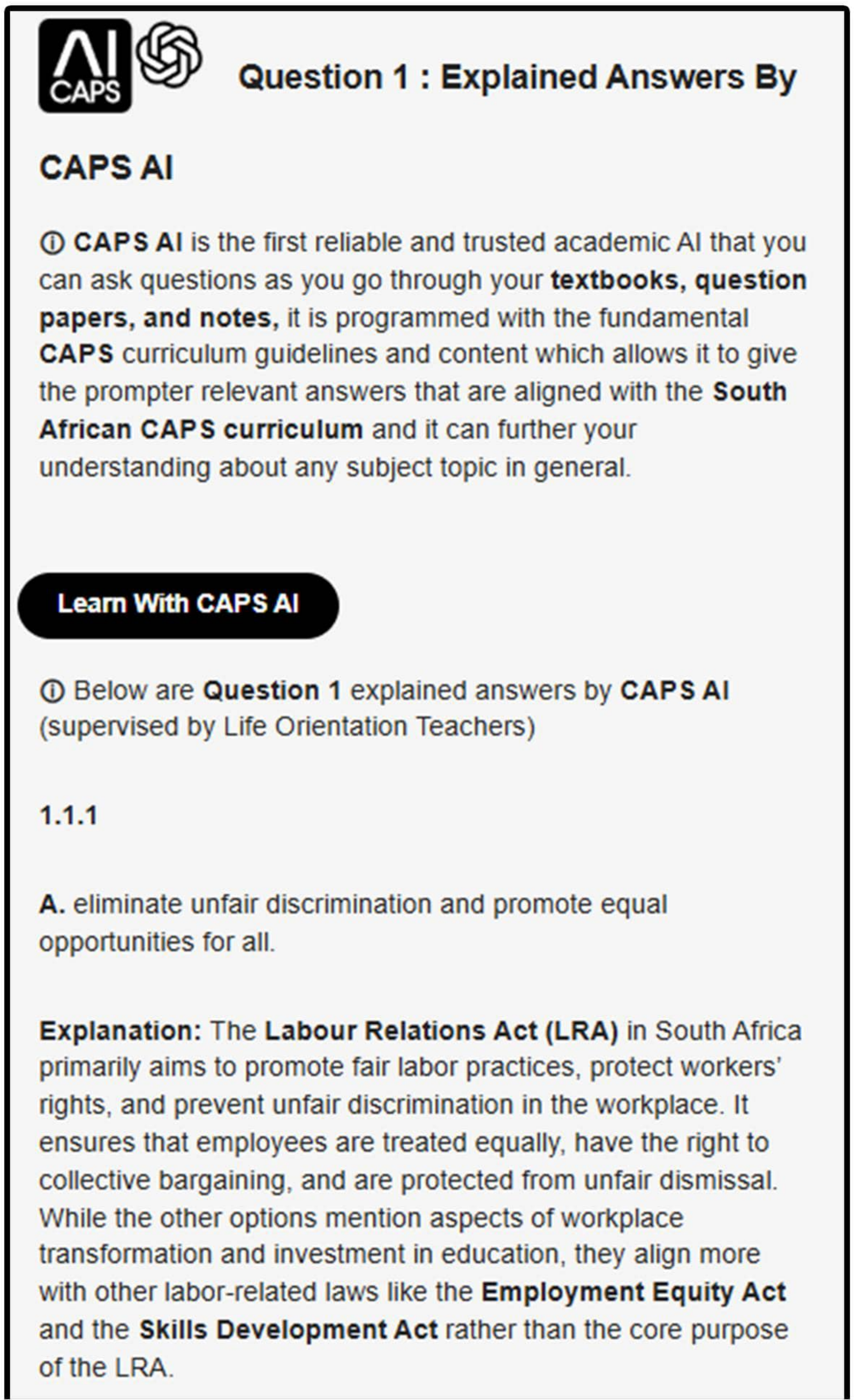


Democracy and Human Rights

Question 1

1.1 Various options are provided as possible answers to the following questions. Choose the answer and write only the letter (A–D) next to the question number (1.1.1–1.1.5), for example, 1.1.6 D.

1.1.1 The purpose of the Labour Relations Act is to …
A. eliminate unfair discrimination and promote equal opportunities for all.
B. improve employment opportunities of disadvantaged people,
C. transform South African companies by encouraging black management.
D. rectify imbalances that exist in the workplace by improving the level of investment in education. (1)



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ⓘ Below are **Question 1** explained answers by **CAPS AI** (supervised by Life Orientation Teachers)

1.1.1

A. eliminate unfair discrimination and promote equal opportunities for all.

Explanation: The **Labour Relations Act (LRA)** in South Africa primarily aims to promote fair labor practices, protect workers' rights, and prevent unfair discrimination in the workplace. It ensures that employees are treated equally, have the right to collective bargaining, and are protected from unfair dismissal. While the other options mention aspects of workplace transformation and investment in education, they align more with other labor-related laws like the **Employment Equity Act** and the **Skills Development Act** rather than the core purpose of the LRA.

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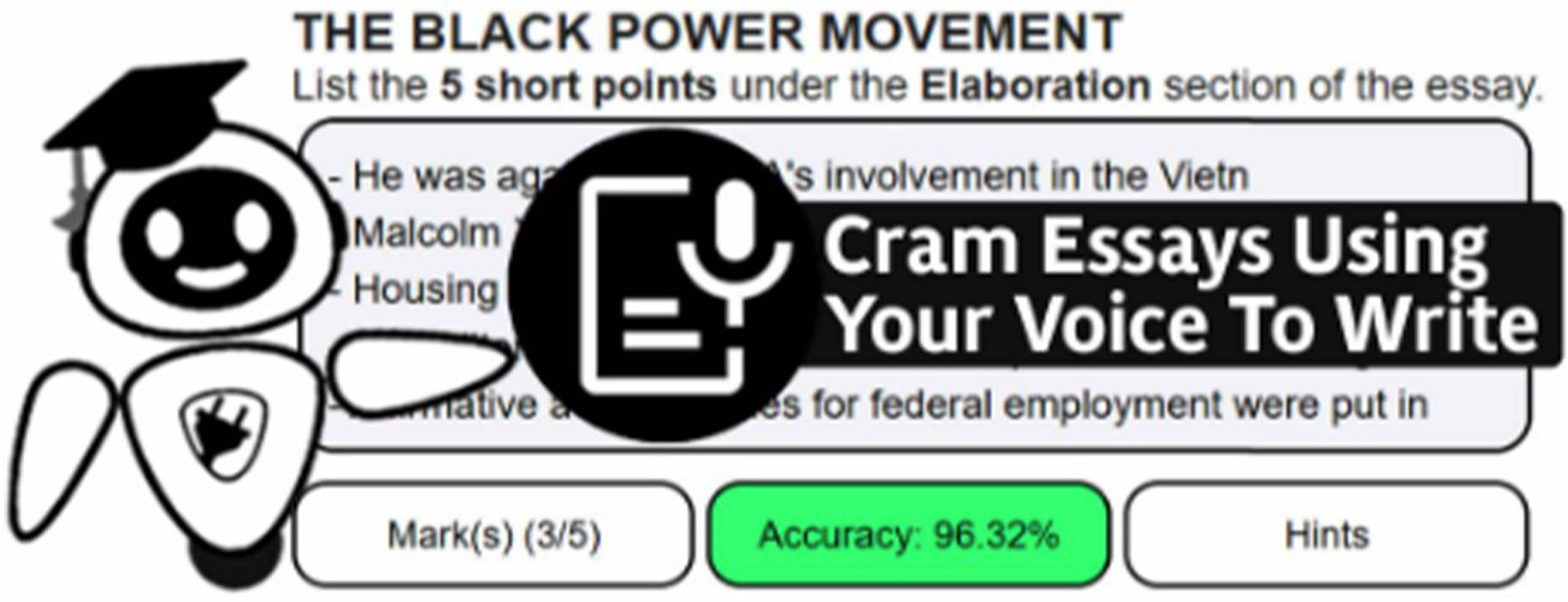

THE BLACK POWER MOVEMENT

List the **5 short points** under the **Elaboration** section of the essay.

- He was ag 's involvement in the Vietn
- Malcolm
- Housing
- mative a es for federal employment were put in

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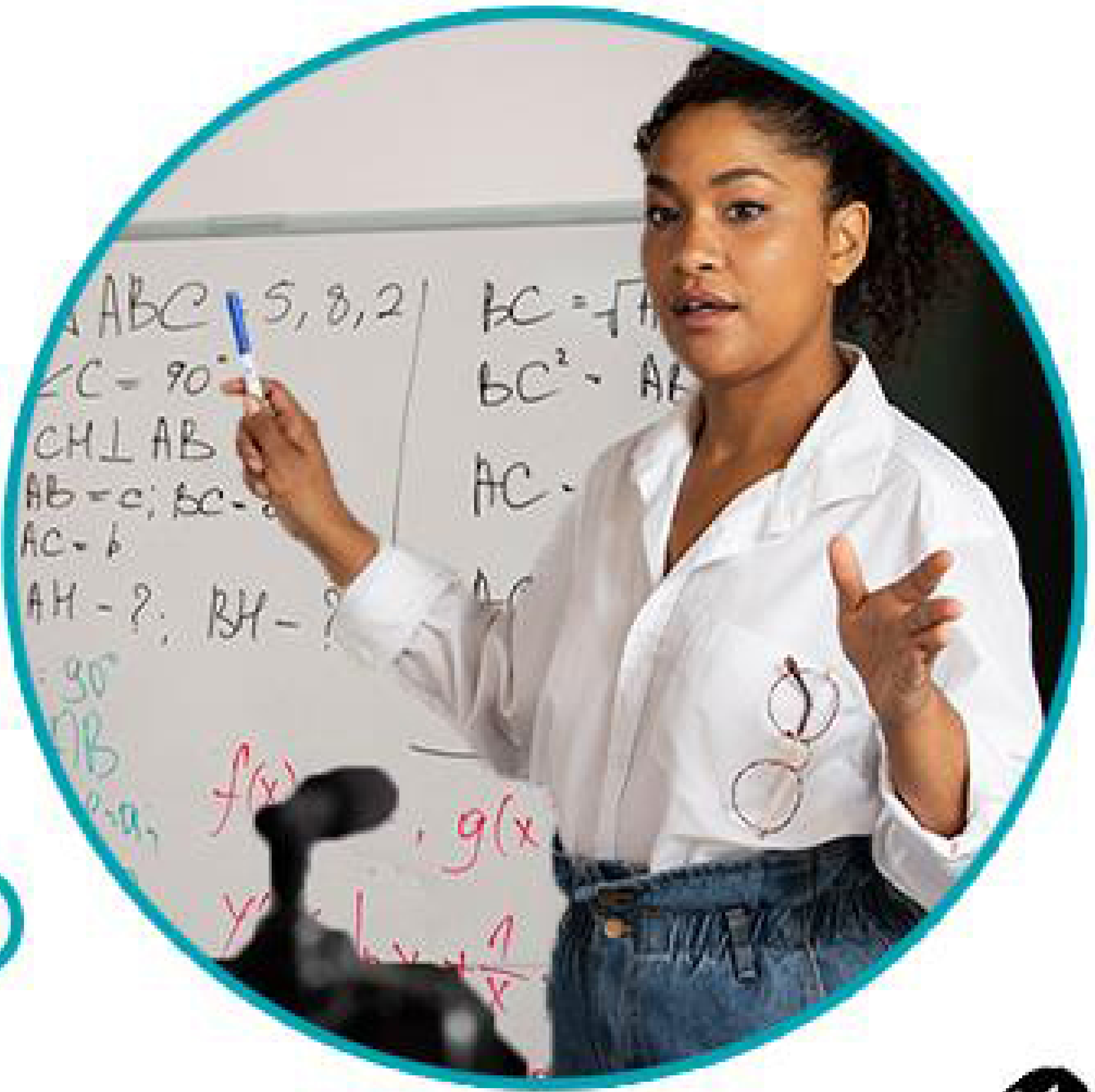

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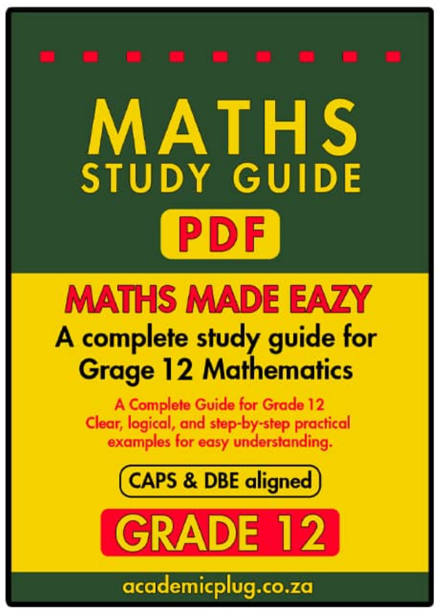
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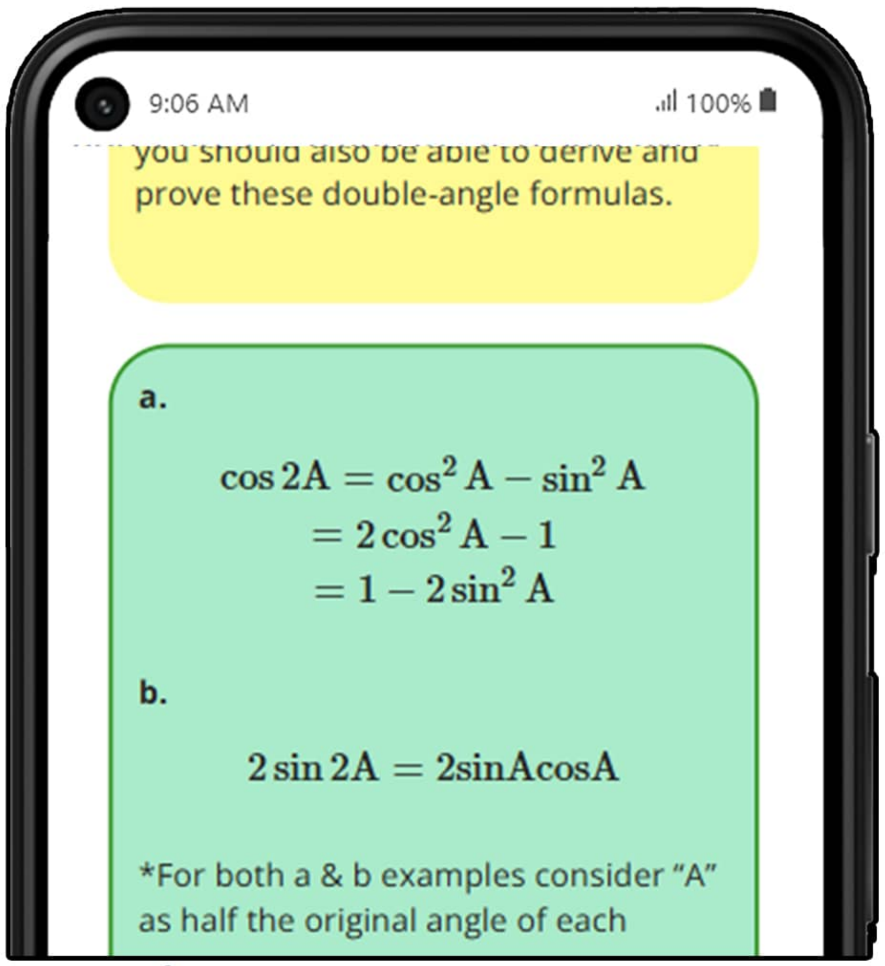

you should also be able to derive and prove these double-angle formulas.

a.

$$\cos 2A = \cos^2 A - \sin^2 A$$
$$= 2\cos^2 A - 1$$
$$= 1 - 2\sin^2 A$$

b.

$$2\sin 2A = 2\sin A\cos A$$

*For both a & b examples consider "A" as half the original angle of each

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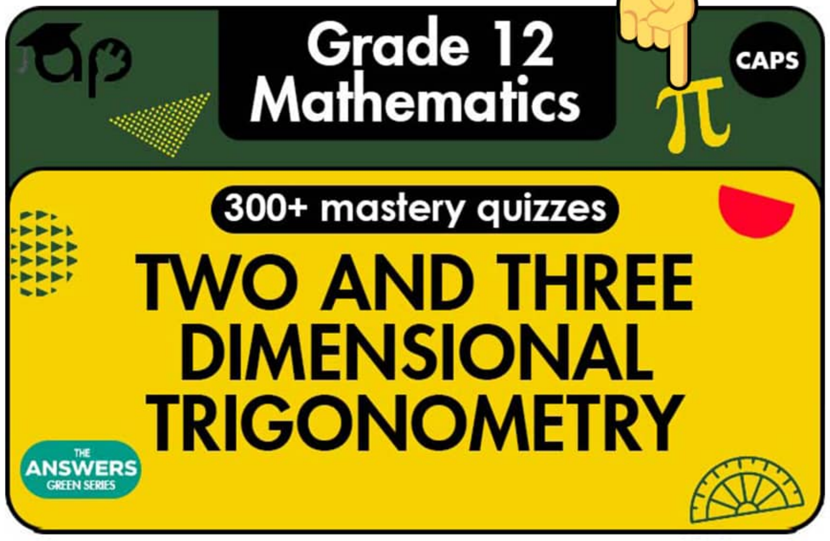
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CHAPTER 6
TWO AND THREE DIMENSIONAL TRIGONOMETERY

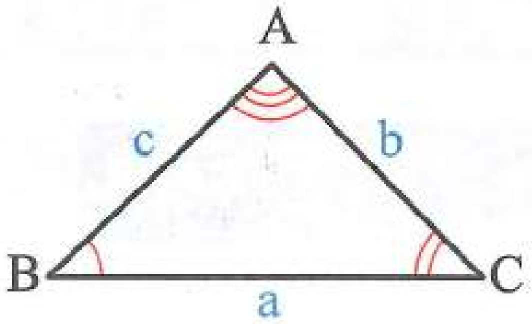
WHAT YOU WILL LEARN IN THIS CHAPTER

PAGE	CONTENT	CAPS	ASSESSMENTS YOU WILL BE ABLE TO:
119	**A. Grade 11 Revision**	8	• Solve problems in two and three dimensions.
119	• Sine Rule		
119	• Cosine Rule		
119	• Area Rule		
120	• The Ambiguous Case of the Sine Rule		
123	**B. Three Dimensional Trigonometry**		

A. GRADE 11 REVISION

Two and three dimensional trigonometry utilises all the trigonometry knowledge that you have learnt up until now. It is essential that you know the area, sine and cosine rules before proceeding with this section.

SINE RULE

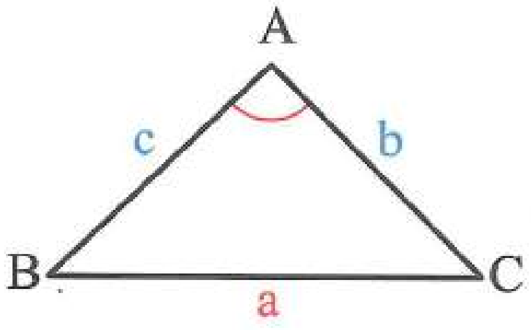


$$\frac{a}{\sin \hat{A}} = \frac{b}{\sin \hat{B}} = \frac{c}{\sin \hat{C}} \quad \text{or} \quad \frac{\sin \hat{A}}{a} = \frac{\sin \hat{B}}{b} = \frac{\sin \hat{C}}{c}$$

Use the Sine Rule when given two sides and an angle that is not between the given sides or if two angles and a side are given.

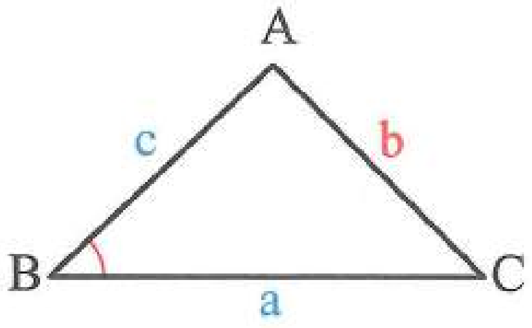
When using the Sine Rule place whatever you are looking for (angle or side) as the numerator of the fraction (on top of the fraction).

COSINE RULE



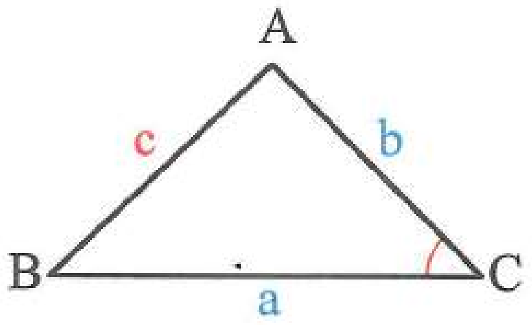
Angle A known

$$a^2 = b^2 + c^2 - 2(b)(c)\cos \hat{A}$$



Angle B known

$$b^2 = a^2 + c^2 - 2(a)(c)\cos \hat{B}$$

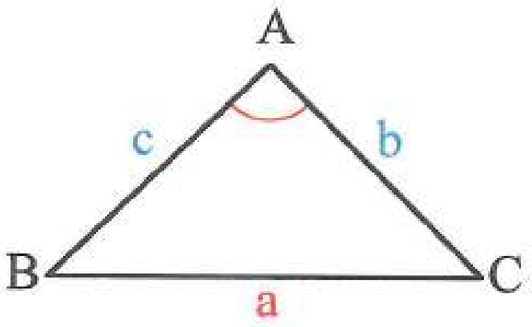


Angle C known

$$c^2 = a^2 + b^2 - 2(a)(b)\cos \hat{C}$$

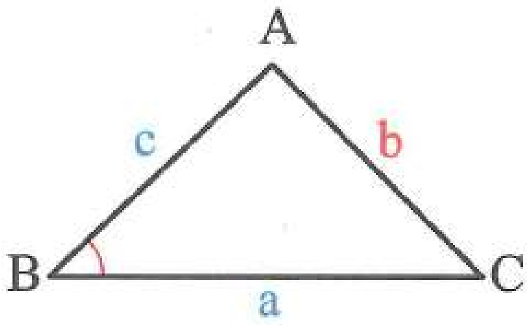
Use the Cosine Rule if you have a triangle where you are given all three sides or two sides and an included angle. If a side of a triangle has a square root in it, the Cosine Rule is the most likely rule to use.

AREA RULE



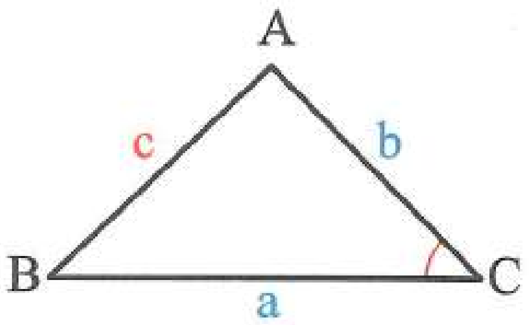
Angle A known

$$A_{\triangle ABC} = \frac{1}{2} b c \sin \hat{A}$$



Angle B known

$$A_{\triangle ABC} = \frac{1}{2} a c \sin \hat{B}$$



Angle C known

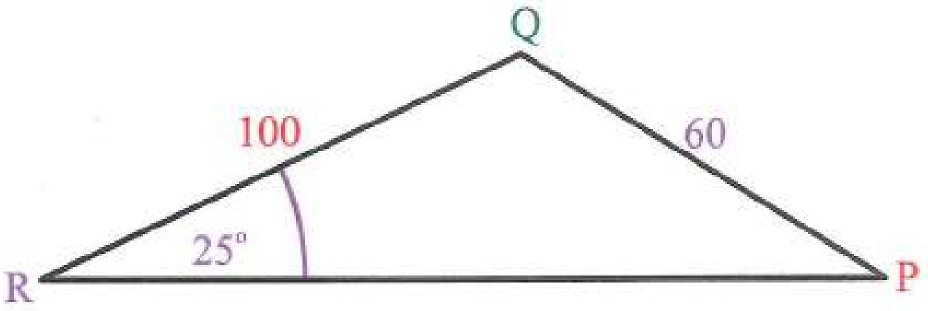
$$A_{\triangle ABC} = \frac{1}{2} a b \sin \hat{C}$$

Use the Area Rule when given two sides and an included angle.

THE AMBIGUOUS CASE OF THE SINE RULE

When solving for an angle in a triangle it is possible for there to be 2 solutions for an angle.
This occurs when the angle you are solving for is opposite the longest given side.

Illustrative Example: Solve for $\hat{P}$ in the $\triangle PQR$ below:



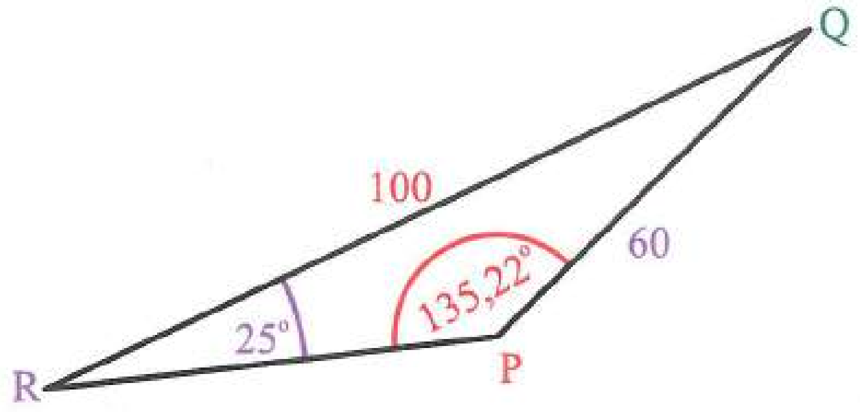
$$\frac{\sin \hat{P}}{RQ} = \frac{\sin \hat{R}}{60}$$

$$\therefore \frac{\sin \hat{P}}{100} = \frac{\sin 25^\circ}{60}$$

$$\therefore \sin \hat{P} = \frac{100 \sin 25^\circ}{60} \rightarrow \text{use shift } \sin^{-1}\left(\frac{100 \sin 25^\circ}{60}\right) \text{ to solve for the angle}$$

$$\therefore \hat{P} = 44{,}78^\circ$$

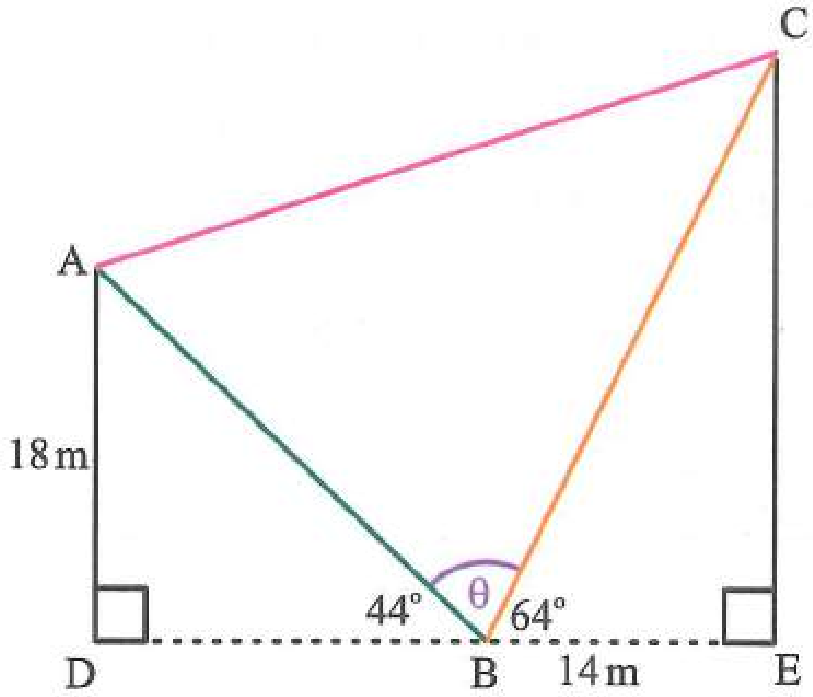
However, we know from trigonometric equations that it is possible to have another solution to $\hat{P}$. It is also possible for $\hat{P} = 180^\circ - 44{,}78^\circ = 135{,}22^\circ$. This is illustrated in the triangle below:



If the question does not state whether the triangle is obtuse or acute, always give both possible angles when using the Sine rule.

Remember that the ambiguous case will only occur when you are solving for the angle opposite the longest given side. An acute triangle is one where all angles are less or equal to 90°, whilst an obtuse triangle is one which has an angle that is greater than 90°.

Example 1: **A triangular advertising banner is erected between 2 poles and secured to a point on the floor as shown in the diagram below. For stability a steel wire is run through the perimeter of the banner.**



a) **If one side of the banner is to be painted and one can of paint covers a surface area of 35m², how many cans are required?**

b) **Determine, to the nearest metre, the length of steel wire required.**

a) $\text{Area} = \frac{1}{2}\text{AB}.\text{BC}\sin\theta$

$\sin 44^\circ = \frac{18}{\text{AB}}$

$\therefore \text{AB} = \frac{18}{\sin 44^\circ} = 25{,}912...$

$\cos 64^\circ = \frac{14}{\text{BC}}$

$\therefore \text{BC} = \frac{14}{\cos 64^\circ} = 31{,}936...$

$\therefore \text{Area} = \frac{1}{2}(25{,}912...)(31{,}936...)\sin(180^\circ - 44^\circ - 64^\circ) \rightarrow$ adj. ∠'s on a straight line

$= \frac{1}{2}(25{,}912...)(31{,}936...)\sin(72^\circ)$

$= 393{,}52\,\text{m}^2$

$\text{Number of cans} = \frac{\text{Total area}}{35} = \frac{393{,}52}{35} = 11{,}24$

$\therefore$ 12 cans are required. → don't round down, 11 cans will not cover the entire banner

b) $\text{Perimeter} = \text{AB} + \text{BC} + \text{AC}$

$\therefore \text{AC}^2 = \text{AB}^2 + \text{BC}^2 - 2(\text{AB})(\text{BC})\cos\theta$

$= (25{,}91)^2 + (31{,}94)^2 - 2(25{,}91)(31{,}94)\cos 72^\circ$

$= 1180{,}028...$

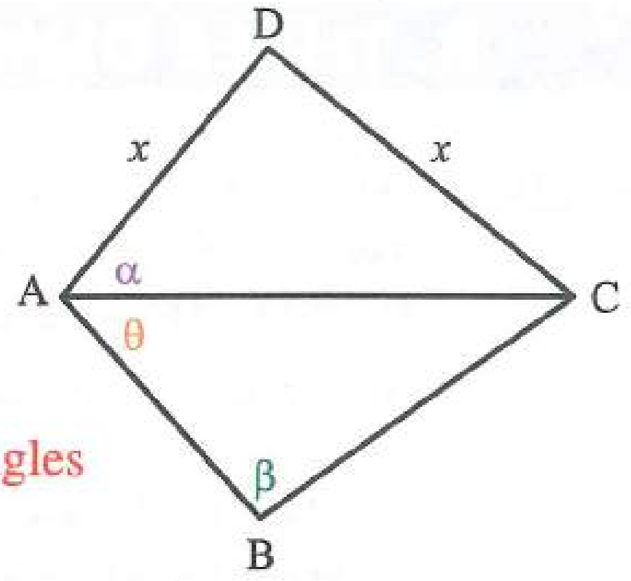
$\therefore \text{AC} = 34{,}35$

$\therefore \text{Perimeter} = 25{,}91 + 31{,}94 + 34{,}35 = 92{,}2$

$\therefore$ 93 m of steel wire is required → 92 m will not cover the entire perimeter

Example 2: **Show that the length of BC in the diagram alongside is:**

$$BC=\frac{2x\cos\alpha\sin\theta}{\sin\beta}$$



Start solving in the triangle which has BC in it:

$$\frac{BC}{\sin\theta}=\frac{AC}{\sin\beta}\rightarrow \text{use the side that is common to both triangles}$$

$$\therefore BC=\frac{AC\sin\theta}{\sin\beta}$$

Now solve for AC in triangle ADC:

$$\frac{AC}{\sin\hat{D}}=\frac{DC}{\sin\alpha}\rightarrow \hat{D}=180^\circ-\hat{A}-\hat{C}=180^\circ-\alpha-\alpha=180^\circ-2\alpha$$

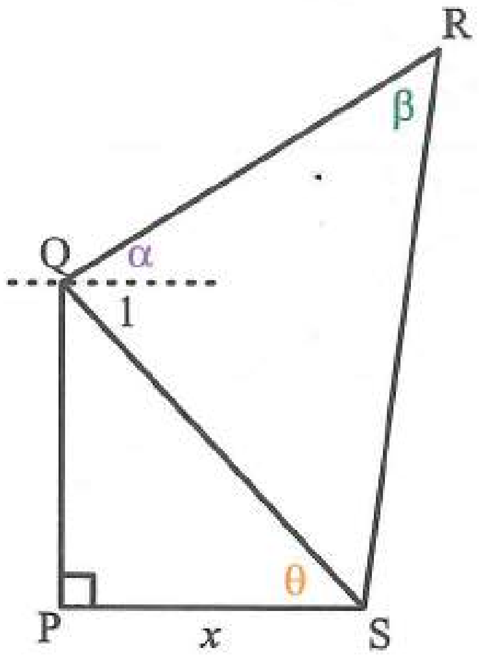
$$\therefore AC=\frac{x\sin\left(180^\circ-2\alpha\right)}{\sin\alpha}\rightarrow DC=x$$

$$\therefore AC=\frac{x\sin 2\alpha}{\sin\alpha}\rightarrow \sin\left(180^\circ-2\alpha\right)=\sin 2\alpha$$

$$\therefore BC=\frac{x\sin 2\alpha\sin\theta}{\sin\alpha\sin\beta}=\frac{2x\sin\alpha\cos\alpha\sin\theta}{\sin\alpha\sin\beta}=\frac{2x\cos\alpha\sin\theta}{\sin\beta}$$

Example 3: **In the diagram below, the angle of elevation of R from Q is α and the angle of elevation of Q from S is θ. If $PS=x$ and $\hat{R}=\beta$, prove that**

$$SR=\frac{x\left(\sin\theta\cos\alpha+\cos\theta\sin\alpha\right)}{\cos\theta\sin\beta}.$$



Start solving in the triangle which has SR in it:

$$\hat{Q}_1=\hat{S}=\theta\rightarrow \text{alternate angles}$$

$$\frac{SR}{\sin\hat{Q}}=\frac{QS}{\sin\beta}\rightarrow \text{use the side that is common to both triangles}$$

$$\therefore\frac{SR}{\sin\left(\theta+\alpha\right)}=\frac{QS}{\sin\beta}$$

$$\therefore SR=\frac{QS\sin\left(\theta+\alpha\right)}{\sin\beta}\rightarrow \text{use compound angle formulas on page 94}$$

$$\therefore SR=\frac{QS\left(\sin\theta\cos\alpha+\cos\theta\sin\alpha\right)}{\sin\beta}$$

Now solve for QS in triangle PQS:

$$\cos\theta=\frac{x}{QS}\rightarrow\frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\therefore QS=\frac{x}{\cos\theta}$$

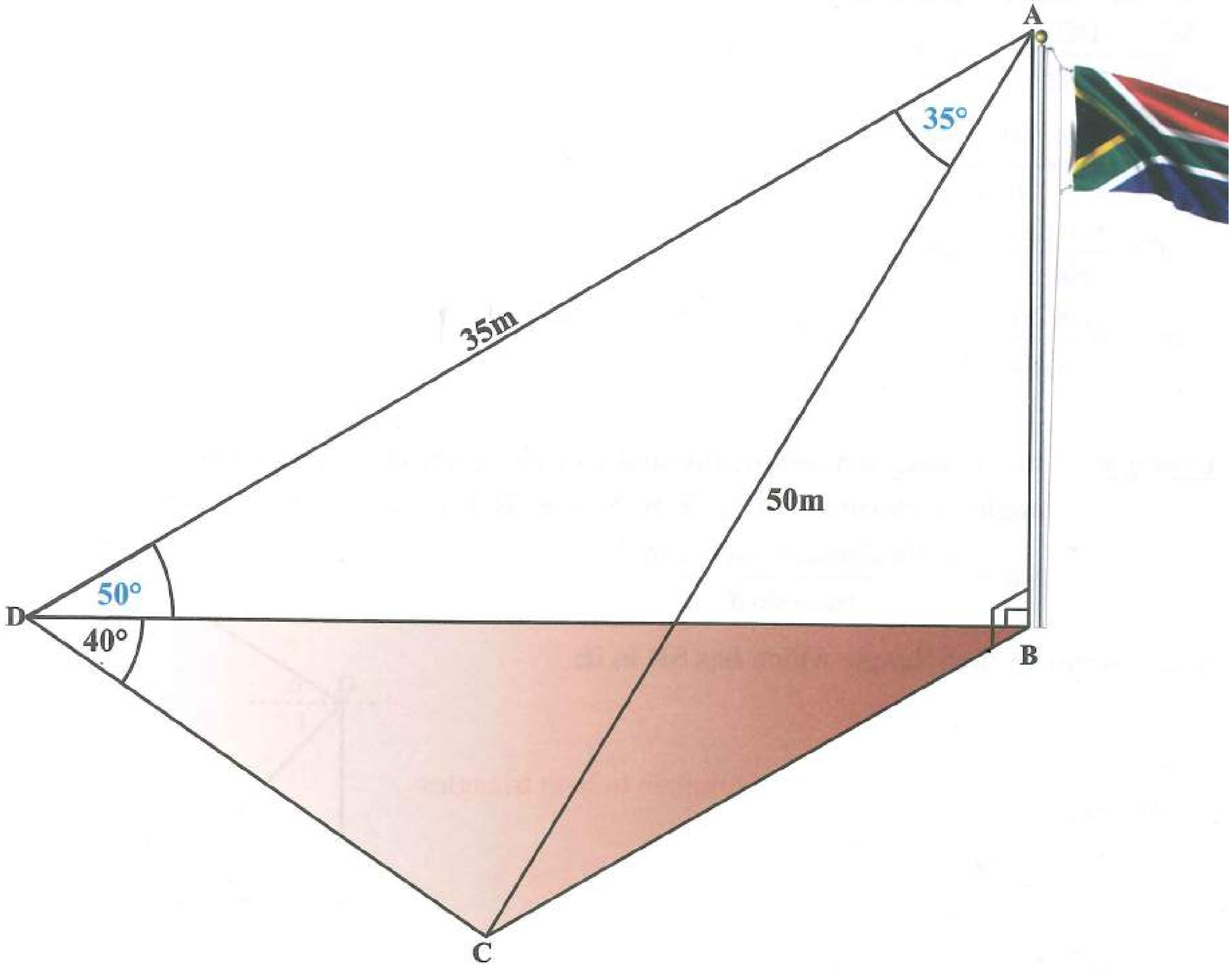
$$\therefore SR=\frac{x\left(\sin\theta\cos\alpha+\cos\theta\sin\alpha\right)}{\cos\theta\sin\beta}\rightarrow SR=\frac{QS\left(\sin\theta\cos\alpha+\cos\theta\sin\alpha\right)}{\sin\beta};QS=\frac{x}{\cos\theta}$$

Complete Exercise 1 on page 127.

B. THREE DIMENSIONAL TRIGONOMETRY

Example 1: AB is a vertical flag pole with the points B, C and D in the same horizontal plane. There are two people looking at the flag from points D and C. DA= 35m, AC = 50m. The angle of elevation of A from D is 50°. Angle $D\hat{A}C = 35^\circ$ and $B\hat{D}C = 40^\circ$.

a) How far is observer D standing from the flag pole?
b) How far are the observers from each other?
c) Calculate the area that the observers form with the flag pole.



With the majority of 3D trig problems, you will have to solve a right angled triangle first (whether or not the it is asked in the question).

$$\sin\theta = \frac{O}{H}\ (\text{Oh Heaven}) \rightarrow \text{opposite over hypotenuse}$$

$$\cos\theta = \frac{A}{H}\ (\text{Another Hour}) \rightarrow \text{adjacent over hypotenuse}$$

$$\tan\theta = \frac{O}{A}\ (\text{Of Algebra}) \rightarrow \text{opposite over adjacent}$$

a) Solve for the length of DB:

$$\cos\theta = \frac{\text{A}}{\text{H}}$$

$$\therefore \cos 50^\circ = \frac{\text{DB}}{35}$$

$$\therefore \text{DB} = 35\cos 50^\circ$$

$$\therefore \text{DB} = 22{,}50\,\text{m}$$

b) Using the cosine rule:

$$\text{DC}^2 = \text{AD}^2 + \text{AC}^2 - 2(\text{AD})(\text{AC})\cos\left(\text{D}\hat{\text{A}}\text{C}\right)$$

$$\therefore\ \text{DC}^2 = 35^2 + 50^2 - 2(35)(50)\cos 35^\circ$$

$$\therefore \text{DC} = \sqrt{857{,}97}$$

$$\therefore \text{DC} = 29{,}29\,\text{m}$$

c) Using the area rule:

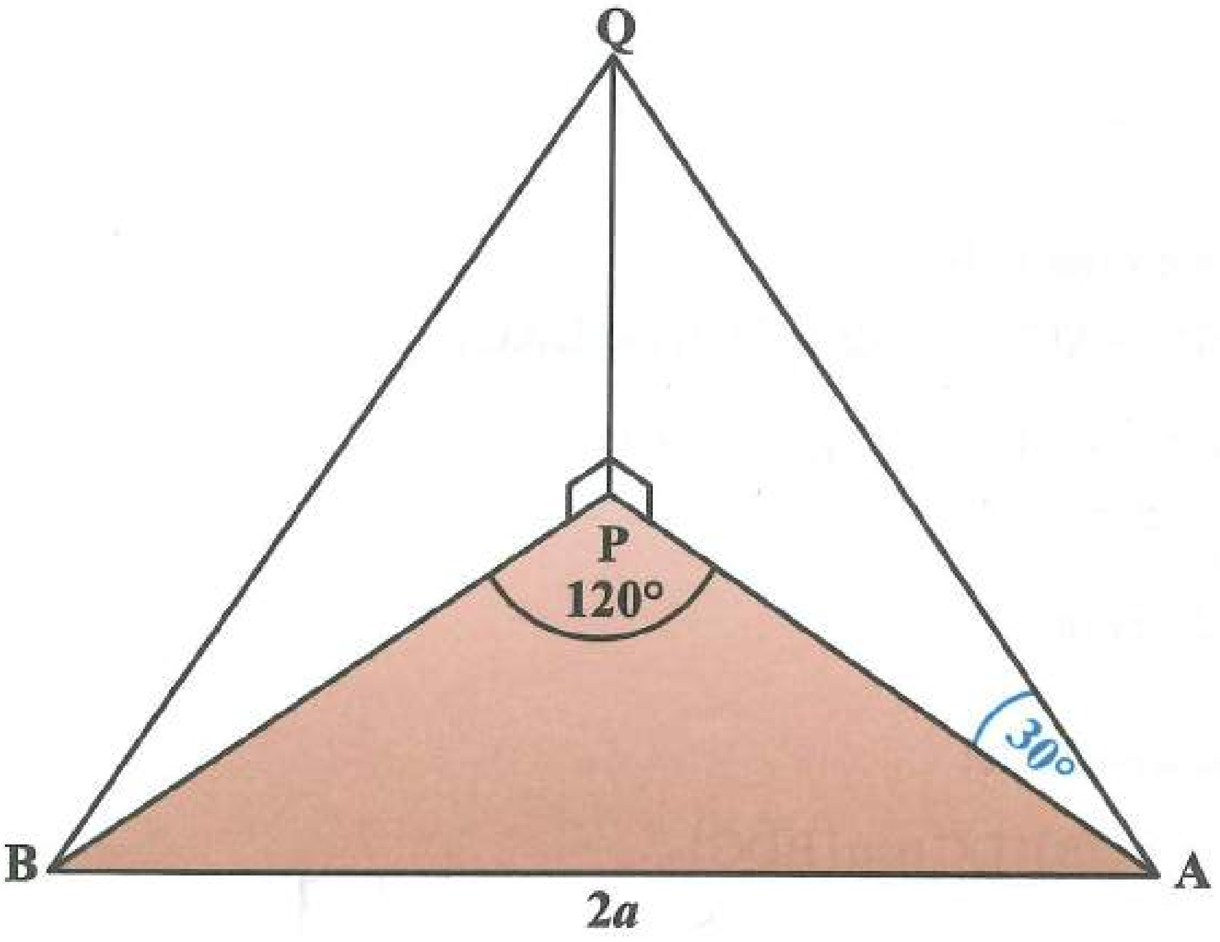
$$A_{\triangle BCD} = \frac{1}{2}(\text{DB})(\text{DC})\sin\left(\text{B}\hat{\text{D}}\text{C}\right)$$

$$\therefore\ A_{\triangle BCD} = \frac{1}{2}(22{,}50)(29{,}29)\sin\left(40^\circ\right) \rightarrow \text{given}$$

$$\therefore\ A_{\triangle BCD} = 211{,}81\,\text{m}^2$$

In a test, even if you could not solve for the lengths of DB and DC, write down any answers for their lengths and use those answers to calculate the area in question (c). By doing this you will still receive method marks for calculating the area and using the correct sides.

Example 2: Two boys are standing at points A and B with slingshots. They are trying to hit a target at the top of tower PQ. The boys are standing at a distance of $2a$ from each other and are equidistant from the tower. The angle of elevation from A to Q is 30°. The boys make an angle of 120° with the tower. Determine the height of the tower in terms of a.



In this question, you cannot find the height of the tower straight away. You first need to find the length of one of the sides of triangle APQ. In order to do this, use the Sine rule in triangle PBA, to determine the length of PA.

$$\therefore \frac{\text{PA}}{\sin 30^\circ} = \frac{2a}{\sin 120^\circ} \rightarrow \text{the boys are equidistant from the pole}$$

$$\therefore \text{ PBA is an isoceles triangle with } \text{P}\hat{\text{B}}\text{A} = \text{P}\hat{\text{A}}\text{B} = \frac{180^\circ - 120^\circ}{2} = 30^\circ$$

$$\therefore \text{PA} = \frac{2a\sin 30^\circ}{\sin 120^\circ} \rightarrow \text{use reduction formulas, } \sin 120^\circ = \sin\left(180^\circ - 60^\circ\right) = \sin 60^\circ = \frac{\sqrt{3}}{2}$$

$$\therefore \text{PA} = \frac{2a\left(\frac{1}{2}\right)}{\frac{\sqrt{3}}{2}}$$

$$\therefore \text{PA} = \frac{2a}{\sqrt{3}} \text{ units} \rightarrow \text{ always put in units}$$

Now solve for the height of the tower PQ:

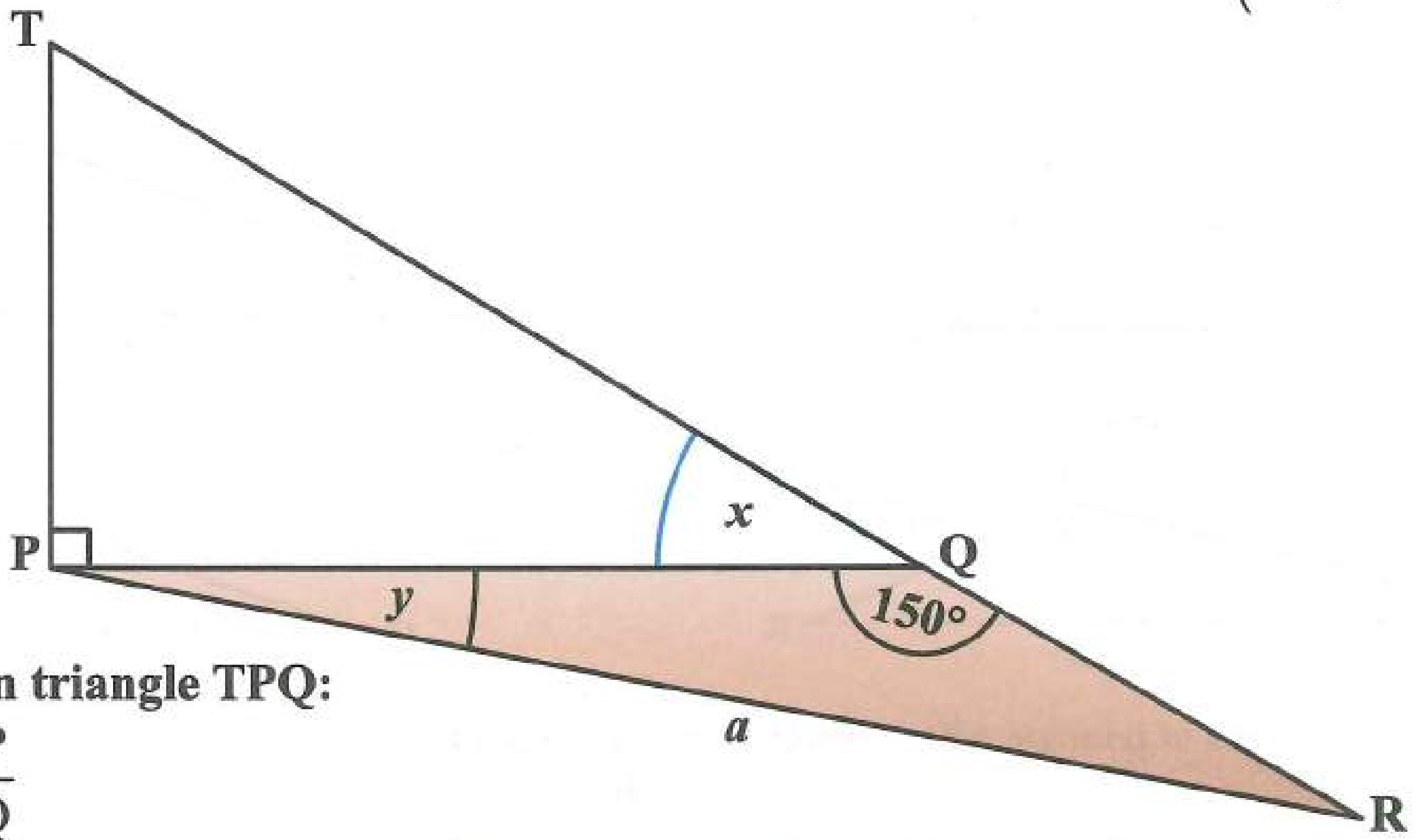
$$\tan\left(\text{Q}\hat{\text{A}}\text{P}\right) = \frac{\text{QP}}{\text{PA}} \rightarrow \tan\theta = \frac{\text{O}}{\text{A}}$$

$$\therefore \text{QP} = \text{PA}\tan\left(\text{Q}\hat{\text{A}}\text{P}\right)$$

$$\therefore \text{QP} = \frac{2a}{\sqrt{3}}\tan\left(30^\circ\right)$$

$$\therefore \text{QP} = \frac{2a}{\sqrt{3}} \times \frac{1}{\sqrt{3}} = \frac{2a}{3} \text{ units} \rightarrow \text{ always put in units}$$

Example 3: TP is a tower. Its foot, P, and the points Q and R are on the same horizontal plane. From Q the angle of elevation to the top of the building is x. It is further given that $P\hat{Q}R = 150^\circ$, $Q\hat{P}R = y$ and the distance between P and R is a. Prove that $TP = a\tan x\left(\cos y - \sqrt{3}\sin y\right)$.



Starting in triangle TPQ:

$$\tan x = \frac{TP}{PQ}$$

$$\therefore TP = PQ\tan x \rightarrow \text{now solve for TP in the triangle on the ground}$$

Solve for $\hat{R}$ in $\triangle PQR$:

$$\hat{R} = 180^\circ - \left(150^\circ + y\right)$$

$$\therefore \hat{R} = 30^\circ - y$$

$$\frac{PQ}{\sin\hat{R}} = \frac{PR}{\sin 150^\circ} \rightarrow \text{sine rule}$$

$$\therefore PQ = \frac{PR\sin\hat{R}}{\sin 150^\circ}$$

$$\therefore PQ = \frac{a\sin\left(30^\circ - y\right)}{\sin\left(180^\circ - 30^\circ\right)} \rightarrow \text{compound angles}$$

$$\therefore PQ = \frac{a\left(\sin 30^\circ \cos y - \cos 30^\circ \sin y\right)}{\sin 30^\circ} \rightarrow \sin 30^\circ = \frac{1}{2} \text{ and } \cos 30^\circ = \frac{\sqrt{3}}{2}$$

$$\therefore PQ = \frac{a\left(\frac{1}{2}\cos y - \frac{\sqrt{3}}{2}\sin y\right)}{\frac{1}{2}} \rightarrow \text{take out a HCF of } \frac{1}{2}$$

$$\therefore PQ = \frac{\frac{1}{2}a\left(\cos y - \sqrt{3}\sin y\right)}{\frac{1}{2}}$$

$$\therefore PQ = a\left(\cos y - \sqrt{3}\sin y\right)$$

$$\therefore TP = a\left(\cos y - \sqrt{3}\sin y\right)\tan x = a\tan x\left(\cos y - \sqrt{3}\sin y\right)$$

Complete Exercise 2 on pages 127-129 and the Mixed Exercise on pages 129 and 130.

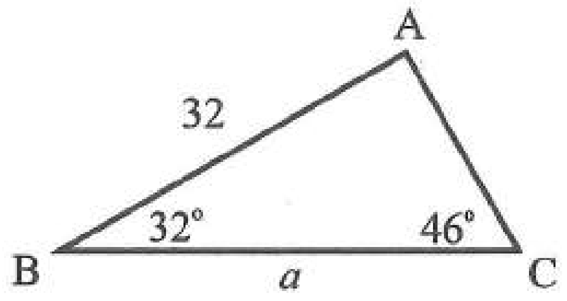
Chapter 6 Two and Three Dimensional Trigonometry (Exercises)

Grade 11 Revision

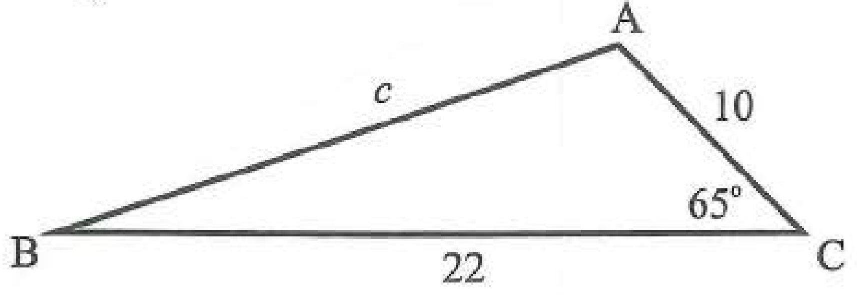
Exercise 1: (answers on page 352)

1) **Determine the value of the unknown variable in each of the following triangles, correct to 1 decimal place:**

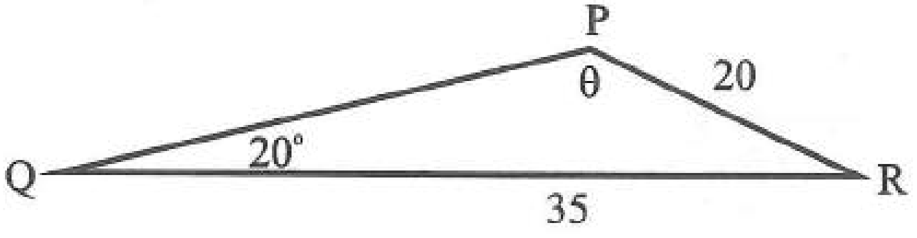
a)


A
32
32°
46°
B
a
C

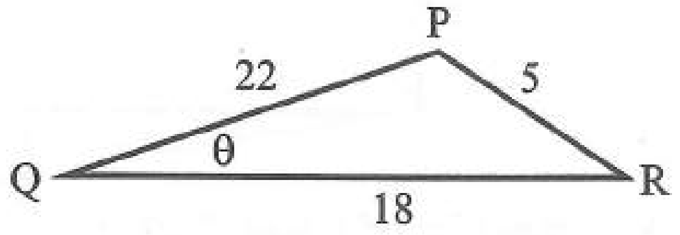
c)


A
c
10
65°
B
22
C

b)

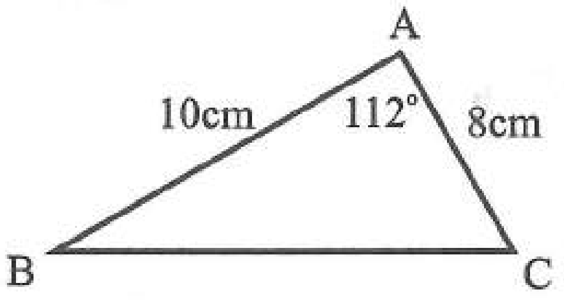

P
θ
20
20°
Q
35
R

d)

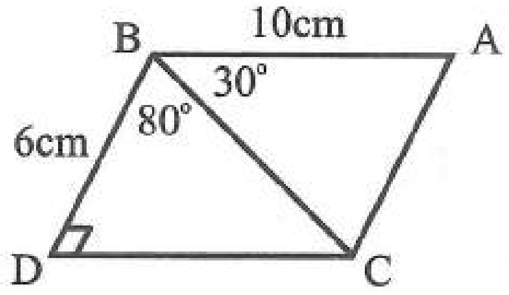


2) **Determine the area of triangle ABC correct to one decimal place:**

a)


A
10cm
112°
8cm
B
C

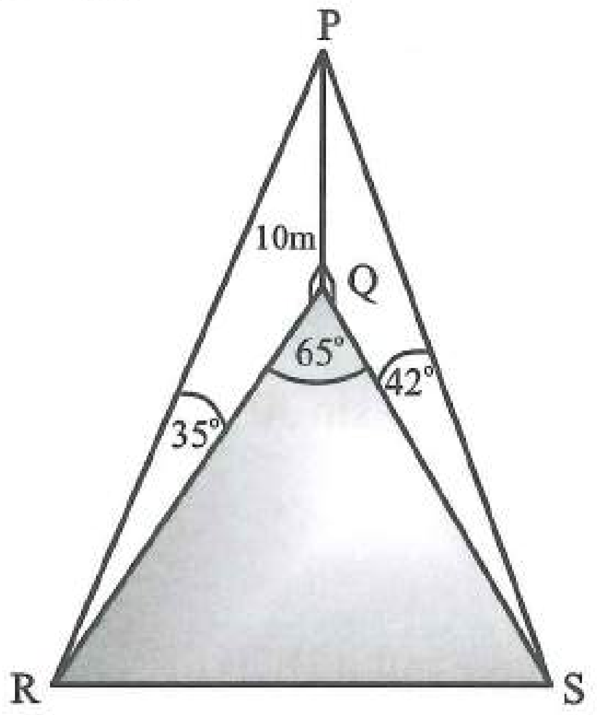
b)


B
10cm
A
30°
80°
6cm
D
C

Three Dimensional Trigonometry

Exercise 2: (answers on page 352)

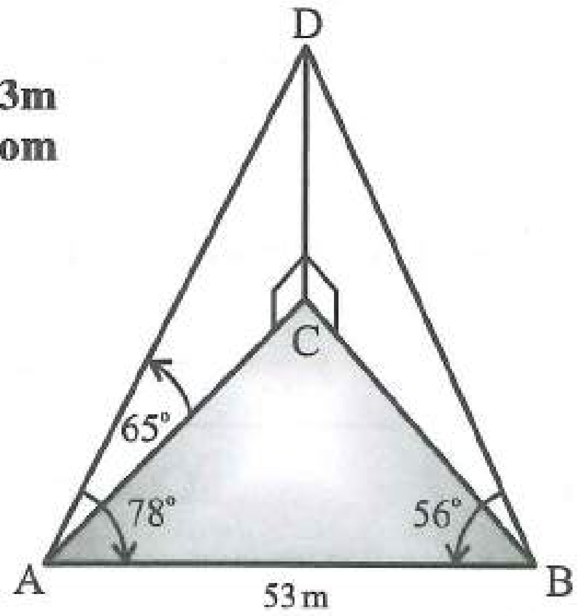
1) **In the diagram below PQ is 10m in length and perpendicular to the horizontal plane. The angle of elevation of P from R is 35° and the angle of elevation of P from S is 42°. R, Q and S all lie in the same horizontal plane and $R\hat{Q}S = 65°$.**


P
10m
Q
65°
42°
35°
R
S

a) Determine the length of RQ and SQ.
b) Determine the length of RS.

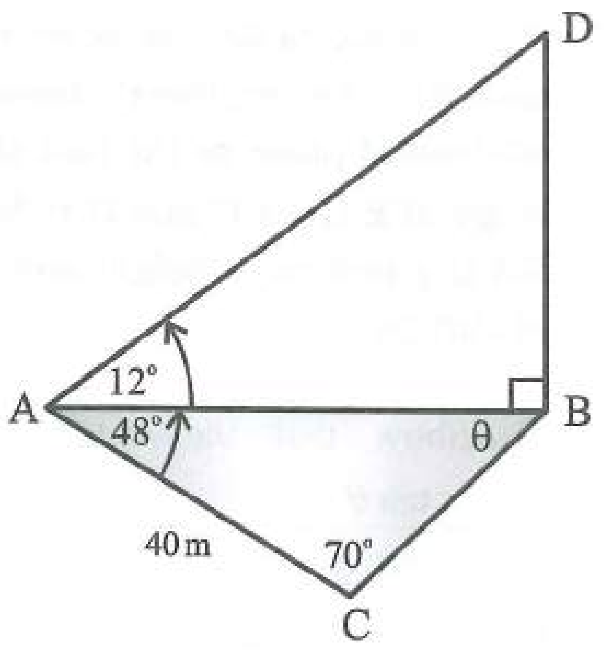
2) **A, B and C are 3 points in the same horizontal plane and AB is 53m in length. CD is a vertical tower and the angle of elevation of D from A is 65°. $D\hat{A}B = 78°$ and $D\hat{B}A = 56°$. Determine:**

a) The size of $A\hat{D}B$.
b) The length of AD.
c) The height of tower CD.


D
C
65°
78°
56°
A
53 m
B

3) A, B and C are 3 points in the same horizontal plane and AC is 40m in length. BD is a lightning pole which is secured to the ground at A by cable AD. The angle of elevation of D from A is 12°.

$B\hat{A}C = 48^\circ$ and $A\hat{C}B = 70^\circ$. Determine:

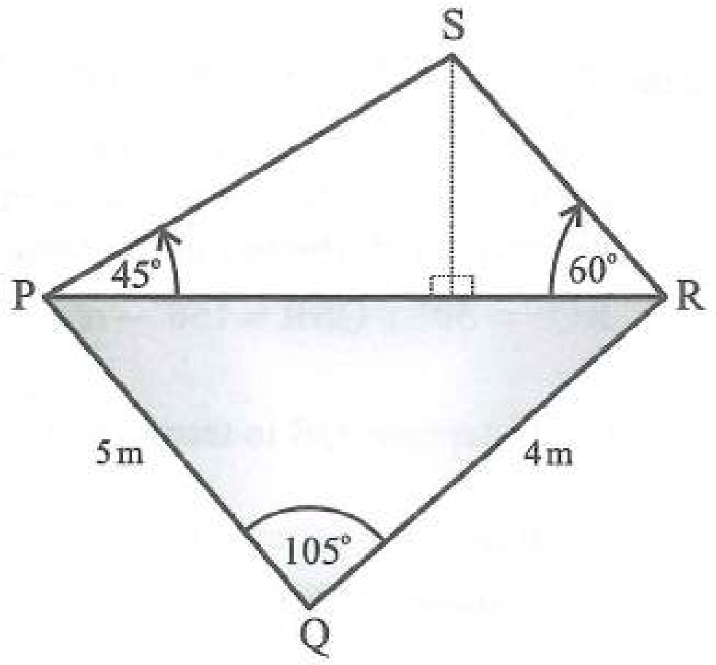


a) The size of θ.

b) The horizontal distance between the bottom of the pole and the point at which it is secured to the ground.

c) The length of cable AD.

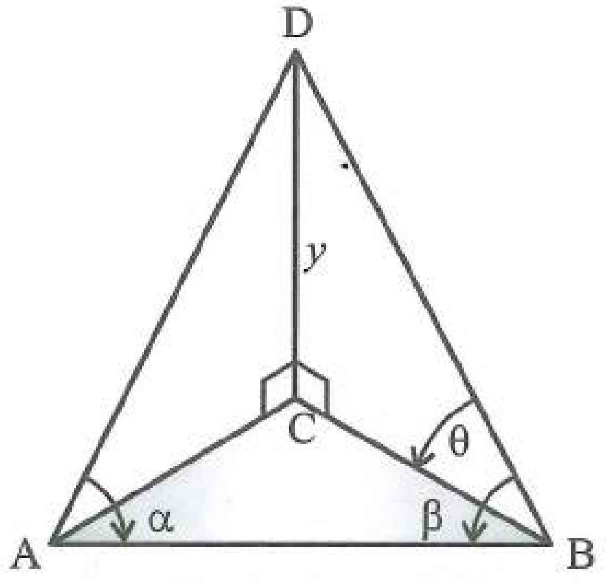
4) A spotlight for a theatre production is designed to illuminate a triangular area on stage. Actors are to stand at the corners of the illuminated area at P, Q and R. The actors at P and R have to stand 5m and 4m away from the actor at Q respectively. The angle of elevation of S from P is 45° and the angle of elevation of S from R is 60°. If the spotlight is placed at a point vertically above PR.



a) Determine how far the actors at P and R must stand from each other.

b) The actor at P makes his entrance on stage by sliding down a wire from S to P. How long is the wire that the actor slides along?

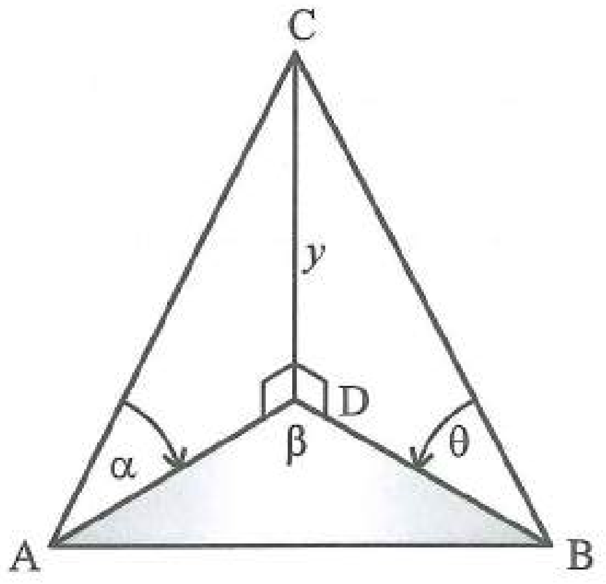
5) In the diagram alongside, D is a point vertically above C. DC is y meters in length. The angle of elevation of D from B is θ. Angle $D\hat{A}B = \alpha$ and $D\hat{B}A = \beta$.



a) Determine the length of DB in terms y and θ.

b) Show that $AB = \dfrac{y\sin(\alpha+\beta)}{\sin\theta\sin\alpha}$.

6) In the diagram alongside, C is a point vertically above D. CD is y meters in length. The angle of elevation of C from B is θ and the angle of elevation of C from A is α. Angle $A\hat{D}B = \beta$.

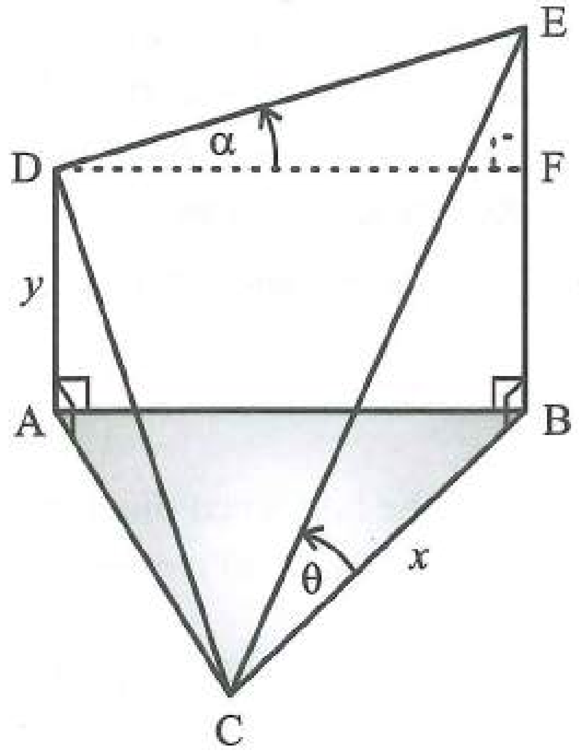


a) Determine the length of AD in terms y and α.

b) Determine the length of BD in terms y and θ.

c) Hence, show that the area of $\triangle ABD = \dfrac{y^2\sin\beta}{2\tan\alpha\tan\theta}$

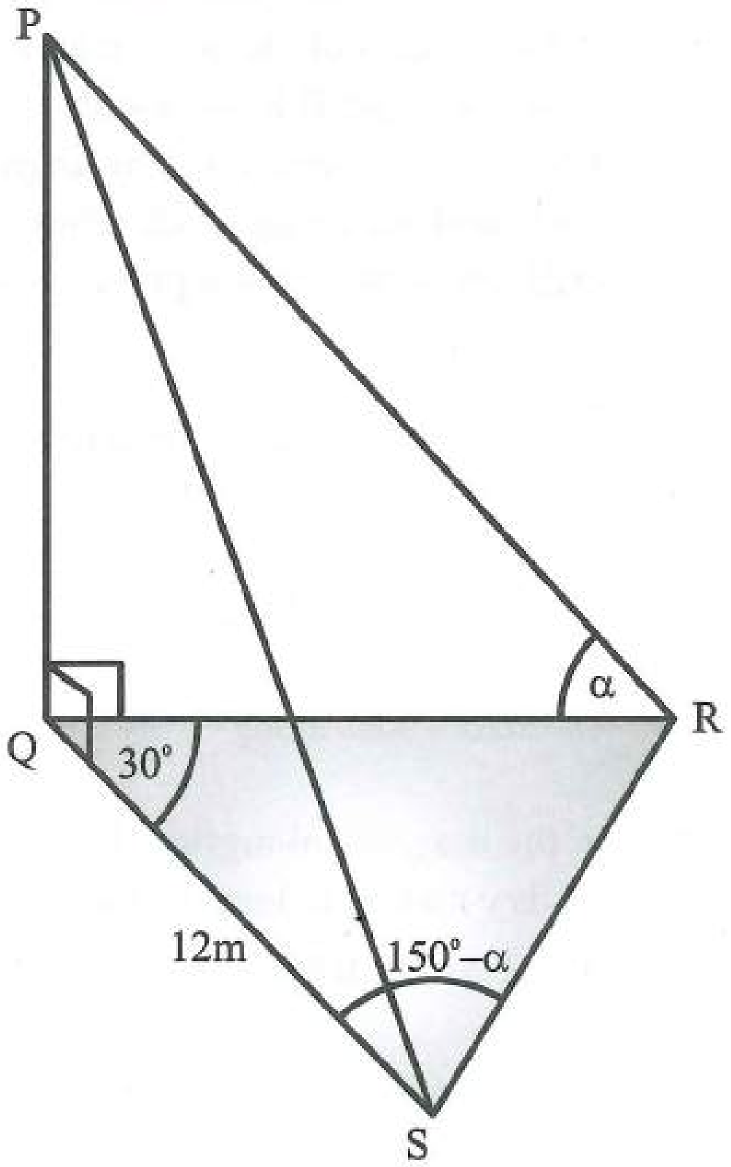
7) A telephone cable is to be erected between 2 cliff sides AD and BE. An engineer stands at point C in the same horizontal plane as the foot of the cliffs. He measures the angle of E from C and D to be θ and α respectively. Cliff DA is y meters in height and C is x meters from the foot of cliff BE.



- **a)** Show that the length of the telephone cable is $\dfrac{x\tan\theta - y}{\sin\alpha}$
- **b)** If $x = 1000\,\text{m}$, $y = 250\,\text{m}$ and $\theta = \alpha = 45^\circ$, what is the distance between the cliffs?

Mixed Exercise: (answers on page 352)

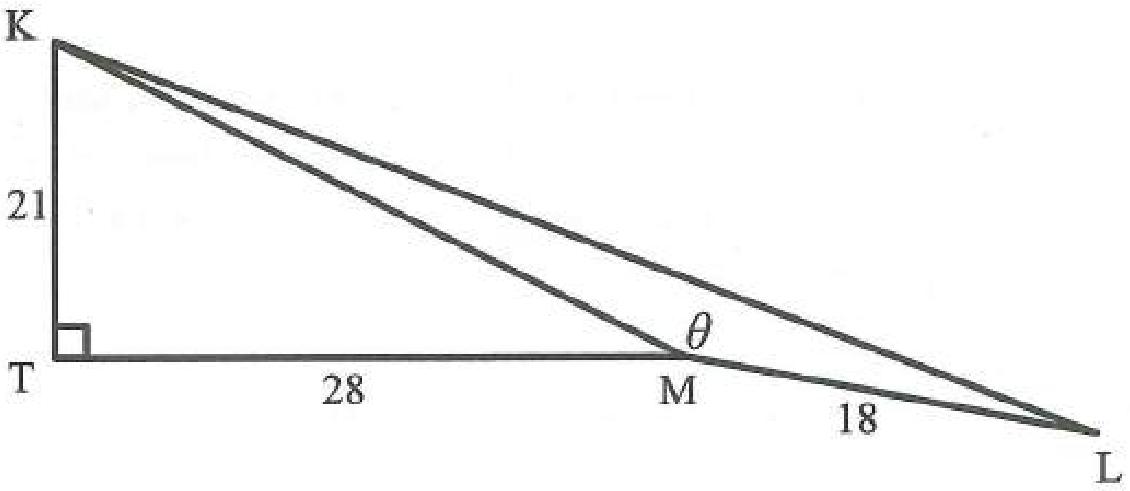
1) In the sketch alongside, PQ is a vertical tower. Q, R and S are points in the same horizontal plane. The angle of elevation of P, the top of the tower, from R is α.
$R\hat{Q}S = 30^\circ$, $Q\hat{S}R = 150^\circ - \alpha$ and $QS = 12\,\text{m}$.



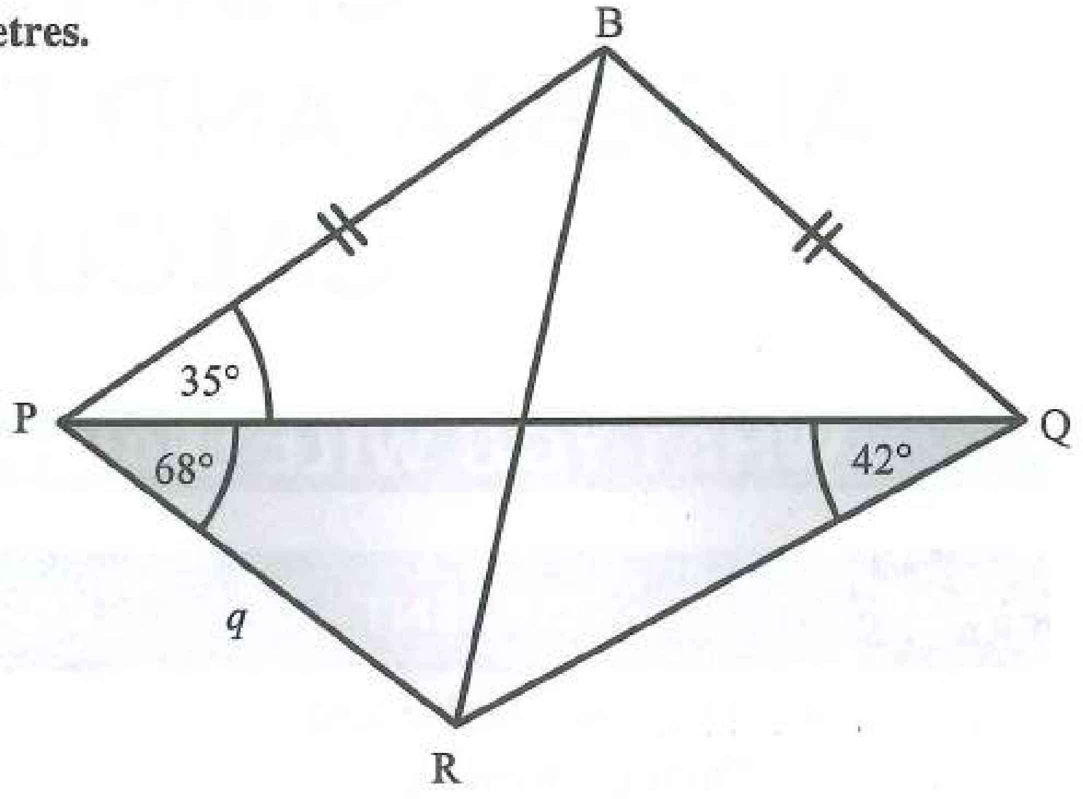
- **a)** Determine QR in terms of $\sin\alpha$ and $\cos\alpha$.
- **b)** Hence, or otherwise, show that the height PQ of the building is $PQ = 6 + 6\sqrt{3}\tan\alpha$.
- **c)** Hence, or otherwise, calculate α, the angle of elevation of P from R, if $PQ = 23\,\text{m}$.

2) In the diagram below the area of ΔKLM is equal to the area of ΔTKM. $K\hat{M}L = \theta$ and θ is an obtuse angle.

- **a)** Determine the area of ΔKLM.
- **b)** Show that $\sin\theta = \dfrac{14}{15}$
- **c)** Determine the value of θ.



3) **In the figure below, $BP = BQ$ and Q, P and R are in the same horizontal plane. $Q\hat{P}R = 68^\circ$, $P\hat{Q}R = 42^\circ$, $B\hat{P}Q = 35^\circ$ and $PR = q$ metres.**



a) Show that $PB = \dfrac{q \sin 35^\circ}{\sin 42^\circ}$.

b) Calculate q if $PB = 54{,}5\,\text{m}$.

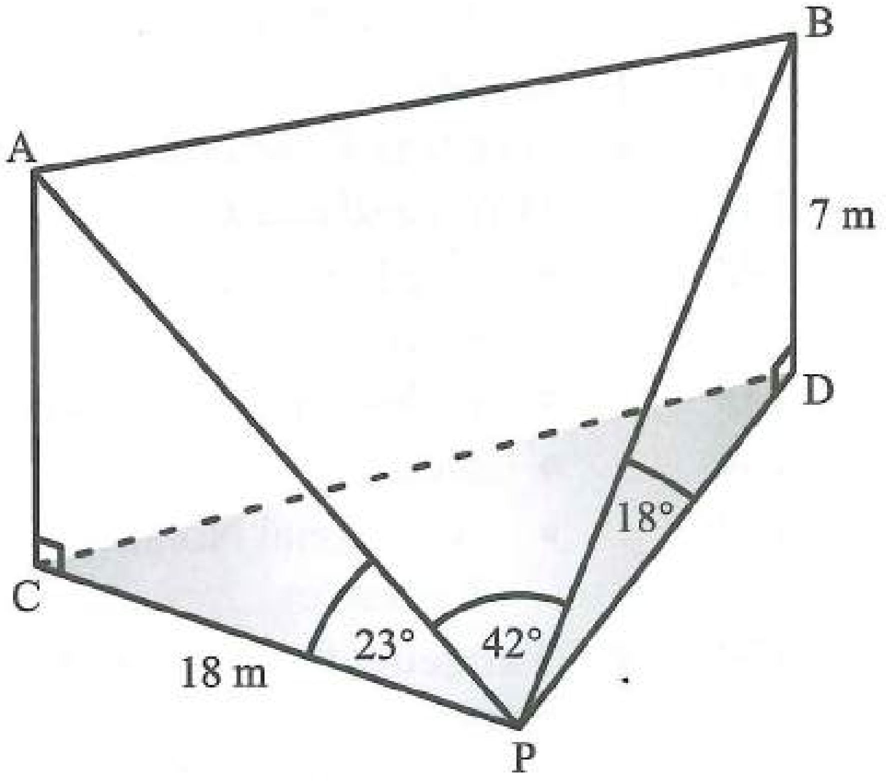
4) **An advertising banner is to be erected between two poles AC and BD. AC and BD are not the same height. An observer stands at point P which is in the same horizontal plane as C and D. From P the angles of inclination to the top of poles A and B are 23° and 18° respectively. P is 18m from the base of Pole AC. The height of pole BD is 7m.**

Calculate, correct to two decimal places:

a) The distance from point P to the top of pole BD

b) The distance from P to the top of pole AC.

c) The distance between the tops of the poles (AB) if $A\hat{P}B = 42^\circ$.

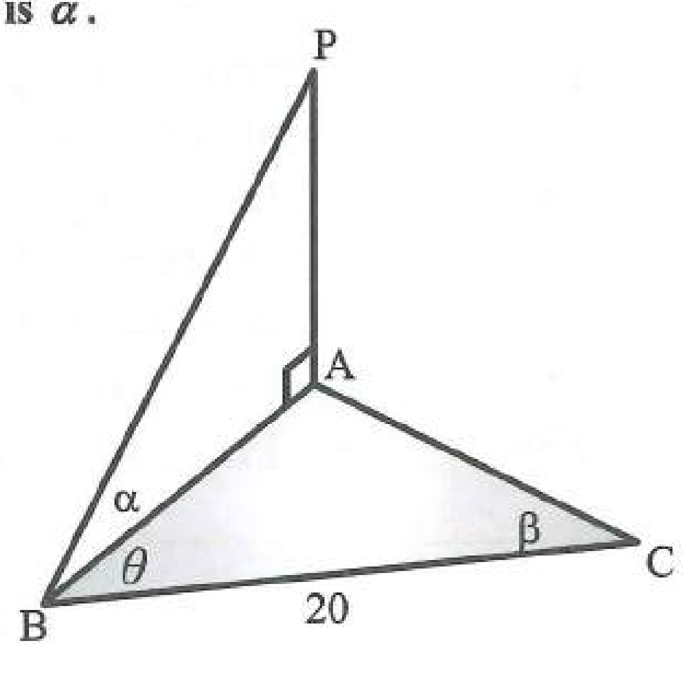


5) **In the figure below A, B and C are in the same horizontal plane. P is a point vertically above A and B and C are 20 units apart. The angle of elevation from B to P is α.**
$A\hat{B}C = \theta$ and $A\hat{C}B = \beta$.

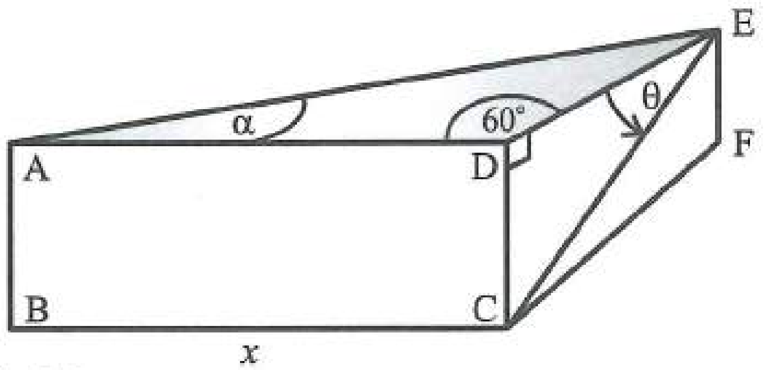
a) Prove that $AP = \dfrac{20 \sin \beta \tan \alpha}{\sin(\theta + \beta)}$.

b) Given that $AB = AC$, show that $AP = \dfrac{10 \tan \alpha}{\cos \beta}$.

c) If $AB = AC$, determine AP if $\beta = 55^\circ$ and $\alpha = 75^\circ$.



6) **An aeroplane wing is constructed as shown below. It is attached to the plane at rectangle ABCD. Prove that the width of the wing (DC) can be given by $DC = \dfrac{2x \sin \alpha \tan \theta}{\sqrt{3} \cos \alpha + \sin \alpha}$.**



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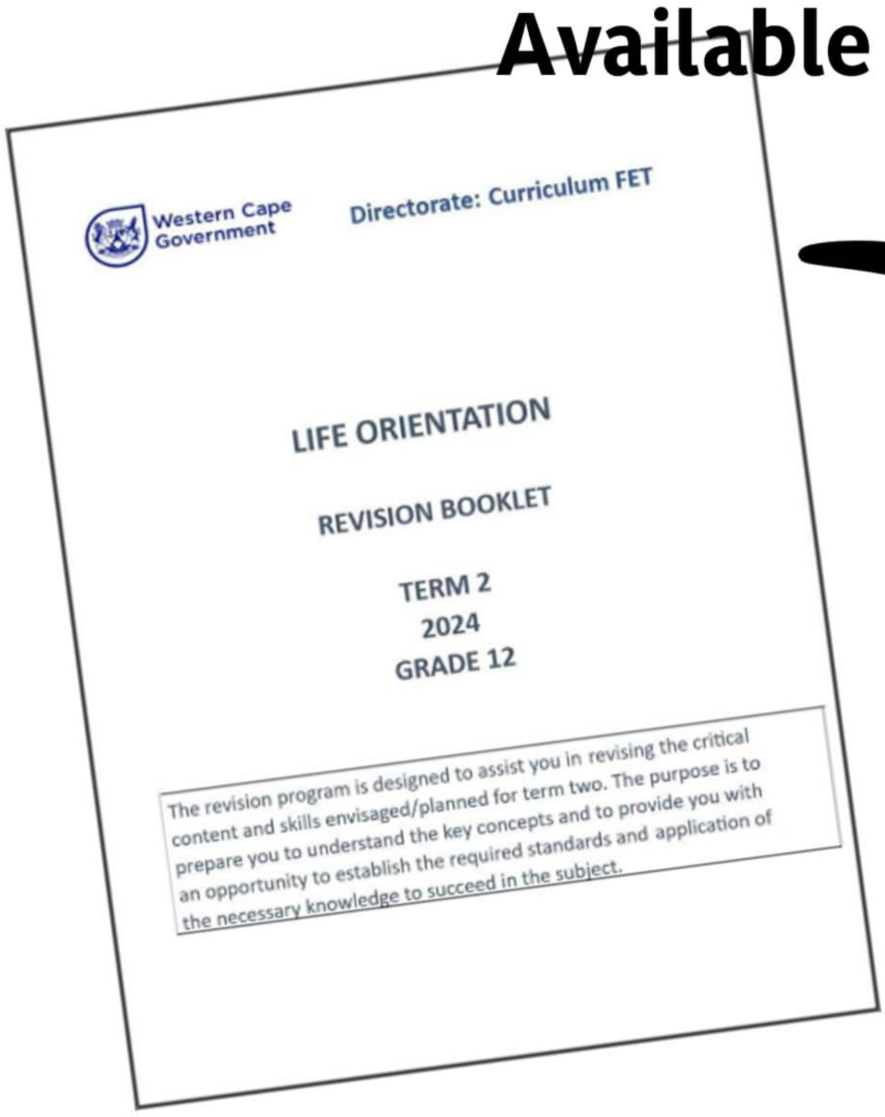


Western Cape Government

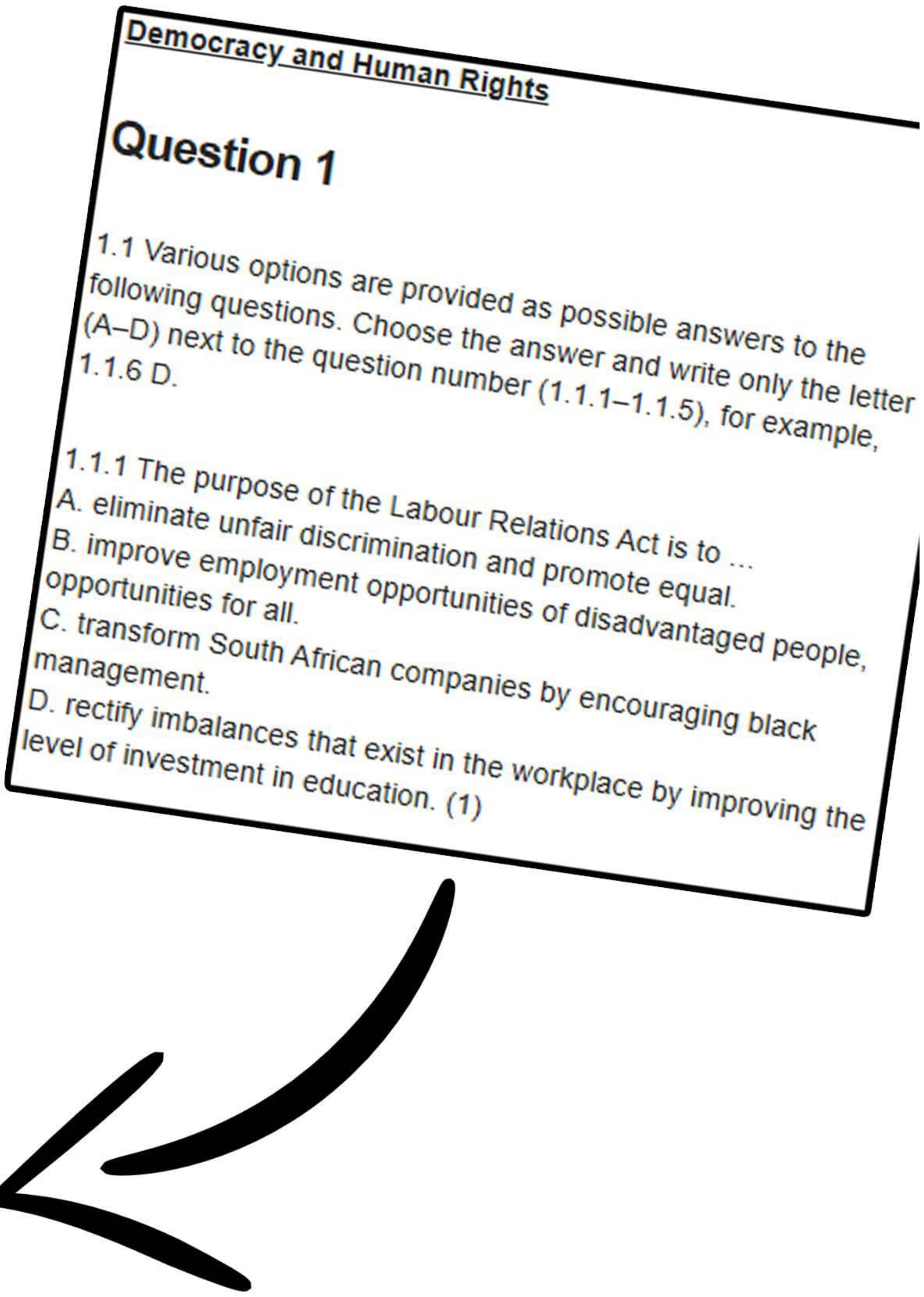
Directorate: Curriculum FET

LIFE ORIENTATION

REVISION BOOKLET

TERM 2
2024
GRADE 12

The revision program is designed to assist you in revising the critical content and skills envisaged/planned for term two. The purpose is to prepare you to understand the key concepts and to provide you with an opportunity to establish the required standards and application of the necessary knowledge to succeed in the subject.

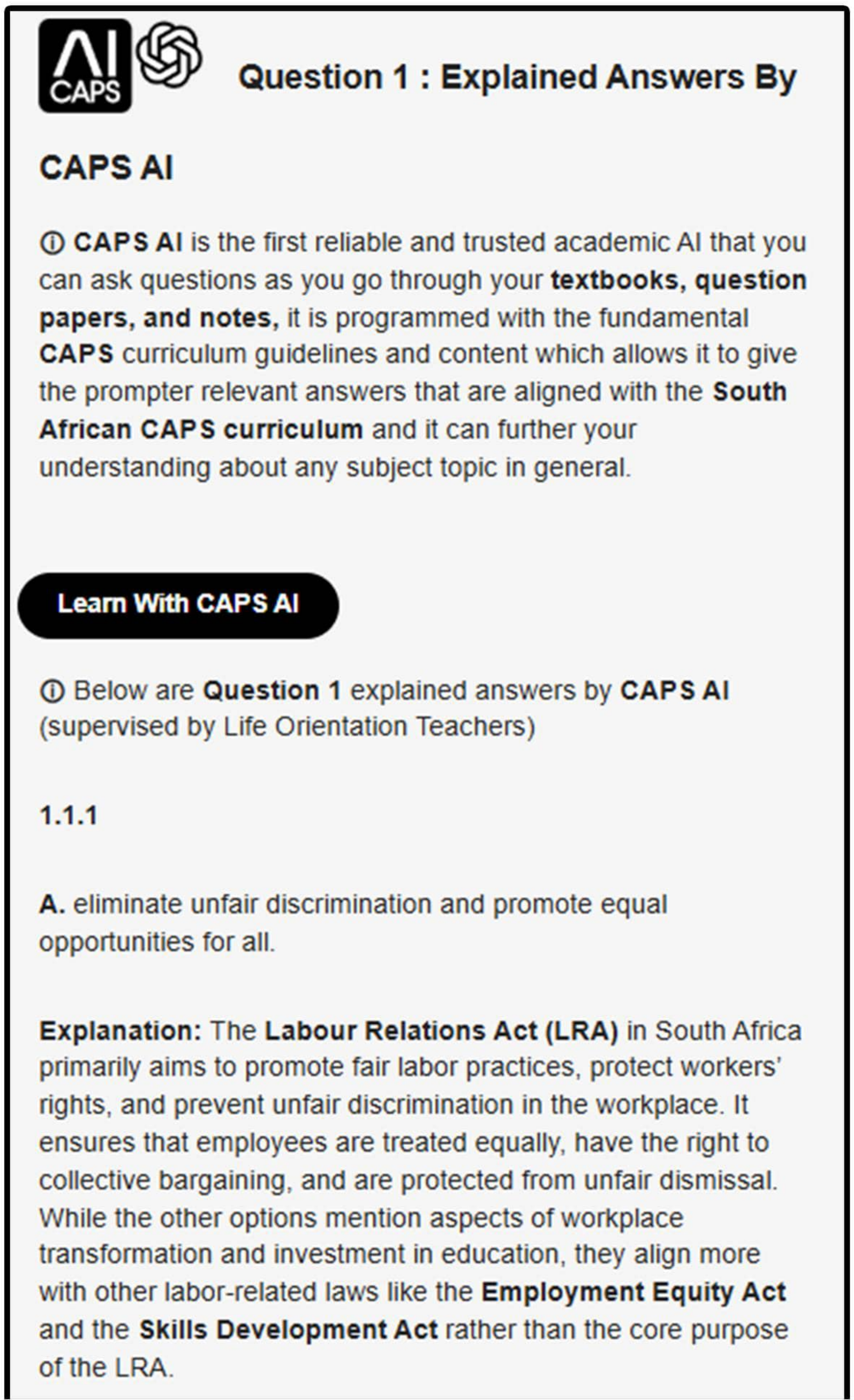


Democracy and Human Rights

Question 1

1.1 Various options are provided as possible answers to the following questions. Choose the answer and write only the letter (A–D) next to the question number (1.1.1–1.1.5), for example, 1.1.6 D.

1.1.1 The purpose of the Labour Relations Act is to …
A. eliminate unfair discrimination and promote equal opportunities for all.
B. improve employment opportunities of disadvantaged people, opportunities for all.
C. transform South African companies by encouraging black management.
D. rectify imbalances that exist in the workplace by improving the level of investment in education. (1)



Question 1 : Explained Answers By CAPS AI

ⓘ **CAPS AI** is the first reliable and trusted academic AI that you can ask questions as you go through your **textbooks, question papers, and notes,** it is programmed with the fundamental **CAPS** curriculum guidelines and content which allows it to give the prompter relevant answers that are aligned with the **South African CAPS curriculum** and it can further your understanding about any subject topic in general.

ⓘ Below are **Question 1** explained answers by **CAPS AI** (supervised by Life Orientation Teachers)

1.1.1

A. eliminate unfair discrimination and promote equal opportunities for all.

Explanation: The **Labour Relations Act (LRA)** in South Africa primarily aims to promote fair labor practices, protect workers' rights, and prevent unfair discrimination in the workplace. It ensures that employees are treated equally, have the right to collective bargaining, and are protected from unfair dismissal. While the other options mention aspects of workplace transformation and investment in education, they align more with other labor-related laws like the **Employment Equity Act** and the **Skills Development Act** rather than the core purpose of the LRA.

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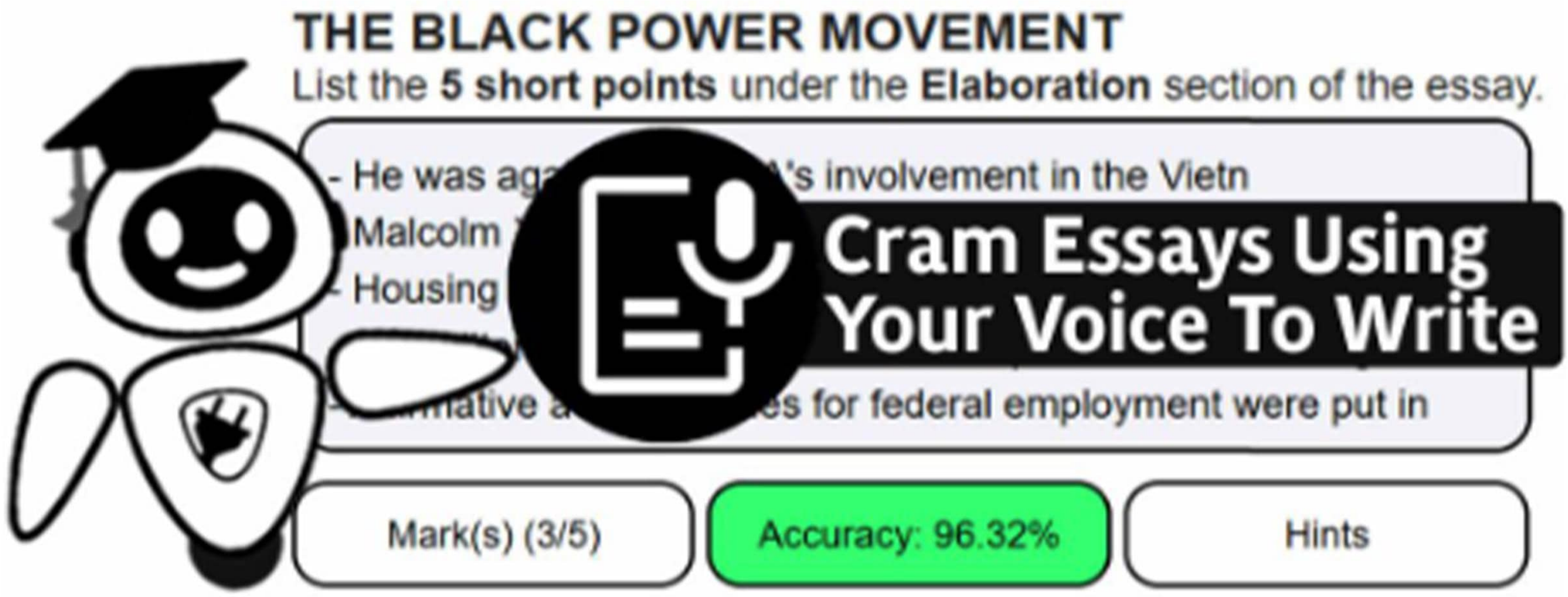

THE BLACK POWER MOVEMENT

List the **5 short points** under the **Elaboration** section of the essay.

- He was ag A's involvement in the Vietn
- Malcolm
- Housing
- mative a es for federal employment were put in

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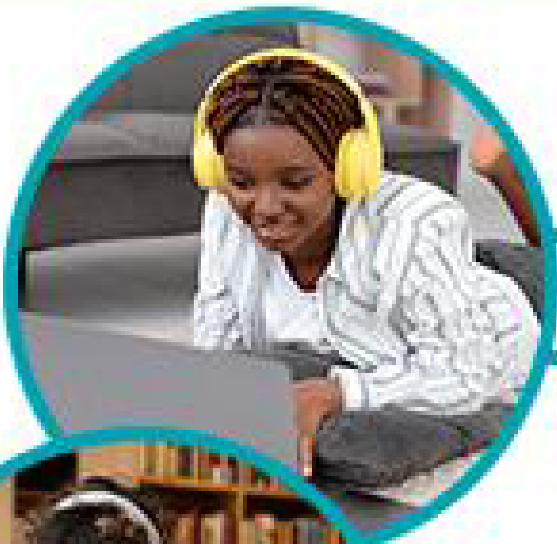

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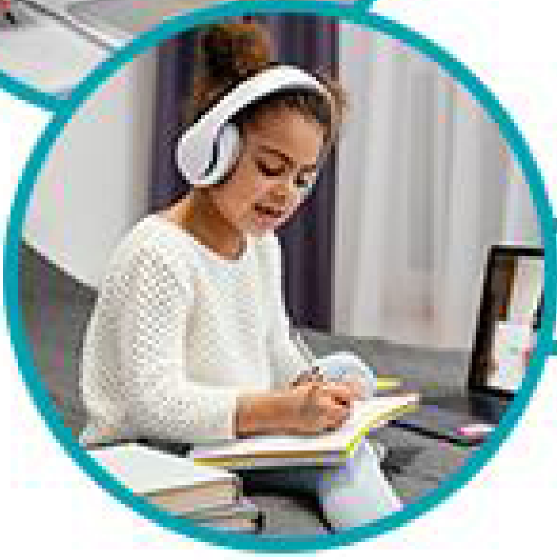

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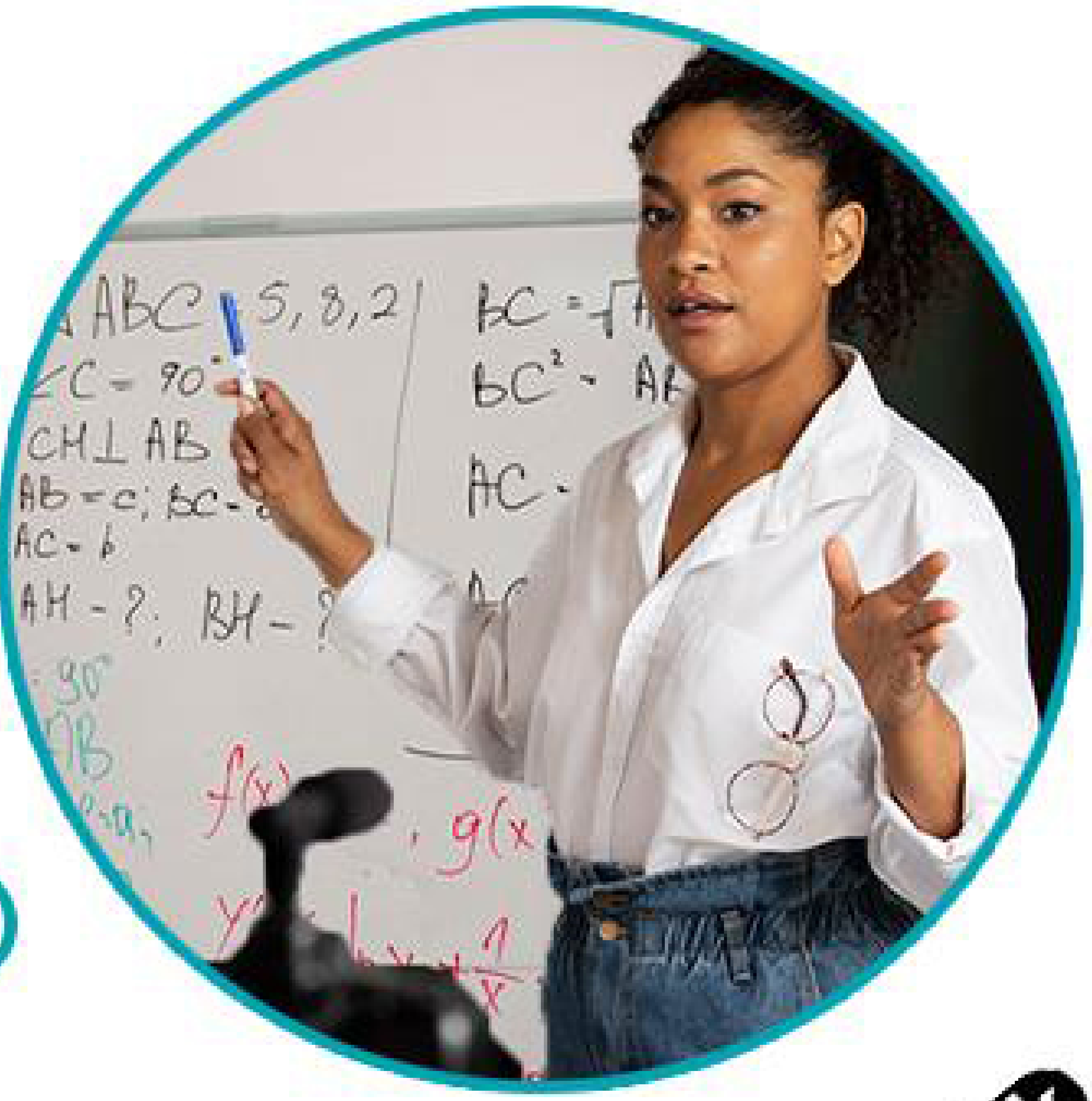

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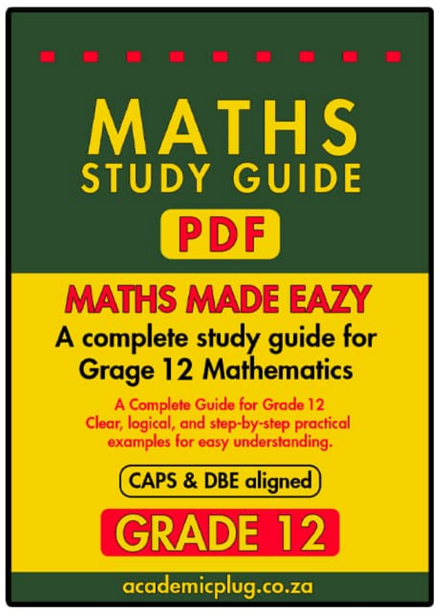
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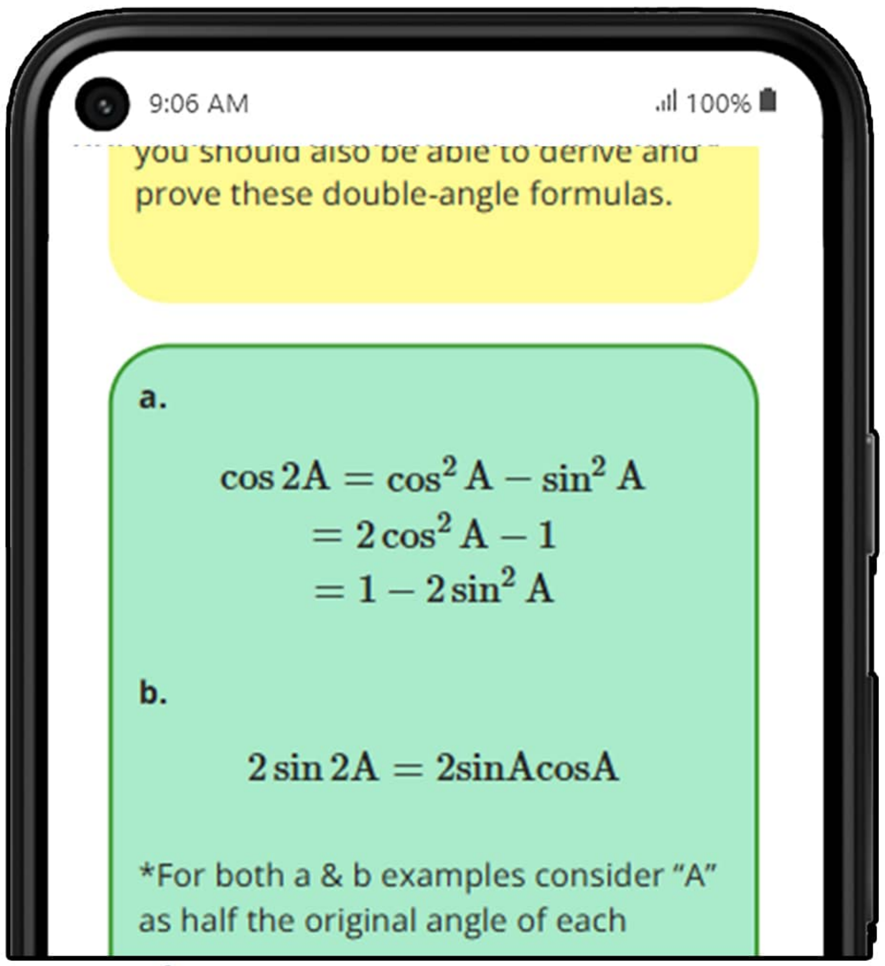

you should also be able to derive and prove these double-angle formulas.

a.

$$\cos 2A = \cos^2 A - \sin^2 A$$
$$= 2\cos^2 A - 1$$
$$= 1 - 2\sin^2 A$$

b.

$$2\sin 2A = 2\sin A\cos A$$

*For both a & b examples consider "A" as half the original angle of each

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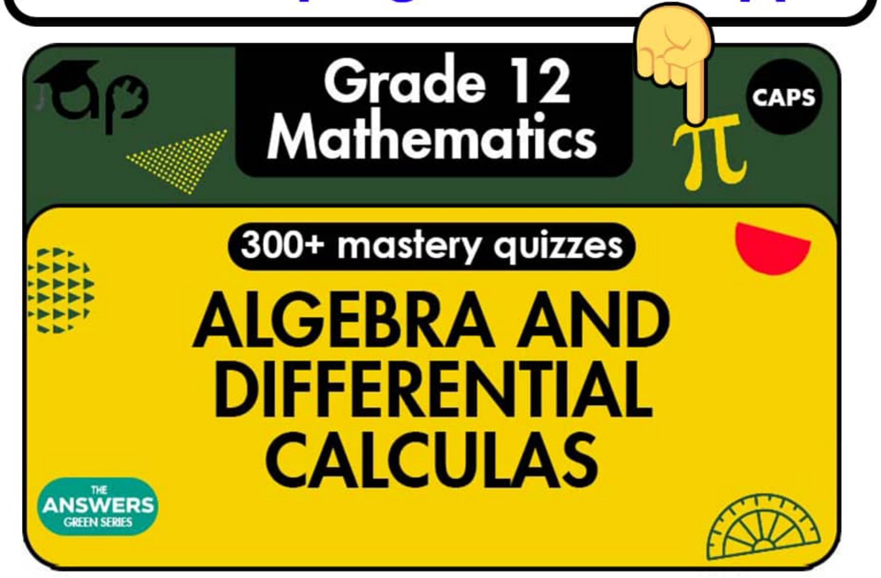
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CHAPTER 7
ALGEBRA AND DIFFERENTIAL CALCULAS

WHAT YOU WILL LEARN IN THIS CHAPTER

PAGE	CONTENT	CAPS	ASSESSMENTS YOU WILL BE ABLE TO:
132	**A. The Remainder and Factor Theorem**	4	• Take note and understand the Remainder Theorem for polynomials up to the third degree.
132	• The Remainder Theorem	5	
134	• The Factor Theorem		
135	• Factorisation of Cubic Polynomials		• Factorise cubic polynomials.
139	**B. Limits**		• Work with and have an intuitive understanding of the concept of a limit.
141	**C. Average Gradient**		
142	**D. Differentiation**		
142	• Gradient at a Point (The Derivative)		• Differentiate specified functions from first principles.
145	• Rules for Differentiation		
146	• Notation		
149	• Additional Notation Examples		• Use the specified rules for differentiation.
150	**E. Tangents to Graphs of Functions**		• Determine the equations of tangents to graphs.
150	• Determining the Equation of a Straight Line		• Work with an increasing and decreasing functions and identify stationary points and points of inflection.
156	**F. Stationary Points and Points of Inflection**		
156	• Increasing and Decreasing Functions and Stationary Points		
158	• Concavity		
158	• Points of Inflection		

A. THE REMAINDER AND FACTOR THEOREM

THE REMAINDER THEOREM

The Remainder Theorem states that if a polynomial $f(x)$ is divided by a linear polynomial $ax+b$, then the remainder $r = f\left(-\frac{b}{a}\right)$.

Example 1: Use the Remainder Theorem to determine the remainder when $f(x)=4x^3-8x^2+x+2$ is divided by:

a) $x+4$ b) $x-2$ c) $2x+1$ d) $2x-5$

a) $f(x)$ is being divided by $x+4$. Therefore, the remainder is equal to $f(-4)$:

$$\begin{aligned} f(-4) &= 4(-4)^3-8(-4)^2+(-4)+2 \\ &= -256-128-4+2 \\ &= -386 \end{aligned}$$

b) $f(x)$ is being divided by $x-2$. Therefore, the remainder is equal to $f(2)$:

$$\begin{aligned} f(2) &= 4(2)^3-8(2)^2+(2)+2 \\ &= 32-32+2+2 \\ &= 4 \end{aligned}$$

c) $f(x)$ is being divided by $2x+1$. Therefore, the remainder is equal to $f\left(-\frac{1}{2}\right)$:

$$\begin{aligned} f\left(-\frac{1}{2}\right) &= 4\left(-\frac{1}{2}\right)^3-8\left(-\frac{1}{2}\right)^2+\left(-\frac{1}{2}\right)+2 \\ &= -\frac{1}{2}-2-\frac{1}{2}+2 \\ &= -1 \end{aligned}$$

d) $f(x)$ is being divided by $2x-5$. Therefore, the remainder is equal to $f\left(\frac{5}{2}\right)$:

$$\begin{aligned} f\left(\frac{5}{2}\right) &= 4\left(\frac{5}{2}\right)^3-8\left(\frac{5}{2}\right)^2+\left(\frac{5}{2}\right)+2 \\ &= \frac{125}{2}-50+\frac{5}{2}+2 \\ &= 17 \end{aligned}$$

Example 2: When $g(x)=x^3-px^2+x+1$ is divided $(x-3)$ the remainder is 4. Determine the value of p.

$g(3)=4 \rightarrow$ the remainder is 4 when $g(x)$ is divided by $x-3$

$\therefore (3)^3-p(3)^2+(3)+1=4$

$\therefore 27-9p+3+1=4$

$\therefore -9p=-27$

$\therefore p=3$

Example 3: When $f(x) = 8x^3 + ax^2 + bx + 10$ is divided $(2x-1)$ the remainder is 11. When f is divided by $(x+1)$ the remainder is –4. Determine the value of a and b, as well as the equation of f.

$f\left(\frac{1}{2}\right) = 11 \rightarrow$ the remainder is 11 when $f(x)$ is divided by $2x-1$

$\therefore 11 = 8\left(\frac{1}{2}\right)^3 + a\left(\frac{1}{2}\right)^2 + b\left(\frac{1}{2}\right) + 10$

$\therefore 11 = 1 + \frac{a}{4} + \frac{b}{2} + 10$

$\therefore 0 = \frac{a}{4} + \frac{b}{2}$

$\therefore -\frac{a}{4} = \frac{b}{2} \rightarrow$ multiply both sides by -4

$\therefore a = -2b \rightarrow \boxed{1}$

$f(-1) = -4 \rightarrow$ the remainder is -4 when $f(x)$ is divided by $x+1$

$\therefore -4 = 8(-1)^3 + a(-1)^2 + b(-1) + 10$

$\therefore -4 = -8 + a - b + 10$

$\therefore -6 = a - b \rightarrow \boxed{2}$

Sub $\boxed{1}$ into $\boxed{2}$:

$\therefore -6 = -2b - b$

$\therefore 3b = 6$

$\therefore b = 2 \rightarrow \boxed{3}$

Sub $\boxed{3}$ into $\boxed{1}$:

$\therefore a = -2(2)$

$\therefore a = -4$

$\therefore f(x) = 8x^3 - 4x^2 + 2x + 10$

Example 4: Given that $f(x) = x^3 - 3x + 2$:

a) Determine the remainder when f is divided by $(x+2)$ and $(x-1)$.

b) What can you deduce from your answers in (a)?

a) $f(-2) = (-2)^3 - 3(-2) + 2 = 0$

$f(1) = (1)^3 - 3(1) + 2 = 0$

b) When f is divided by $(x+2)$ and $(x-1)$ the remainder is 0. If there is no remainder, $(x+2)$ and $(x-1)$ must be **factors** of f.

Example 4 above leads us into the factor theorem on the next page.

THE FACTOR THEOREM

The Factor Theorem states that:

1. **If $f(x)$ is a polynomial and $ax+b$ is a factor then $f\left(-\frac{b}{a}\right)=0$.**
2. **If $f\left(-\frac{b}{a}\right)=0$ then $ax+b$ is a factor of $f(x)$.**

Example 1: **If $x+2$ is a factor of $f(x)=3x^3+px^2-2x+8$, determine the value of p.**

$f(x)=3x^3+px^2-2x+8$

According to the factor theorem, as $x+2$ is a factor, $f(-2)=0$

$f(-2)=3(-2)^3+p(-2)^2-2(-2)+8$

$\therefore 0=-24+4p+4+8 \rightarrow f(-2)=0$

$\therefore 0=4p-12$

$\therefore 4p=12$

$\therefore p=3$

Example 2: **Determine the value of a and b if x^2+x-2 is a factor of x^3+ax^2+2x+b.**

$f(x)=x^3+ax^2+2x+b$

You first need to find the factors of x^2+x-2:

$\therefore x^2+x-2=(x+2)(x-1)$

$f(-2)=(-2)^3+a(-2)^2+2(-2)+b$ and $f(1)=(1)^3+a(1)^2+2(1)+b$

$=-8+4a-4+b$ $\qquad$ $=1+a+2+b$

$=4a+b-12$ $\qquad$ $=a+b+3$

$f(-2)=0$ as $x+2$ is a factor $\qquad$ $f(1)=0$ as $x-1$ is a factor

$\therefore\ 4a+b-12=0$ $\qquad$ $\therefore\ a+b+3=0$

Now solve simultaneously:

$\therefore 4a+b-12=0 \rightarrow \boxed{1}$ and $b=-3-a \rightarrow \boxed{2}$

Substitute $\boxed{2}$ into $\boxed{1}$:

$\therefore 4a+(-3-a)-12=0$

$\therefore 3a=15$

$\therefore a=5 \rightarrow \boxed{3}$

Substitute $\boxed{3}$ into $\boxed{2}$:

$\therefore b=-3-5$

$\therefore b=-8$

FACTORISATION OF CUBIC POLYNOMIALS

Two methods of factorising cubic polynomials will be shown in this textbook.

1) **Using inspection**
2) **Using synthetic division**

Steps to factorising cubic polynomials using inspection:

1) **Determine a factor of the polynomial using the factor theorem.**
2) **Determine the coefficients of x^2 and the constant by inspection.**
3) **Determine coefficient of x by equating coefficients.**
4) **Factorise the quadratic factor in order to obtain the other factors of the cubic polynomial (the quadratic formula may have to be used).**

Illustrative Example: Factorise $f(x) = x^3 - 4x^2 + x + 6$ as far as possible.

1) Substitute factors of the constant (6), until you obtain $f(b) = 0$:

$f(1) = (1)^3 - 4(1)^2 + (1) + 6 = 4$

$\therefore x - 1$ is not a factor

Now try -1: $f(-1) = (-1)^3 - 4(-1)^2 + (-1) + 6 = 0$

$\therefore x + 1$ is a factor of $f(x). \rightarrow f(-1) = 0$

2) Determine the coefficient of x^2 and the constant by inspection:

The cubic equation will factorise into a linear factor and a quadratic factor.

$\therefore 1x^3 - 4x^2 + x + 6 = (x+1)(ax^2 + bx + c) \rightarrow (\text{linear factor})(\text{quadratic factor})$

The coefficient of x^3 is obtained by multiplying x from the linear factor with ax^2 from the quadratic factor. By inspection the value of a has to be equal to 1 as the coefficient of x^3 has to be 1.

The constant is obtained by multiplying 1 from the linear factor with c from the quadratic factor. By inspection the value of c has to be equal to 6.

$\therefore x^3 - 4x^2 + x + 6 = (x+1)(x^2 + bx + 6)$

3) Determine the coefficient of x by equating coefficients:

The coefficient of x^2 in the cubic is obtained by multiplying x from the linear factor with bx from the quadratic factor as well as from the constant (1) from the linear factor being multiplied by x^2 from the quadratic factor. By inspection the coefficient of x^2 has to be equal to –4. The value of b can then be calculated.

$\therefore -4x^2 = bx^2 + x^2$

$\therefore -4x^2 = x^2(b+1)$

By equating coefficients it can be seen that $-4 = b + 1$

$\therefore b = -5$

$\therefore f(x) = (x+1)(x^2 - 5x + 6)$

4) Now factorise the quadratic factor:

$f(x) = (x+1)(x-2)(x-3)$

Steps to factorising cubic polynomials using synthetic division:

1) **Determine a factor of the polynomial using the factor theorem.**
2) **Use synthetic division to obtain a quadratic factor of the cubic polynomial.**
3) **Factorise the quadratic in order to obtain the other factors of the cubic polynomial (the quadratic formula may have to be used).**

Illustrative Example: Factorise $f(x) = x^3 - 4x^2 + x + 6$ as far as possible.

1) Substitute factors of the constant (6), until you obtain $f(b) = 0$:

$f(1) = (1)^3 - 4(1)^2 + (1) + 6 = 4$

$\therefore x - 1$ is not a factor

Now try -1: $f(-1) = (-1)^3 - 4(-1)^2 + (-1) + 6 = 0$

$\therefore x + 1$ is a factor of $f(x)$. $\rightarrow f(-1) = 0$

2) Now use synthetic division to obtain the other factors:

- Draw a 'backwards L' with the coefficients of x and the constant of the cubic polynomial in it. Bring down the coefficient of x^3 (1). Write down the root (–1) from the factor theorem above next to the 'backwards L' as shown below:

$$\begin{array}{cccc|c} 1 & -4 & 1 & 6 & -1 \\ \hline 1 & & & & \end{array}$$

- The '1' is then multiplied by the root '–1'. The answer $(1 \times -1 = -1)$ is then added to the coefficient of x^2 (–4). The sum of the numbers $(-1 - 4 = -5)$ is then written under the coefficient of x^2 as shown below:

$$\begin{array}{cccc|c} 1 & \overset{-1}{-4} & 1 & 6 & -1 \\ \hline 1 & -5 & & & \end{array}$$

- The process is then repeated, –5 is multiplied by the factor (–1) and the answer (5) is then added to the coefficient of x as shown below:

$$\begin{array}{cccc|c} 1 & \overset{-1}{-4} & \overset{+5}{1} & 6 & -1 \\ \hline 1 & -5 & 6 & & \end{array}$$

- The process is then repeated again as shown below. The last number is the remainder and is always be equal to 0 as you are dividing f by a factor.

$$\begin{array}{cccc|c} 1 & \overset{-1}{-4} & \overset{+5}{1} & \overset{-6}{6} & -1 \\ \hline 1 & -5 & 6 & 0 & \end{array}$$

- The numbers under the 'backwards L' are the quotient (answer) when f is divided by the factor. The 1st number (1) is the coefficient of x^2, –5 is the coefficient of x, and 6 is the constant.

$\therefore f(x) = (x+1)(1x^2 - 5x + 6)$

3) Now factorise fully:

$f(x) = (x+1)(x-2)(x-3)$

Always remember to write polynomials in descending powers of x.

Example 1: Use the Factor Theorem to factorise $g(x)=6x^3-11x^2-3x+2$.

In order to determine a factor, substitute factors of the constant (2) into $g(x)$.

Try 1:	Try 2:
$g(1)=6(1)^3-11(1)^2-3(1)+2$ $\therefore g(1)=-6\neq 0$ $\therefore x-1$ is not a factor	$g(2)=6(2)^3-11(2)^2-3(2)+2$ $\therefore g(2)=48-44-6+2$ $\therefore g(2)=0$ $\therefore x-2$ is a factor

Now use inspection or synthetic division:

Using inspection	Using synthetic division
$\therefore 6x^3-11x^2-3x+2=(x-2)(6x^2+bx-1)$ $\therefore -11x^2=bx^2-12x^2$ $\therefore -11x^2=x^2(b-12)$ $\therefore -11=b-12$ $\therefore b=1$ $\therefore g(x)=(x-2)(6x^2+x-1)$ $\therefore g(x)=(x-2)(3x-1)(2x+1)$	$\begin{array}{rrrr\|l} & 12 & 2 & -2 & \\ 6 & -11 & -3 & 2 & 2 \\ \hline 6 & 1 & -1 & 0 & \end{array}$ $\therefore g(x)=(x-2)(6x^2+1x-1)$ $\therefore g(x)=(x-2)(3x-1)(2x+1)$

Example 2: Solve for x if $x^3+5x^2+2x-8=0$.

You have to factorise $x^3+5x^2+2x-8=0$ in order to solve the equation.
Determine a factor by substituting factors of the constant (–8) into LHS of the equation.

Try 1:

$(1)^3+5(1)^2+2(1)-8=1+5+2-8=0$

$\therefore x-1$ is a factor

Now use inspection or synthetic division:

Using inspection	Using synthetic division
$\therefore x^3+5x^2+2x-8=(x-1)(x^2+bx+8)$ $\therefore 5x^2=bx^2-x^2$ $\therefore 5x^2=x^2(b-1)$ $\therefore 5=b-1$ $\therefore b=6$ $\therefore x^3+5x^2+2x-8=(x-1)(x^2+6x+8)$ $\therefore (x-1)(x^2+6x+8)=0$ $\therefore (x-1)(x+2)(x+4)=0$ $\therefore x=1$ or $x=-2$ or $x=-4$	$\begin{array}{rrrr\|l} & 1 & 6 & 8 & \\ 1 & 5 & 2 & -8 & 1 \\ \hline 1 & 6 & 8 & 0 & \end{array}$ $x^3+5x^2+2x-8=(x-1)(1x^2+6x+8)$ $\therefore (x-1)(x^2+6x+8)=0$ $\therefore (x-1)(x+2)(x+4)=0$ $\therefore x=1$ or $x=-2$ or $x=-4$

Example 3: **Solve for x if $x^3-5x+2=0$. Leave your answers in simplest surd form.**

You have to factorise $x^3-5x+2=0$ in order to solve the equation.
It is important to realise that the coefficient of x^2 is equal to 0 in this example.
Determine a factor by substituting factors of the constant (2) into LHS of the equation.
Try 2:

$(2)^3-5(2)+2=8-10+2=0$

$\therefore x-2$ is a factor

Now use inspection or synthetic division:

Using inspection	Using synthetic division
$x^3-5x+2=(x-2)(x^2+bx-1)$ $\therefore 0x^2=bx^2-2x^2$ $\therefore 0x^2=x^2(b-2)$ $\therefore 0=b-2$ $\therefore b=2$ $\therefore x^3-5x+2=(x-2)(x^2+2x-1)$ As the quadratic factor (x^2+2x-1) does not factorise, you have to use the quadratic formula. $\therefore (x-2)(x^2+2x-1)=0$ $\therefore x=2 \quad \text{or} \quad x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ $=\frac{-2\pm\sqrt{(2)^2-4(1)(-1)}}{2(1)}$ $=\frac{-2\pm2\sqrt{2}}{2}$ $=\frac{2(-1\pm\sqrt{2})}{2}$ $=-1\pm\sqrt{2}$	$\begin{array}{rrrrl} & 2 & 4 & -2 & \\ 1 & 0 & -5 & 2 & \rvert\, 2 \\ \hline 1 & 2 & -1 & \mathbf{0} & \end{array}$ $\therefore x^3-5x+2=(x-2)(1x^2+2x-1)$ As the quadratic factor (x^2+2x-1) does not factorise, you have to use the quadratic formula. $\therefore (x-2)(x^2+2x-1)=0$ $\therefore x=2 \quad \text{or} \quad x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ $=\frac{-2\pm\sqrt{(2)^2-4(1)(-1)}}{2(1)}$ $=\frac{-2\pm2\sqrt{2}}{2}$ $=\frac{2(-1\pm\sqrt{2})}{2}$ $=-1\pm\sqrt{2}$

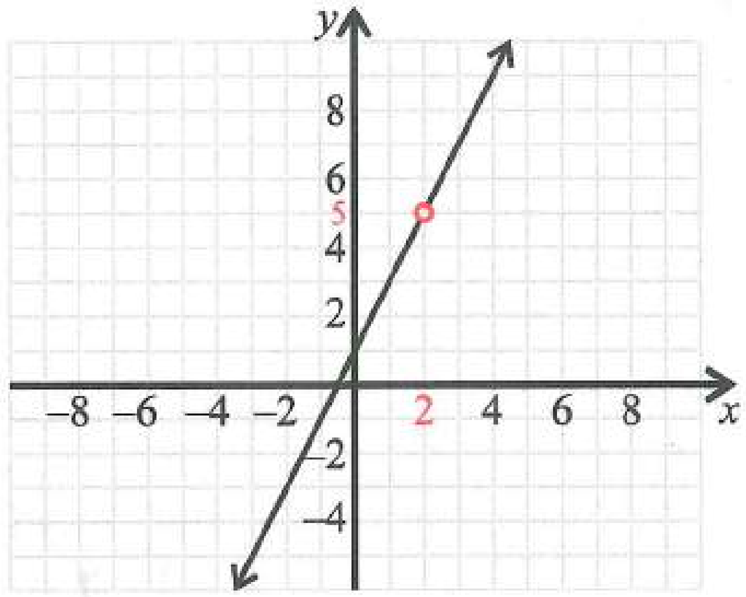
Complete Exercise 1 on page 160.

B. LIMITS

The limit of a function is denoted as: $\lim\limits_{x \to a} f(x)$

The limit of a function is the value that the function tends to (approaches) as a variable in the function (for example x) tends to a specific value (for example a).

Illustrative Example: The graph of $f(x) = \dfrac{(2x+1)(x-2)}{x-2}$ is sketched below:



From the equation and the graph it can be seen that $f(x)$ is undefined when $x = 2$, as division by 0 would occur. However, if the function is examined starting from the **left moving right towards 2**, as the x-value gets closer to 2, the y-value gets closer to 5. If the function is examined from the **right moving left towards 2**, as the x-value gets closer to 2, the y-value also gets closer to 5. **It can be said that as x tends to 2, the function approaches a value of 5.**

So, although the function does not have a value at $x = 2$, **as x tends to 2 the value of the function tends to 5.**

Example 1: Determine the value of $\lim\limits_{h \to 2} (2h+3)$

$\lim\limits_{h \to 2} (2h+3) \rightarrow$ reads as: the limit of $2h+3$ as h tends to 2

$= 2(2)+3 \rightarrow$ substitute the value that h tends to

$= 7$

Example 2: Determine the value of $\lim\limits_{h \to 0} \left(\dfrac{2h^2+5h}{h} \right)$

$\lim\limits_{h \to 0} \dfrac{2h^2+5h}{h} \rightarrow$ you cannot substitute $h = 0$ as this would cause a 0 in the denominator

$= \lim\limits_{h \to 0} \dfrac{h(2h+5)}{h} \rightarrow$ factorise

$= \lim\limits_{h \to 0} 2h+5 \rightarrow$ you have to keep writing $\lim\limits_{h \to 0}$ until you substitute $h = 0$

$= 2(0)+5$

$= 5$

Example 3: Determine the value of $\lim\limits_{h \to -2}\left(\dfrac{h^2+5h+6}{h+2}\right)$.

$\lim\limits_{h\to -2} \dfrac{h^2+5h+6}{h+2} \rightarrow$ you can't substitute $h=-2$ as it would cause a 0 in the denominator

$= \lim\limits_{h\to -2} \dfrac{(h+2)(h+3)}{h+2} \rightarrow$ factorise

$= \lim\limits_{h\to -2} \ h+3 \rightarrow$ you have to keep writing $\lim\limits_{h\to -2}$ until you substitute $h=0$

$=-2+3=1$

Example 4: Determine the value of $\lim\limits_{h\to 0}\dfrac{\frac{3}{h}+2}{\frac{1}{h}+3}$.

You can't substitute $h=0$ into the expression as it would cause a 0 in the denominator.

$$\therefore \lim_{h\to 0}\frac{\frac{3}{h}+2}{\frac{1}{h}+3} = \lim_{h\to 0}\left(\frac{3}{h}+2\right)\div\left(\frac{1}{h}+3\right) \rightarrow \text{find a LCD of } h$$

$$= \lim_{h\to 0}\left(\frac{3+2h}{h}\right)\div\left(\frac{1+3h}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{3+2h}{\cancel{h}}\right)\times\left(\frac{\cancel{h}}{1+3h}\right)$$

$$= \lim_{h\to 0}\frac{3+2h}{1+3h} \rightarrow \text{now you can substitute } h=0$$

$$= \frac{3+2(0)}{1+3(0)} = 3$$

Example 5: If $f(x)=5x^2$, **find the value of** $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$.

Before starting to solve the this problem, determine the value of $f(x+h)$

$$f(x+h)=5(x+h)^2 \rightarrow \text{replace every } x \text{ with } x+h$$

$$=5\left(x^2+2xh+h^2\right) \rightarrow \text{don't forget the middle term}$$

$$=5x^2+10xh+5h^2$$

$$\therefore \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{5x^2+10xh+5h^2-5x^2}{h}$$

$$= \lim_{h\to 0}\frac{10xh+5h^2}{h} \rightarrow \text{take out a HCF of } h$$

$$= \lim_{h\to 0}\frac{\cancel{h}(10x+5h)}{\cancel{h}}$$

$$= \lim_{h\to 0} 10x+5h = 10x+5(0) = 10x$$

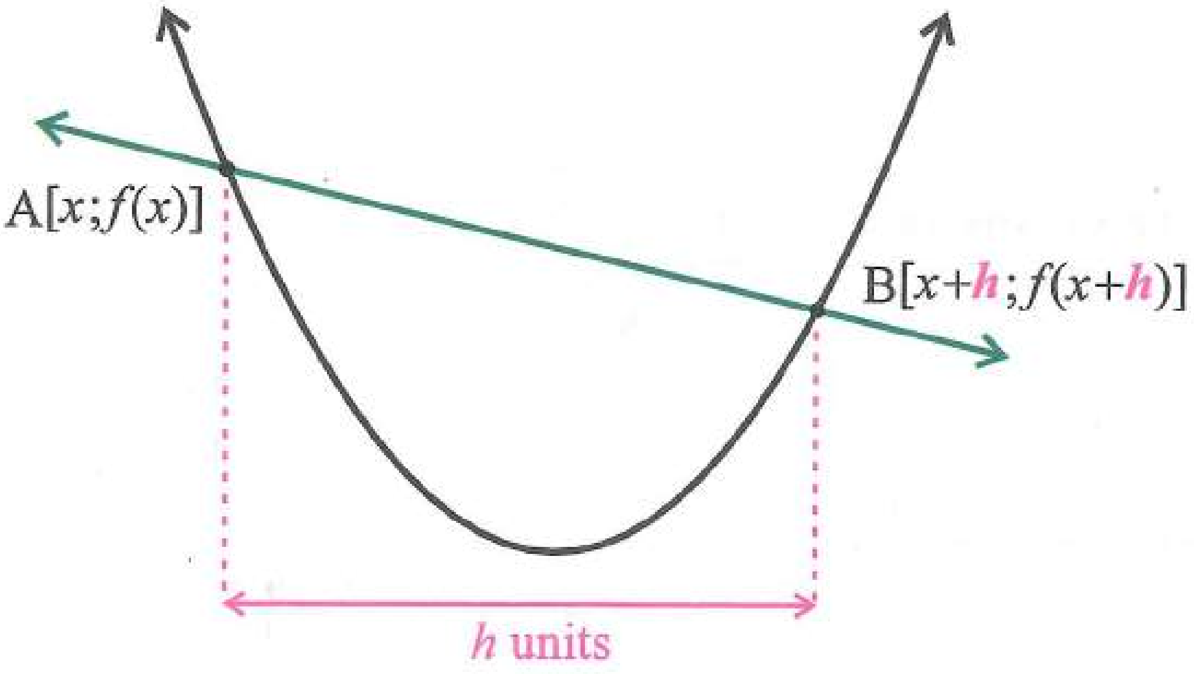
Complete Exercise 2 on page 160.

C. AVERAGE GRADIENT

The average gradient between two points on a curve is the gradient of the straight line joining the two points.

Illustrative Example:

Consider the function $f(x)$ shown below. A is the point $[x; f(x)]$. B is a point h units away from A such that B is the point $[x+h; f(x+h)]$. Determine the average gradient between A and B.



$$\text{average gradient} = m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}$$

From the above example, it can be seen that the two formulas for average gradient are:

Average Gradient:

$$m = \frac{y_2 - y_1}{x_2 - x_1} \text{ or } m = \frac{f(x+h) - f(x)}{h}$$

It is important to note that h is the difference between the x-values.
In other words: $h = x_2 - x_1$

Example: **Determine the average gradient of $f(x) = x^2 + 2$ between the points where $x = 3$ and $x = 4$.**

Using $m = \frac{y_2 - y_1}{x_2 - x_1}$

Determine the y-values by substituting x back into $f(x)$ for each point:

$f(3) = (3)^2 + 2 = 11$

$f(4) = (4)^2 + 2 = 18$

$\therefore$ the points are: $(3;11)$ and $(4;18)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{18 - 11}{4 - 3}$$
$$= 7$$

Using $\frac{f(x+h) - f(x)}{h}$

$f(x+h) = (x+h)^2 + 2 = x^2 + 2xh + h^2 + 2$

$$m = \frac{f(x+h) - f(x)}{h}$$
$$= \frac{x^2 + 2xh + h^2 + 2 - (x^2 + 2)}{h}$$
$$= \frac{2xh + h^2}{h} = \frac{\cancel{h}(2x+h)}{\cancel{h}} = 2x + h$$

$\therefore m = 2x + h \rightarrow h = x_2 - x_1 = 4 - 3$

$\therefore m = 2(3) + (4 - 3) \rightarrow x_1 = 3$

$\therefore m = 7$

D. DIFFERENTIATION

Up until now you have only dealt with the concept of average gradient.
You will now be introduced to the concept of the gradient at a point.
Using differentiation it is possible to determine the gradient at a specific point.

There are two methods that can be used to determine the derivative of a function:

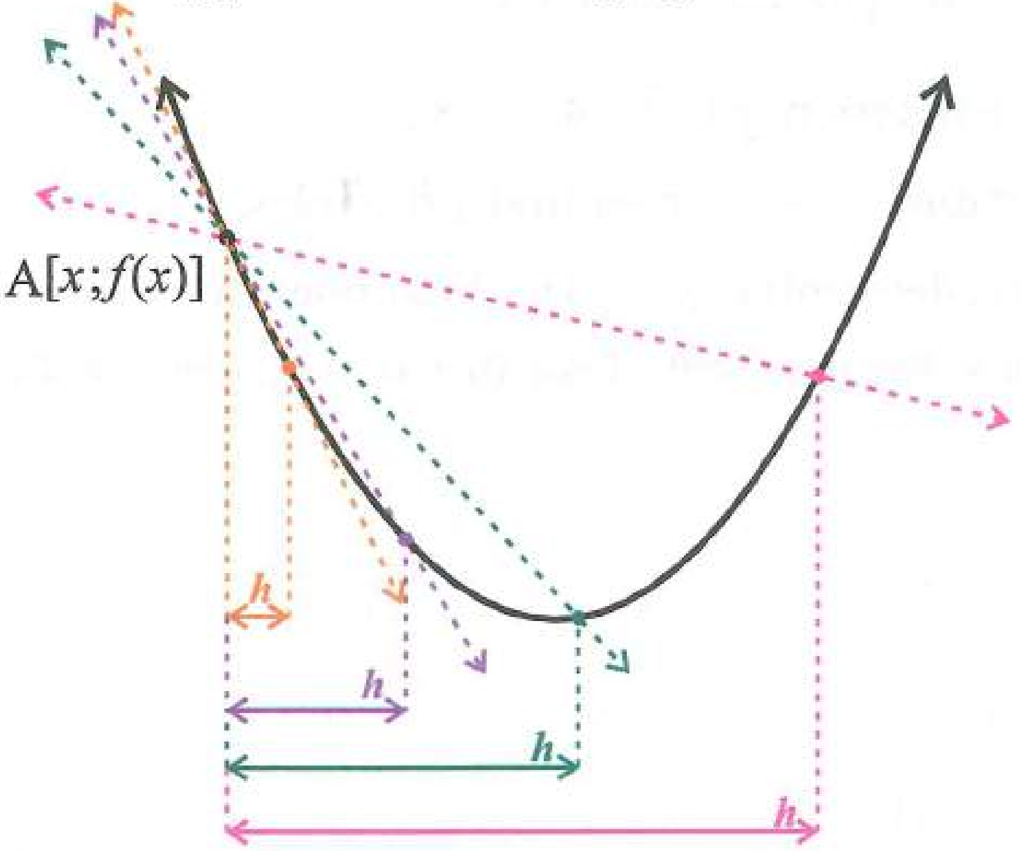
1. **Determine the derivative from first principles.**
2. **Determine the derivative using the rules for differentiation.**

GRADIENT AT A POINT (THE DERIVATIVE)

Illustrative Example:

In the illustrative example on the previous page, it was shown that the average gradient between two points can be calculated using: $f(x)=\dfrac{f(x+h)-f(x)}{h}$

We will now look at what happens to the average gradient as the value of *h* decreases.



It can be seen that as *h* decreases, the gradient of the line starts to get closer to the gradient of the graph at point A. We can say that as *h* tends to 0, the gradient of the line approaches the gradient of the graph at point A. This is known as the derivative. The value of the derivative is the gradient at a point.

There are many different notations used to represent the derivative. For now, the derivative will be denoted as $f'(x)$. Further notations which represent the derivative will be introduced when we deal with the rules for differentiation on page 145. The formula block below shows the definition of the derivative.

Formula for the Gradient at a Point (derivative from first principles):

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$f'(x)$ is the notation used for the derivative, which represents the gradient at a point.
Do not confuse the derivative with $f(x)$, which is the *y*-value of a function at a specific *x*-value, or with $f^{-1}(x)$ which is the inverse of $f(x)$.

Example 1: Using first principles, determine the gradient of $f(x)=8x$ at an
point.

$$f(x)=8x$$

$$\therefore f(x+h)=8(x+h) \rightarrow \text{ replace every } x \text{ with } x+h$$

$$=8x+8h$$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \rightarrow \text{ write the } \lim_{h\to 0} \text{ until you substitute in } h=0$$

$$\therefore f'(x)=\lim_{h\to 0}\frac{8x+8h-[8x]}{h}$$

$$=\lim_{h\to 0}\frac{8h}{h}$$

$$=\lim_{h\to 0} 8$$

$$=8$$

$\therefore$ The gradient of f (at any point) is equal to 8.

Example 2: Given the function $f(x)=4x^2+8x$:

- **a) Determine $f'(x)$ from first principles.**
- **b) Hence, determine $f'(-4)$. What does this answer mean?**
- **c) What is the gradient of the function where $x=4$?**

a) $f(x)=4x^2+8x$

$$\therefore f(x+h)=4(x+h)^2+8(x+h) \rightarrow \text{ replace every } x \text{ with } x+h$$

$$=4(x^2+2xh+h^2)+8x+8h=4x^2+8xh+4h^2+8x+8h$$

$$\therefore f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \rightarrow \text{ write the } \lim_{h\to 0} \text{ until you substitute in } h=0$$

$$\therefore f'(x)=\lim_{h\to 0}\frac{4x^2+8xh+4h^2+8x+8h-\left[4x^2+8x\right]}{h}$$

$$=\lim_{h\to 0}\frac{4x^2+8xh+4h^2+8x+8h-4x^2-8x}{h}$$

$$=\lim_{h\to 0}\frac{8xh+4h^2+8h}{h} \rightarrow \text{ take out a common factor of } h$$

$$=\lim_{h\to 0}\frac{\cancel{h}(8x+4h+8)}{\cancel{h}}$$

$$=8x+8 \rightarrow \text{ this is the gradient at any point on } f$$

b) $f'(-4)=8(-4)+8$

$$=-24$$

This means at the point where $x=-4$ the gradient of f is equal to –24.

c) $f'(4)=8(4)+8$

$$=40 \rightarrow \text{ this is the gradient at the point where } x=4$$

Example 3: **Determine the derivative of $f(x) = -x^3$ at the point where $x = -1$, using the definition of the derivative.**

$f(x) = -x^3$

$\therefore f(x+h) = -(x+h)^3 \rightarrow$ replace every x with $x+h$

$= -(x+h)^2(x+h)$

$= -\left(x^2 + 2hx + h^2\right)(x+h) \rightarrow$ don't forget the middle term

$= -\left(x^3 + hx^2 + 2hx^2 + 2h^2x + h^2x + h^3\right)$

$= -x^3 - 3hx^2 - 3h^2x - h^3$

$f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} \rightarrow$ write the $\lim\limits_{h \to 0}$ until you substitute in $h = 0$

$= \lim\limits_{h \to 0} \dfrac{-x^3 - 3hx^2 - 3h^2x - h^3 - \left(-x^3\right)}{h}$

$= \lim\limits_{h \to 0} \dfrac{-3hx^2 - 3h^2x - h^3}{h} \rightarrow$ take out a common factor of h

$= \lim\limits_{h \to 0} \dfrac{\cancel{h}\left(-3x^2 - 3hx - h^2\right)}{\cancel{h}}$

$= \lim\limits_{h \to 0} \left(-3x^2 - 3hx - h^2\right) = -3x^2$

$\therefore f'(-1) = -3(-1)^2 = -3$

Example 4: **Determine the derivative of $f(x) = \dfrac{2}{x}$ at the point where $x = 2$, using the definition of the derivative.**

$f(x) = \dfrac{2}{x}$

$\therefore f(x+h) = \dfrac{2}{x+h} \rightarrow$ replace every x with $x+h$

$f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} \rightarrow$ write the $\lim\limits_{h \to 0}$ until you substitute in $h = 0$

$= \lim\limits_{h \to 0} \left(\dfrac{2}{x+h} - \dfrac{2}{x}\right) \div h \rightarrow$ split the numerator and denominator apart

$= \lim\limits_{h \to 0} \left(\dfrac{2x - 2(x+h)}{x(x+h)}\right) \times \dfrac{1}{h} \rightarrow$ 'tip and times'

$= \lim\limits_{h \to 0} \dfrac{(2x - 2x - 2h)}{h\left(x^2 + hx\right)}$

$= \lim\limits_{h \to 0} \dfrac{-2\cancel{h}}{\cancel{h}\left(x^2 + hx\right)} = -\dfrac{2}{x^2}$

$\therefore f'(2) = -\dfrac{2}{(2)^2} = -\dfrac{1}{2}$

Complete Exercises 3 on page 160 and Exercise 4 on page 161.

RULES FOR DIFFERENTIATION

The rules for differentiation allow you to differentiate functions without going through the process of differentiating from first principles.

Unless you are asked to differentiate from first principles, use the rules for differentiation.

Rule 1: The derivative of a constant is equal to 0.

If $f(x)=k$, where k is a constant real number, then $f'(x)=0$.

Example: If $f(x)=10$ determine $f'(x)$.

$f(x)=10$

$\therefore f'(x)=0$

Rule 2: The Power Rule

If $f(x)=x^n$, where n is a constant real number, then $f'(x)=n\,x^{n-1}$.

Example: If $f(x)=x^{10}$ determine $f'(x)$.

$f(x)=x^{10}$

$\therefore f'(x)=10x^{10-1}$

$\therefore f'(x)=10x^{9}$

Rule 3: The derivative of a function multiplied by a constant

If $f(x)=k.g(x)$, where k is a constant real number, then $f'(x)=k.g'(x)$.

Example: If $f(x)=3x^4$ determine $f'(x)$.

$f(x)=3x^4$

$\therefore f'(x)=3\times 4x^{4-1}$

$\therefore f'(x)=12x^{3}$

Rule 4: The Sum Rule

If $f(x)=g(x)+h(x)$, then $f'(x)=g'(x)+h'(x)$.

Example: If $f(x)=x^3+2x^{-3}$ determine $f'(x)$.

$f(x)=x^3+2x^{-3}$

$\therefore f'(x)=3x^2+2\left(-3x^{-3-1}\right)$

$\therefore f'(x)=3x^2-6x^{-4}$

Rule 5: The Difference Rule

If $f(x)=g(x)-h(x)$, then $f'(x)=g'(x)-h'(x)$.

Example: If $f(x)=x^3-2x$ determine $f'(x)$.

$f(x)=x^3-2x^1$ → if a power is not given, the power is 1

$\therefore f'(x)=3x^2-2x^{1-1}$

$\therefore f'(x)=3x^2-2x^0$ → any number raised to the power of 0 is equal to 1

$\therefore f'(x)=3x^2-2$

In Grade 12, there are no differentiation rules for multiplication or division.

NOTATION

There are various ways to represent a function and its derivative.
There are four different notations that you are expected to know when calculating the derivative. In this section blue will be used to denote the function and green will be used to denote the derivative.

1. **You are given the function in the form $f(x)=...$**

   Example: $f(x)=5x^2+3x$

   The derivative is written as: $f'(x)=10x+3$

2. **You are given the function in the form $y=...$**

   Example: $y=5x^2+3x$

   The derivative is written as: $\frac{dy}{dx}=10x+3$

3. **You are given the function in the form $D_x[\textit{function}]=...$**

   Example: $D_x\left[5x^2+3x\right]$

   The derivative is written as: $D_x\left[5x^2+3x\right]=10x+3$

4. **You are given the function in the form $\frac{d}{dx}[\textit{function}]=...$**

   Example: $\frac{d}{dx}\left[5x^2+3x\right]$

   The derivative is written as: $\frac{d}{dx}\left[5x^2+3x\right]=10x+3$

In each step of the solving process, keep the original function notation that you are given until you actually determine the derivative.

Example 1: **Given that $f(x) = 2x^3 - 3x^2 + 5$ determine $f'(x)$.**

$$f(x) = 2x^3 - 3x^2 + 5$$

$$\therefore f'(x) = 6x^2 - 6x + 0$$

$$= 6x^2 - 6x$$

Example 2: **Determine $\frac{dy}{dx}$ if $y = (2x+1)(x-3)$.**

$y = (2x+1)(x-3) \rightarrow$ there are no differentiation rules for multiplication, $\therefore$ mulitply out

$\therefore y = 2x^2 - 5x - 3 \rightarrow$ this is not the derivative, don't write $\frac{dy}{dx}$ until you find the derivative

$$\therefore \frac{dy}{dx} = 4x - 5$$

Example 3: **Determine $D_t\left[\frac{t^2+5t+6}{t+2}\right]$.**

$D_t\left[\frac{t^2+5t+6}{t+2}\right] \rightarrow$ if there is more than 1 term in the denominator, factorise the numerator

$= D_t\left[\frac{\cancel{(t+2)}(t+3)}{\cancel{t+2}}\right] \rightarrow$ keep the $D_t[\ldots]$ until you find the derivative

$$= D_t[t+3]$$

$$= 1$$

In the above example *t* is a different variable which has been used instead of *x*.

Example 4: **If $f(x) = \frac{x^2}{3} + \frac{3}{x^2}$ determine $f'(x)$. Write your answer with positive exponents.**

$f(x) = \frac{x^2}{3} + \frac{3}{x^2} \rightarrow$ do not find a LCD if there are single terms in the denominators

$\therefore f(x) = \frac{x^2}{3} + 3x^{-2} \rightarrow$ bring all variables in the denominator to the numerator

$\therefore f'(x) = \frac{2x}{3} - 6x^{-3} \rightarrow$ now write your answer with positive exponents

$$\therefore f'(x) = \frac{2x}{3} - \frac{6}{x^3}$$

Only bring variables (not numbers) to the 'top'. $\frac{x^2}{3}$ is the same as $\frac{1}{3} \times x^2$, which can be differentiated using the rules for differentiation.

As a general rule, you should always write your final answer with positive exponents.

Example 5: **Determine $D_x\left[\frac{x^3-1+x^2}{x}\right]$. Write your answer with positive exponents.**

$$D_x\left[\frac{x^3-1+x^2}{x}\right]\rightarrow\text{if there is a single variable in the denominator split the fraction apart}$$

$$=D_x\left[\frac{x^3}{x}-\frac{1}{x}+\frac{x^2}{x}\right]$$

$$=D_x\left[x^2-\frac{1}{x}+x\right]\rightarrow\text{'bring the variable in the denominator to the top'}$$

$$=D_x\left[x^2-x^{-1}+x\right]$$

$$=2x+x^{-2}+1\rightarrow\text{now write your answer with positive exponents}$$

$$=2x+\frac{1}{x^2}+1$$

Example 6: **Determine $\frac{dy}{dx}$ if $y=\frac{2}{3x}+\sqrt{2}$. Write your answer with positive exponents.**

$$y=\frac{2}{3x}+\sqrt{2}\rightarrow\text{'bring variables to the top'},\sqrt{2}\text{ is a constant}$$

$$\therefore y=\frac{2x^{-1}}{3}+\sqrt{2}$$

$$\therefore\frac{dy}{dx}=-\frac{2x^{-2}}{3}+0\rightarrow\text{write your answer with positive exponents}$$

$$\therefore\frac{dy}{dx}=-\frac{2}{3x^2}$$

Example 7: **Determine $\frac{d}{dx}\left[12\sqrt[3]{x}+\frac{1}{2x}+2\right]$. Write your answer with positive exponents.**

$$\frac{d}{dx}\left[12\sqrt[3]{x}+\frac{1}{2x}+2\right]\rightarrow\text{write surds in exponential form and 'bring variables to the top'}$$

$$=\frac{d}{dx}\left[12x^{\frac{1}{3}}+\frac{x^{-1}}{2}+2\right]$$

$$=12\times\frac{1}{3}x^{\frac{1}{3}-1}-\frac{x^{-2}}{2}+0$$

$$=\frac{4}{x^{\frac{2}{3}}}-\frac{1}{2x^2}$$

Complete Exercises 5 on page 161.

ADDITIONAL NOTATION EXAMPLES

Example 1: If $uv = 1000$, determine:

a) $\dfrac{du}{dv}$ **b)** $\dfrac{dv}{du}$

a) $uv = 1000 \rightarrow$ differentiate u with respect to v, make u the subject of the formula

$$\therefore u = \frac{1000}{v} = 1000v^{-1}$$

$$\therefore \frac{du}{dv} = -1000v^{-2} = -\frac{1000}{v^2}$$

b) $uv = 1000 \rightarrow$ differentiate v with respect to u, make v the subject of the formula

$$\therefore v = \frac{1000}{u} = 1000u^{-1}$$

$$\therefore \frac{dv}{du} = -1000u^{-2} = -\frac{1000}{u^2}$$

Example 2: If $y = x^2$ and $x = \dfrac{1}{3t}$ determine:

a) $\dfrac{dy}{dx}$ **b)** $\dfrac{dx}{dt}$ **c)** $\dfrac{dy}{dt}$

a) $\dfrac{dy}{dx} = 2x$

b) $x = \dfrac{t^{-1}}{3}$

$$\therefore \frac{dx}{dt} = \frac{-t^{-2}}{3} = -\frac{1}{3t^2}$$

c) In order to solve for $\dfrac{dy}{dt}$ you first have to write y in terms of t :

$y = x^2 \rightarrow \boxed{1}$ and $x = \dfrac{1}{3t^2} \rightarrow \boxed{2}$

Sub $\boxed{2}$ into $\boxed{1}$:

$$\therefore y = \left(\frac{1}{3t}\right)^2$$

$$= \frac{1}{9t^2}$$

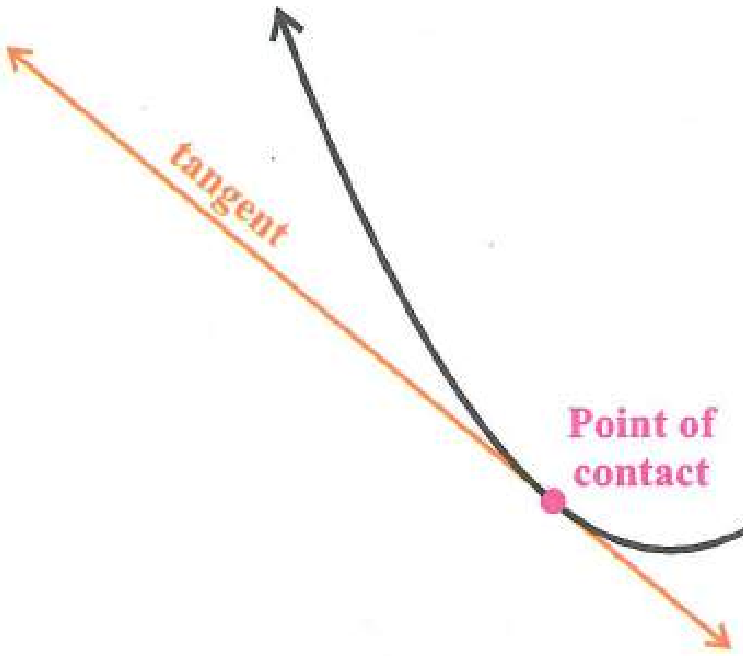
$$= \frac{t^{-2}}{9}$$

$$\therefore \frac{dy}{dt} = \frac{-2t^{-3}}{9}$$

$$= \frac{-2}{9t^3}$$

E. TANGENTS TO GRAPHS OF FUNCTIONS

A tangent is a straight line that touches a graph at one point.
The gradient of the tangent is equal to the gradient of the graph at the point of contact. Thus, the derivative of the graph at the point of contact is equal to the gradient of the tangent.



DETERMINING THE EQUATION OF A STRAIGHT LINE

As a tangent is a straight line, it is important for you to be able to determine the equation of a straight line.

Illustrative Example: Determine the equation of the line with a gradient of 4 which passes through the point $(2;4)$.

To determine the equation of a straight line use one of the formulas below:

Equation of a straight line:

$$y = mx + c \text{ or } (y - y_1) = m(x - x_1)$$

Method 1: using: $y = mx + c$

$y = 4x + c$

Substitute the point $(2;4)$:

$\therefore 4 = 4(2) + c$

$\therefore 4 = 8 + c$

$\therefore c = -4$

$\therefore y = 4x - 4$

Method 2: using: $(y - y_1) = m(x - x_1)$

$y - y_1 = 4(x - x_1)$

Substitute the point $(2;4)$:

$\therefore y - 4 = 4(x - 2) \rightarrow$ do not leave the equation in this form

$\therefore y - 4 = 4x - 8$

$\therefore y = 4x - 4 \rightarrow$ always write the equation in the form $y = mx + c$

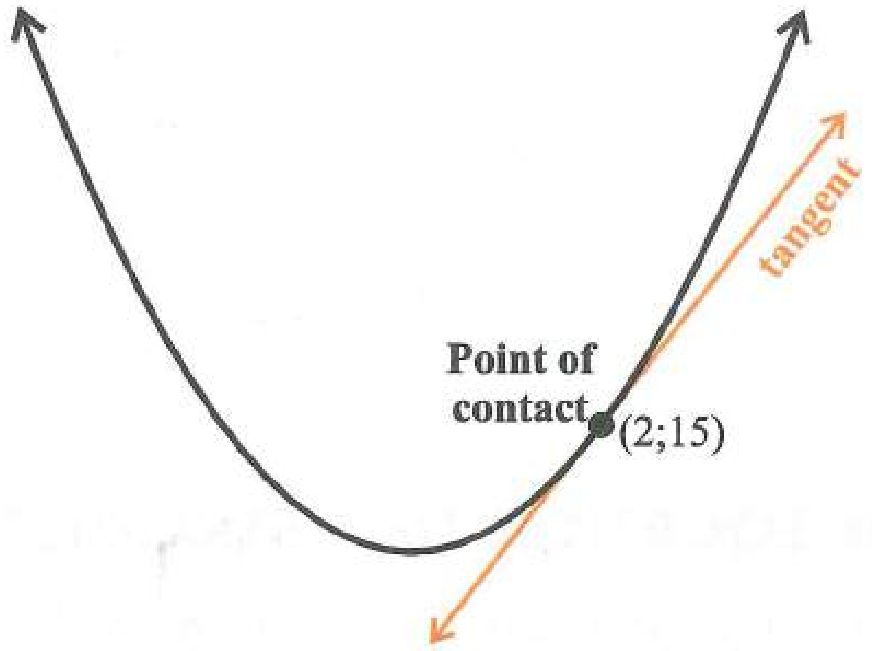
Example 1: **Determine the equation of the tangent to $f(x)=x^2+4x+3$ at the point where $x=2$.**

Step 1: Determine the y-value of the point of contact by substituting $x=2$ into f:

$f(2)=(2)^2+4(2)+3$

$\therefore f(2)=15$

$\therefore$ the point of contact is: $(2;15)$



Step 2: Determine the gradient of f at the point of contact:
The gradient at the point of contact is equal to the gradient of the tangent.

$f(x)=x^2+4x+3$

$\therefore f'(x)=2x+4 \rightarrow$ the derivative is the gradient of f

$\therefore f'(2)=2(2)+4=8$

Step 3: Determine the equation of the tangent. Remember that the gradient of the tangent is equal to the gradient of the graph at the point of contact, which is 8.

Method 1: using: $y=mx+c$	**Method 2: using:** $(y-y_1)=m(x-x_1)$
$y=8x+c$ Substitute the point of contact $(2;15)$: $\therefore 15=8(2)+c$ $\therefore 15=16+c$ $\therefore c=-1$ $\therefore y=8x-1$	$y-y_1=8(x-x_1)$ Substitute the point of contact $(2;15)$: $\therefore y-15=8(x-2)$ $\therefore y-15=8x-16$ $\therefore y=8x-1$

Example 2: **Determine the equation of the tangent to $xy = 8$ at the point where $x = 4$.**

Step 1: Determine the y-value of the point of contact by substituting $x = 4$ into the equation:

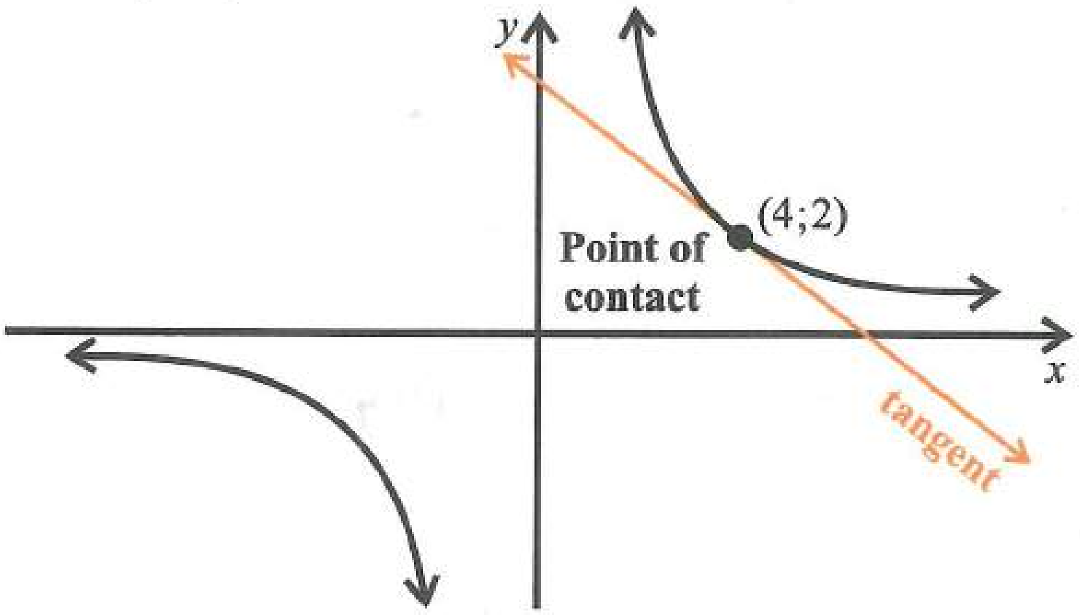
$xy = 8 \rightarrow$ write the equation in terms of $y = \ldots$

$\therefore y = \frac{8}{x} \rightarrow$ this is a hyperbola

Substitute $x = 4$ into the equation

$\therefore y = \frac{8}{4} = 2$

$\therefore$ the point of contact is: $(4;2)$



Step 2: Determine the gradient of the function at the point of contact:
The gradient at the point of contact is equal to the gradient of the tangent.

$y = \frac{8}{x} = 8x^{-1}$

$\therefore \frac{dy}{dx} = -8x^{-2} \rightarrow$ the derivative is the gradient of the function

$\therefore \frac{dy}{dx} = -\frac{8}{x^2}$

$\therefore$ at $x = 4: \frac{dy}{dx} = -\frac{8}{(4)^2} = -\frac{1}{2}$

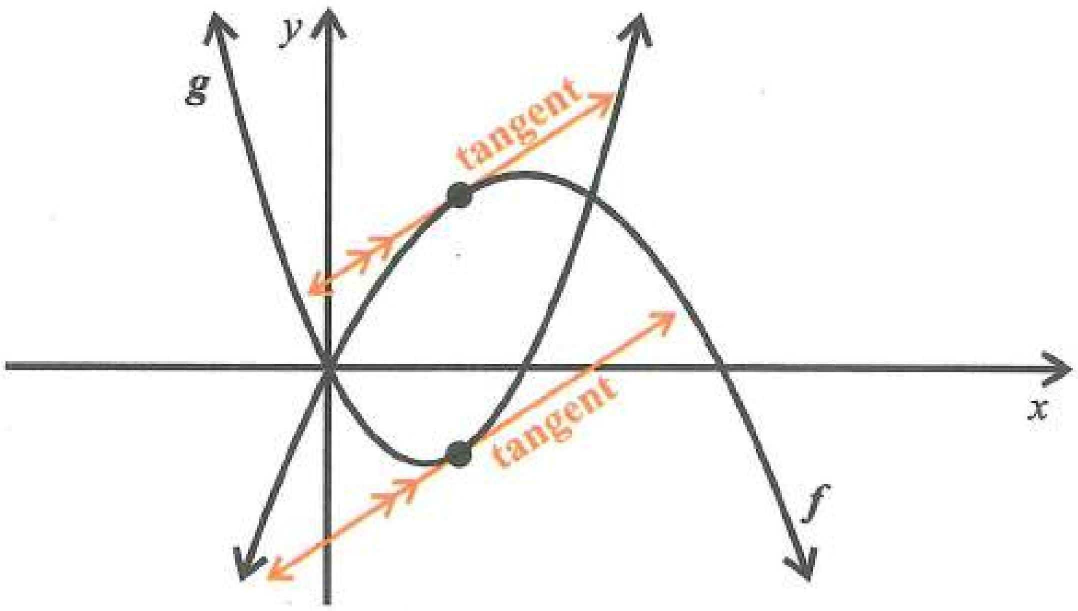
Step 3: Determine the equation of the tangent. The gradient of the tangent is equal to the gradient of the graph at the point of contact, which is $-\frac{1}{2}$.

Method 1: using: $y = mx + c$	**Method 2: using:** $(y - y_1) = m(x - x_1)$
$y = -\frac{1}{2}x + c$ Substitute the point of contact $(4;2)$: $\therefore 2 = -\frac{1}{2}(4) + c$ $\therefore 2 = -2 + c$ $\therefore c = 4$ $\therefore y = -\frac{1}{2}x + 4$	$y - y_1 = -\frac{1}{2}(x - x_1)$ Substitute the point of contact $(4;2)$: $\therefore y - 2 = -\frac{1}{2}(x - 4)$ $\therefore y - 2 = -\frac{1}{2}x + 2$ $\therefore y = -\frac{1}{2}x + 4$

Example 3: **If $f(x)=-x^2+3x$ and $g(x)=2x^2-3x$:**

a) **Determine the value of x for which the tangent to f is parallel to the tangent of g.**

b) **Determine the point on f which has a gradient of –9.**



a) In order for tangents to be parallel their gradients have to be the same:

$f'(x)=-2x+3$ and $g'(x)=4x-3$

$f'(x)=g'(x) \rightarrow$ if tangents are parallel their gradients are equal

$\therefore -2x+3=4x-3$

$\therefore -6x=-6$

$\therefore x=1$

b) In this question you are looking for the point where the gradient is equal to –9. In other words you are looking for the point where the derivative is equal to –9.

$\therefore f'(x)=-9$

$\therefore -2x+3=-9$

$\therefore -2x=-12$

$\therefore x=6 \rightarrow$ this is the x-value where the gradient is equal to -9

You now have to determine the y-value where the gradient is equal to –9:

$f(6)=-(6)^2+3(6) \rightarrow$ substitute into $f(x)$ not $f'(x)$

$=-18$

$\therefore$ The point is $(6;-18)$

When determining a y-value, always substitute into $f(x)$ not $f'(x)$.

Example 4: **Given that $f(x)=x^2-2x$, determine the equation of the tangent to $f(x)$ which is parallel to the line $x+y=7$.**

As the tangent is parallel to the line $x+y=7$, you need to determine the gradient of the line $x+y=7$ by writing it in the form $y=mx+c$:

$\therefore y=-x+7 \rightarrow$ gradient of the line equals -1

$\therefore$ The gradient of the tangent is equal to -1

$\therefore f'(x)=-1 \rightarrow$ parallel lines have the same gradient, $f'(x)$ is the gradient of f

$f'(x)=2x-2$

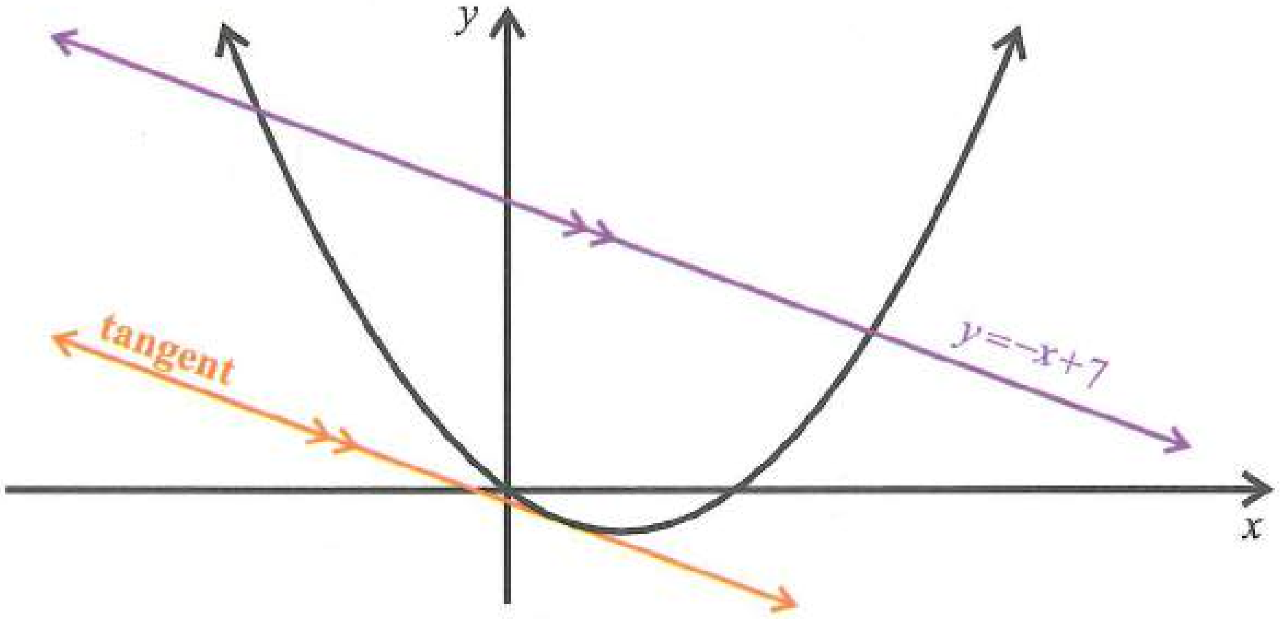
$\therefore 2x-2=-1$

$\therefore x=\frac{1}{2} \rightarrow$ this is the x-value of the point of contact of the tangent

In order to determine the point of contact substitute $x=\frac{1}{2}$ into $f(x)$.

$$f\left(\frac{1}{2}\right)=\left(\frac{1}{2}\right)^2-2\left(\frac{1}{2}\right)$$
$$=-\frac{3}{4}$$

$\therefore$ The point of contact is $\left(\frac{1}{2};-\frac{3}{4}\right)$



You can now determine the equation of the tangent:

Method 1: using: $y=mx+c$	**Method 2: using:** $(y-y_1)=m(x-x_1)$
$y=-x+c$ Substitute the point of contact $\left(\frac{1}{2};-\frac{3}{4}\right)$: $\therefore -\frac{3}{4}=-\left(\frac{1}{2}\right)+c$ $\therefore -\frac{3}{4}=-\frac{1}{2}+c$ $\therefore c=-\frac{1}{4}$ $\therefore y=-x-\frac{1}{4}$	$y-y_1=-(x-x_1)$ Substitute the point of contact $\left(\frac{1}{2};-\frac{3}{4}\right)$: $\therefore y+\frac{3}{4}=-\left(x-\frac{1}{2}\right)$ $\therefore y+\frac{3}{4}=-x+\frac{1}{2}$ $\therefore y=-x-\frac{1}{4}$

Example 5: **Show that $f(x)=x^2$ and $g(x)=-x^2+\frac{1}{2}$ intersect at 90°.**

Determine the points of intersection of the graphs. The graphs intersect where $f(x)=g(x)$.

$\therefore x^2=-x^2+\frac{1}{2}$

$\therefore 2x^2=\frac{1}{2}$

$\therefore x^2=\frac{1}{4}$

$\therefore x=\pm\frac{1}{2}$ → don't forget the '±', these are the x-values of the points of intersection

If the graphs intersect at 90°, it means they are perpendicular to each other at the point of intersection.

Recap: Perpendicular Lines:

If two lines are perpendicular, the product of their gradients is equal to –1

$$m_1\times m_2=-1$$

You now have to determine the gradients of each graph at the points where $x=\pm\frac{1}{2}$:

$f'(x)=2x$ and $g'(x)=-2x$ → these are the gradients of f and g repectively

If $x=\frac{1}{2}$:	**If $x=-\frac{1}{2}$:**
$f'(x)=2x$ and $g'(x)=-2x$ $f'\left(\frac{1}{2}\right)=2\left(\frac{1}{2}\right)$ and $g'\left(\frac{1}{2}\right)=-2\left(\frac{1}{2}\right)$ $=1$ $\quad$ $=-1$ $\therefore f'\left(\frac{1}{2}\right)\times g'\left(\frac{1}{2}\right)=-1$ ∴ The graphs intersect at 90° if $x=\frac{1}{2}$	$f'(x)=2x$ and $g'(x)=-2x$ $f'\left(-\frac{1}{2}\right)=2\left(-\frac{1}{2}\right)$ and $g'\left(-\frac{1}{2}\right)=-2\left(-\frac{1}{2}\right)$ $=-1$ $\quad$ $=1$ $\therefore f'\left(-\frac{1}{2}\right)\times g'\left(-\frac{1}{2}\right)=-1$ ∴ The graphs intersect at 90° if $x=-\frac{1}{2}$

It is important to remember that the derivative represents the gradient of a graph at a point. A tangent to a graph will have the same gradient as the graph at the point of contact.

Complete Exercise 6 on page 162.

F. STATIONARY POINTS AND POINTS OF INFLECTION

INCREASING AND DECREASING FUNCTIONS AND STATIONARY POINTS

The derivative is the gradient at a point (the instantaneous rate of change of y with respect to x) of a function.

If the **gradient of a function is equal to zero,** the tangent will be a straight horizontal line. **At this point the graph will have a stationary point (turning point)**.
This is true for all functions which have turning points.

At a turning point, the slope of the graph will change. The graph will either be increasing or decreasing as you approach the turning point. You can see whether the graph will be increasing or decreasing from the sign of the gradient.

If the **derivative is negative**, the gradient will be negative and hence the **graph will be decreasing**. If the **derivative is positive**, the gradient will be positive and hence the **graph will be increasing.**

Illustrative Example: Consider the function $f(x) = x^2 - 2x + 3$::

$f'(x) = 2x - 2$

In order to determine the turning point (TP) let $f'(x) = 0$:

$\therefore 2x - 2 = 0$

$\therefore x = 1 \rightarrow$ this is the x-value of the stationary point (TP)

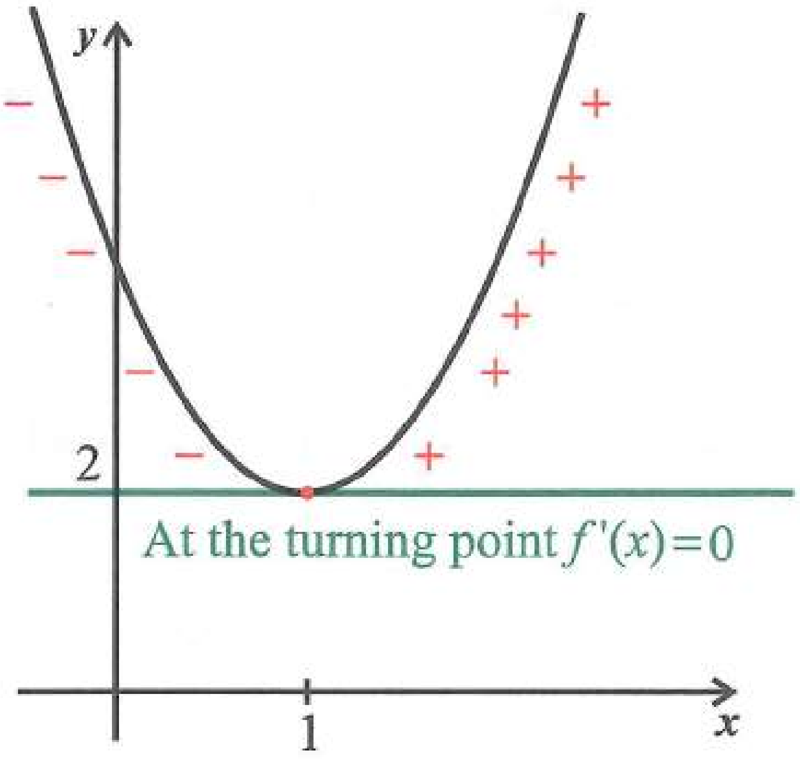
You can now substitute $x = 1$ into f in order to determine the y-value of the stationary point:

$f(1) = (1)^2 - 2(1) + 3 = 2$

$\therefore$ The stationary point (TP) is $(1;2)$.

The table below shows the value of the derivative for different values of x on either side of the stationary point.

x	$f'(x)$
-1	$f'(-1) = -4$
0	$f'(0) = -2$
1	$f'(1) = 0$
2	$f'(2) = 2$
$2{,}5$	$f'(2{,}5) = 3$



From the table it can be seen that:

$f(x)$ is strictly decreasing when: $x < 1$ as $f'(x) < 0$ when $x < 1$

$f(x)$ is strictly increasing when: $x > 1$ as $f'(x) > 0$ when $x > 1$

Example 1: **Given that $f(x) = -x^2 + 4x - 3$:**

- a) **Determine the stationary point of *f*.**
- b) **Is *f* increasing or decreasing at the point where $x = -4$.**
- c) **For which values of *x* is *f* increasing?**

a) $f'(x) = -2x + 4$

For a stationary point $f'(x) = 0$:

$\therefore -2x + 4 = 0$

$\therefore x = 2 \rightarrow$ this is the *x*-value of the stationary point

In order to determine the *y*-value of stationary point, substitute $x = 2$ into $f(x)$:

$$f(2) = -(2)^2 + 4(2) - 3$$
$$= 1$$

$\therefore$ The stationary point is $(2;1)$

b) $f'(-4) = -2(-4) + 4$

$$= 12$$

$\therefore f'(-4) > 0$

$\therefore$ The function is increasing at $x = -4$.

c) For the function to be increasing $f'(x) > 0$:

$\therefore -2x + 4 > 0$

$\therefore -2x > -4 \rightarrow$ change the direction of the inequality when dividing by a negative

$\therefore x < 2 \rightarrow$ this can also be written as $x \in (-\infty;2)$

Example 2: **Determine the stationary points of $f(x) = x^3 - 6x^2 + 9x + 10$.**

$f'(x) = 3x^2 - 12x + 9$

For a stationary point $f'(x) = 0$:

$\therefore 3x^2 - 12x + 9 = 0$

$\therefore x^2 - 4x + 3 = 0$

$\therefore (x-3)(x-1) = 0$

$\therefore x = 3$ or $x = 1 \rightarrow$ these are the *x*-values of the stationary points

To determine the *y*-values of the stationary points you have to substitute the *x*-values into *f*:

$f(1) = (1)^3 - 6(1)^2 + 9(1) + 10$ and $f(3) = (3)^3 - 6(3)^2 + 9(3) + 10$

$= 14$ $\qquad\qquad\qquad\qquad$ $= 10$

$\therefore$ The stationary points are: $(1;14)$ and $(3;10)$

CONCAVITY

A function is said to be concave up on the interval $(a\,;b)$ if the function lies above tangents drawn to the curve at any point between a and b.

A function is said to be concave down on the interval $(a\,;b)$ if the function lies below tangents drawn to the curve at any point between a and b.

Figure 1 below shows a function which is concave up, whilst figure 2 shows a function which is concave down.

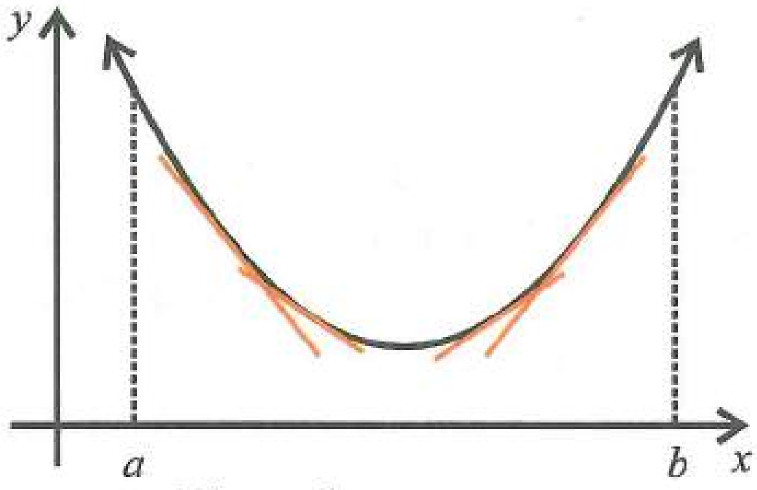


Figure 1: concave up

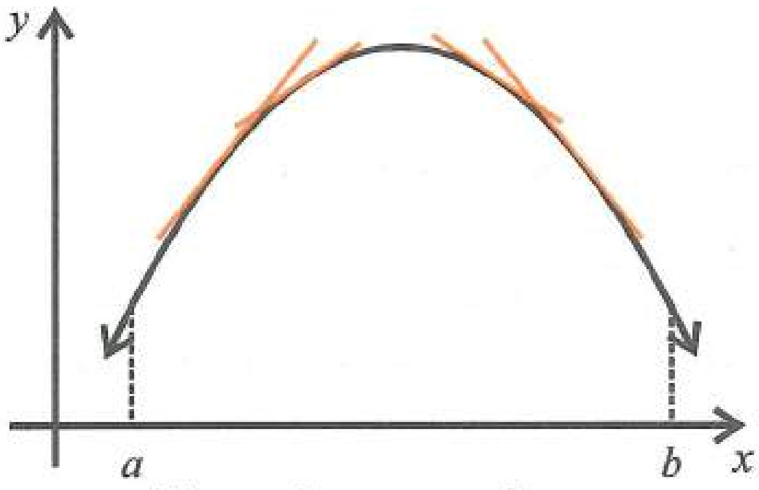


Figure 2: concave down

POINTS OF INFLECTION

A point of inflection is a point on a curve where the concavity changes, either from concave up to concave down or vice versa.
The second derivative can be used to determine the concavity of a function.
The sign of the second derivative shows whether the graph is concave up or concave down. If the **second derivative is greater** than **zero** the curve **is concave up**, if it is **less** than **zero** it is **concave down.**

The first derivative provides an indication of where a stationary point (point of maximum or minimum) lies. However, it is possible for this point to be a point of inflection. In order to determine whether this point is a maximum, minimum or point of inflection, the second derivative can be used.
The second derivative test works as follows:

If $f'(c) = 0$ and:

$f''(c) < 0 \rightarrow$ **then there is a relative maximum at $x = c$**

$f''(c) > 0 \rightarrow$ **then there is a relative minimum at $x = c$**

$f''(c) = 0 \rightarrow$ **then it is possible that there is a point of inflection at $x = c$**

If the concavity changes on either side of c then c is a point of inflection.

A point of inflection which is not a stationary point will always occur between the turning points of a cubic function.

Figure 3 and figure 4 below show graphs which have points of inflection at P, where the graph changes from concave down to concave up and vice versa.

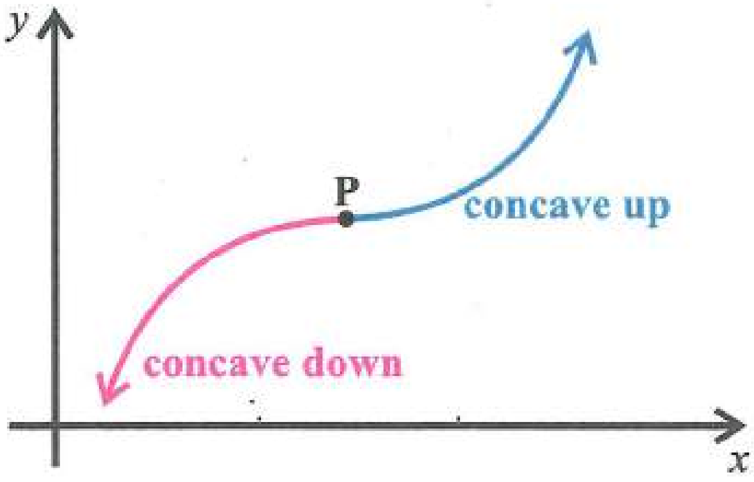


Figure 3

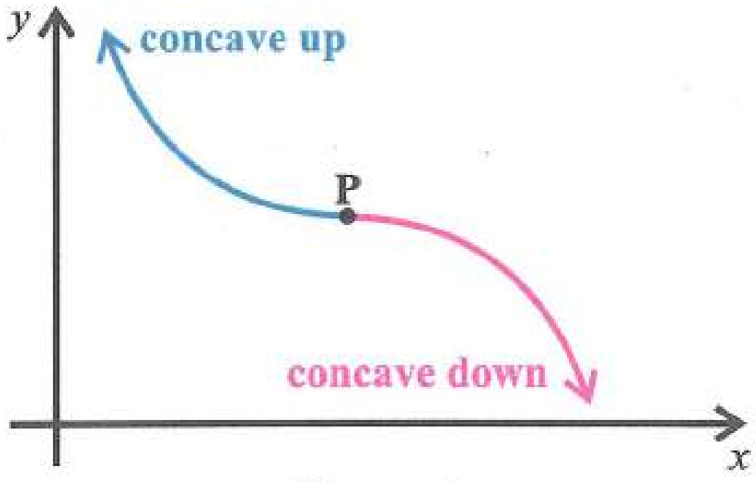


Figure 4

Example: **Given that $f(x) = x^3 + 18x^2 - 4x + 4$, show that there is a point of inflection when $x = -6$.**

$f'(x) = 3x^2 + 36x - 4$

$f''(x) = 6x + 36$

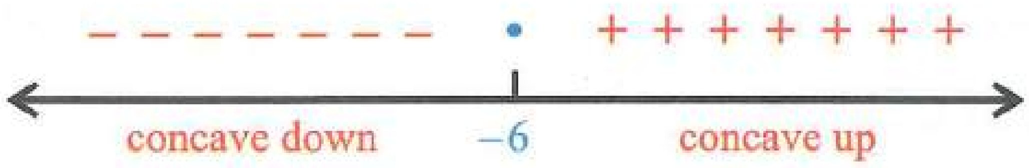
A point of inflection can occur when $f''(x) = 0$:

$\therefore 6x + 36 = 0$

$\therefore 6x = -36$

$\therefore x = -6$

In order to show that there is a point of inflection where $x = -6$, you have to show that the concavity of the graph changes as x increases through -6. In other words, you have to show that the sign of the second derivative changes as x increases through -6. Draw a number line and substitute values on either side of $x = -6$ into the second derivative.



As x increases through -6 the concavity of the function changes.

$\therefore$ A point of inflection occurs at $x = -6$.

In Grade 12, you may assume that if the second derivative is 0 where $x = c$, then there is a point of inflection at c.

Complete Exercise 7 on page 162 and the Mixed Exercise on pages 162 and 163.

The Remainder and Factor Theorem and Factorisation of Cubic Polynomials

Exercise 1: (answers on page 353)

1) Answer the following questions. If:

a) $x-1$ is a factor of $h(x)=x^3+(q-4)x^2+(3-4q)x+3$, determine the value of q.

b) $g(x)=2x^3+mx^2-4x+n$ is divisible by $f(x)=x^2+2x-3$, determine the value of m and n.

2) Factorise the following cubic polynomials:

a) $f(x)=x^3+4x^2+x-6$

b) $f(x)=x^3-2x^2-x+2$

c) $f(x)=x^3+3x^2-4x-12$

d) $f(x)=x^3-3x^2-9x+27$

e) $f(x)=x^3-12x+16$

f) $f(x)=x^3+2x^2-3x$

Limits

Exercise 2: (answers on page 353)

1) Evaluate the following limits:

a) $\lim\limits_{h\to 2}(2h+4)$

b) $\lim\limits_{x\to 0}\dfrac{3x^2+6x}{2x}$

c) $\lim\limits_{x\to 2}\dfrac{x^2-4}{x-2}$

d) $\lim\limits_{h\to -1}\dfrac{1+2h+h^2}{h+1}$

e) $\lim\limits_{x\to 4}\dfrac{x^2-3x-4}{x-4}$

f) $\lim\limits_{x\to \frac{3}{2}}\dfrac{2x^2-x-3}{2x-3}$

2) If $f(x)=2x$ determine $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$.

3) If $f(x)=3x^2$ determine $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$.

Average Gradient, Gradient at a Point and Differentiation from First Principles

Exercise 3: (answers on page 353)

1) Determine the average gradient of $f(x)=2x^2$ between the points where $x=-1$ and $x=5$ using:

a) The formula $m=\dfrac{f(x+h)-f(x)}{h}$.

b) The formula $m=\dfrac{y_2-y_1}{x_2-x_1}$.

2) It is given that $\dfrac{f(x+h)-f(x)}{h}=-2x-h+1$:

a) Determine the average gradient of f between the points $x=3$ and $x=5$.

b) Hence, determine the gradient of f at the point where $x=5$.

Exercise 4: (answers on page 353)

1) **From first principles, determine the derivatives of the following functions at any point:**

   a) $f(x)=x$

   b) $f(x)=2x^2+2$

   c) $g(x)=x^3$

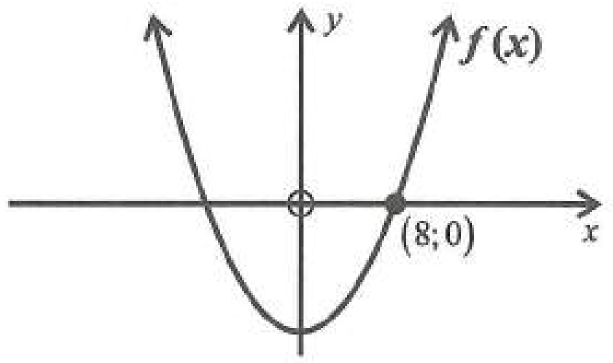
   d) $f(x)=\sqrt{x}$

2) **Given that $f(x)=3+x-x^2$:**

   a) Determine $f'(x)$ from first principles.

   b) Hence, determine $f'(5)$ and explain what this answer represents.

3) **Use the graph below to answer the following questions:**



   a) Is $f'(8)>f(8)$? Explain your answer.

   b) Is $f'(8)$ greater than the slope of the graph at the point where $x=12$?

Differentiation Rules

Exercise 5: (answers on page 354)

1) **Determine the derivative of the following, writing all answers with positive exponents:**

   a) $y=6x$

   b) $f(x)=10$

   c) $g(x)=-8x^2$

   d) $g(x)=\sqrt{x}$

   e) $y=x^4+3x^2+5x+2$

   f) $f(x)=\dfrac{3}{x}+2$

   g) $y=\dfrac{x^2+6x+8}{x+2}$

   h) $y=(2x-3)(x^2-4)$

   i) $f(x)=\dfrac{2x^5+5x^2-8x}{2x}$

   j) $f(y)=\dfrac{27y^3-8}{2-3y}$

2) **Writing all answers with positive exponents, determine:**

   a) $D_x\left[\left(1+\sqrt{x}\right)^2\right]$

   b) $\dfrac{d}{dx}\left[\dfrac{1}{4}x^4+2x^{\frac{1}{2}}\right]$

3) **Given the functions $f(x)=2x^3+3x^2+x+5$ determine:**

   a) The derivative of *f*.

   b) The gradient of f at the point where $x=3$.

Tangents to Graphs of Functions

Exercise 6: (answers on page 354)

1) **Determine:**
   - a) The equation of the tangent to $f(x)=-x^2+x+2$ at point $\mathrm{P}(1;2)$.
   - b) The value of *x*, if the gradient of the tangent to $f(x)=-x^2+2x-1$ is 10.
   - c) The equation of the tangent to $f(x)=x^2+5x+3$ if the tangent touches the curve when $x=3$.

2) **Determine the point of contact of the tangent to:**
   - a) $y=x^2-x$ if the tangent is parallel to the line $y=3x+1$.
   - b) $y=3x^2-5x+6$ if the tangent is perpendicular to the line $y=\frac{1}{2}x+\frac{3}{4}$.

3) **Determine the equation(s) of the tangent(s) to:**
   - a) $f(x)=-x^2+7x-2$ if the tangent is parallel to the line $y+3x-8=0$.
   - b) $f(x)=x^3$ if the tangent is perpendicular to the line $y+\frac{1}{3}x-12=0$.

4) **The gradient of the tangent to $f(x)=ax^3+cx^2$ at the point $(1;4)$ is 12:**
   - a) Determine the value of *a* and *c*.
   - b) Determine the equation of the line perpendicular to *f* if the line and *f* pass through the point $(2;1)$. This line is known as the normal.

Stationary Points and Points of Inflection

Exercise 7: (answers on page 354)

1) **Given that $f(x)=x^2-4x+6$:**
   - a) Determine the stationary point of *f*.
   - b) Determine the interval for which *f* will be increasing.

2) **Consider the function $f(x)=2x^3-3x^2-12x+5$:**
   - a) Determine the stationary point(s) of *f*.
   - b) Determine the point of inflection *f*.
   - c) For what values of *x* is $f(x)$ increasing?

Mixed Exercise: (answers on page 355)

1) **Evaluate the following:**
   - a) $\lim\limits_{x\to-1}\frac{x^2-1}{x+1}$
   - b) $\lim\limits_{x\to0}\frac{x^3+5x^2+6x}{x^2+2x}$

2) **Writing all answers with positive exponents, determine the derivative of the following:**
   - a) $f(x)=6\sqrt[3]{x^2}$
   - b) $y=4\sqrt{x}+\frac{2}{x}$
   - c) $y=\frac{1}{\sqrt[4]{x^{-3}}}+\frac{6x^2+2x}{2x}$
   - d) $h(x)=\frac{3}{2x^2}-\frac{5x+2}{x}+\frac{x^2-x-2}{x-2}$

3) **Given that $f(x) = 3x^2 + qx + 4$:**

a) Determine the value of q for which the equation of the tangent to f at the point $(2; y)$ is $y = 8x - 8$.

b) Determine the equation of $g(x)$, if $g(x)$ is perpendicular to the tangent and also passes through the point $(2; y)$ on f.

4) **If $g(x) = 6 - x^2 + 8x$, determine:**

a) The stationary point of g.

b) Whether g is increasing or decreasing at $(1;13)$

5) **Determine the derivative from first principles of $f(x)$ if:**

a) $f(x) = 1 + x$

b) $f(x) = \frac{2}{x}$

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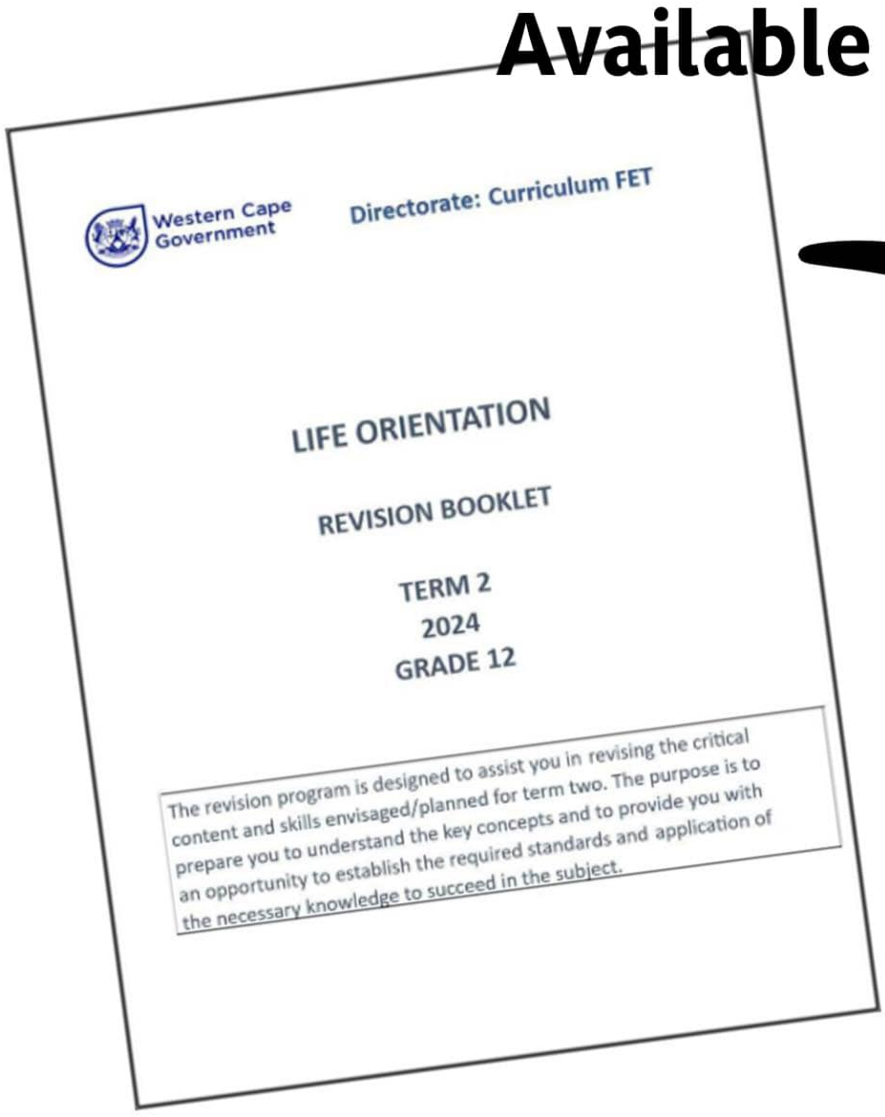

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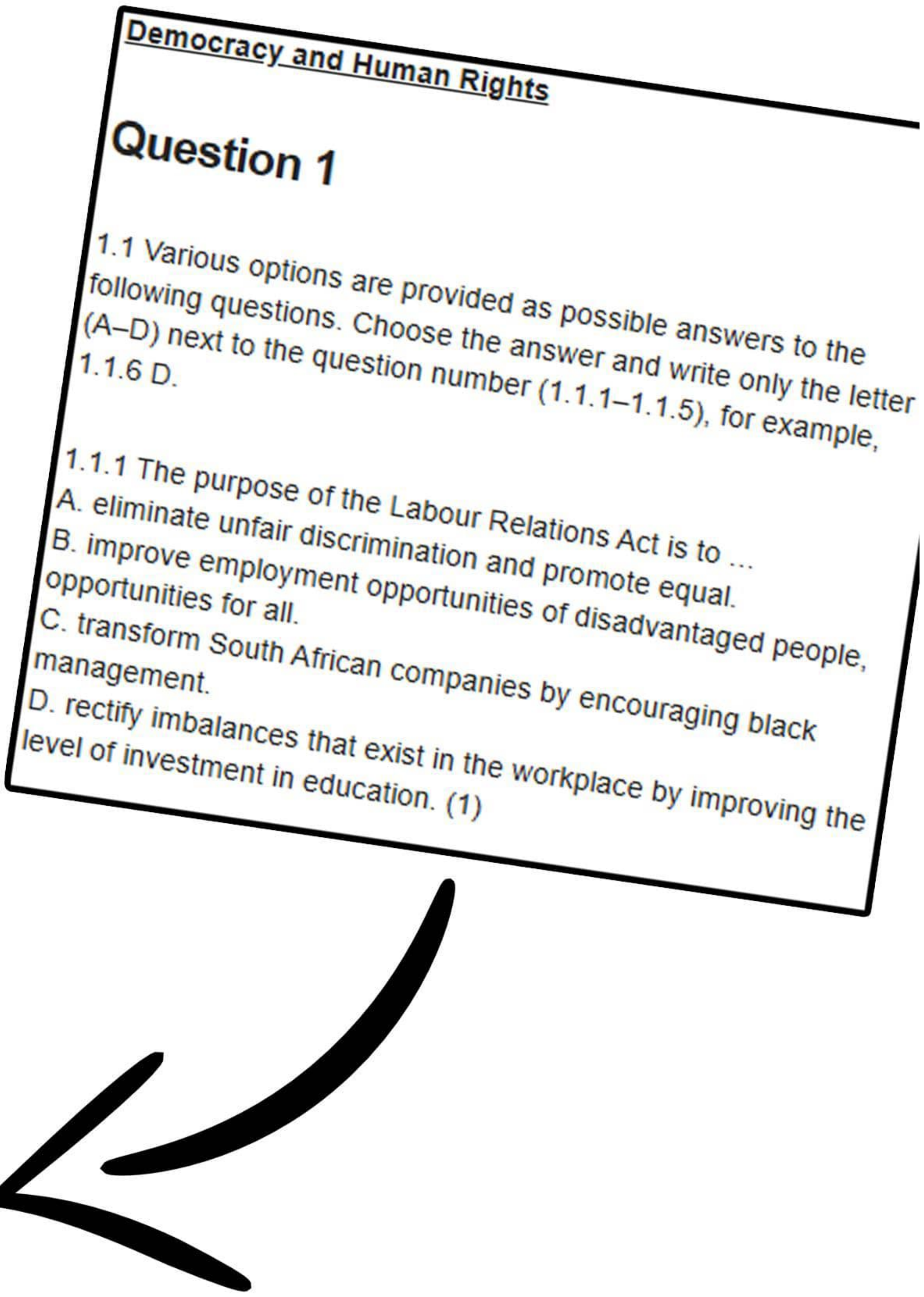
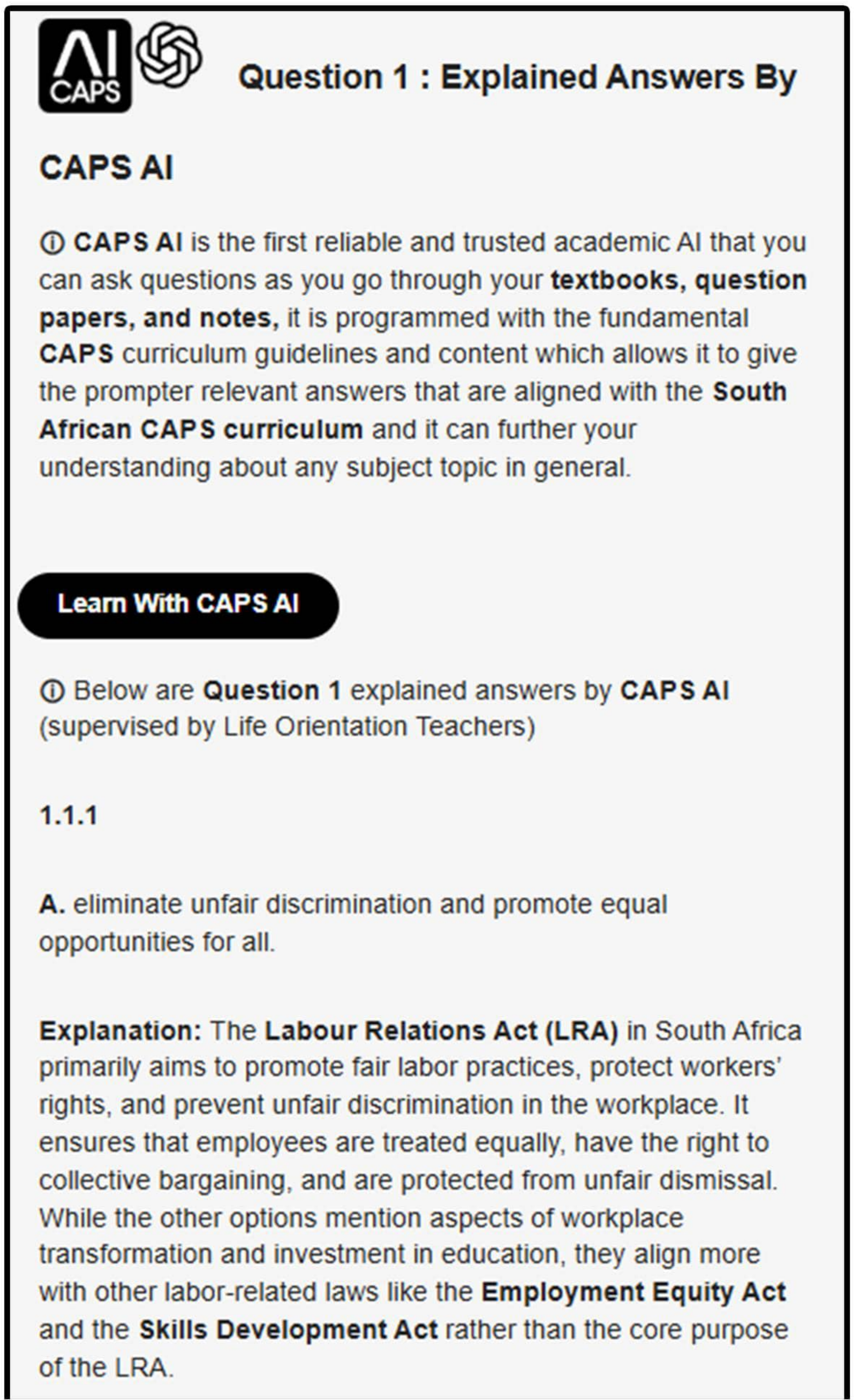


Democracy and Human Rights

Question 1

1.1 Various options are provided as possible answers to the following questions. Choose the answer and write only the letter (A–D) next to the question number (1.1.1–1.1.5), for example, 1.1.6 D.

1.1.1 The purpose of the Labour Relations Act is to …
A. eliminate unfair discrimination and promote equal opportunities for all.
B. improve employment opportunities of disadvantaged people,
C. transform South African companies by encouraging black management.
D. rectify imbalances that exist in the workplace by improving the level of investment in education. (1)



Question 1 : Explained Answers By CAPS AI

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ⓘ Below are **Question 1** explained answers by **CAPS AI** (supervised by Life Orientation Teachers)

1.1.1

A. eliminate unfair discrimination and promote equal opportunities for all.

Explanation: The **Labour Relations Act (LRA)** in South Africa primarily aims to promote fair labor practices, protect workers' rights, and prevent unfair discrimination in the workplace. It ensures that employees are treated equally, have the right to collective bargaining, and are protected from unfair dismissal. While the other options mention aspects of workplace transformation and investment in education, they align more with other labor-related laws like the **Employment Equity Act** and the **Skills Development Act** rather than the core purpose of the LRA.

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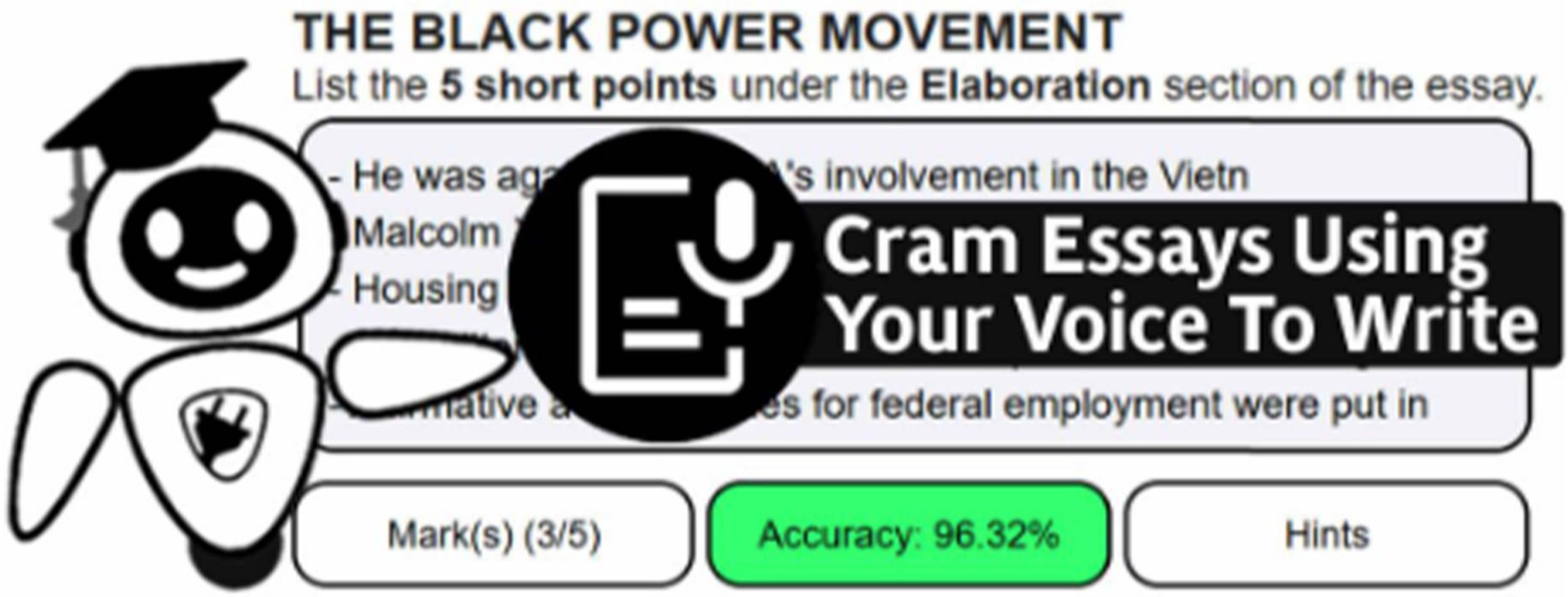

THE BLACK POWER MOVEMENT

List the **5 short points** under the **Elaboration** section of the essay.

- He was ag 's involvement in the Vietn
- Malcolm
- Housing
- mative a es for federal employment were put in

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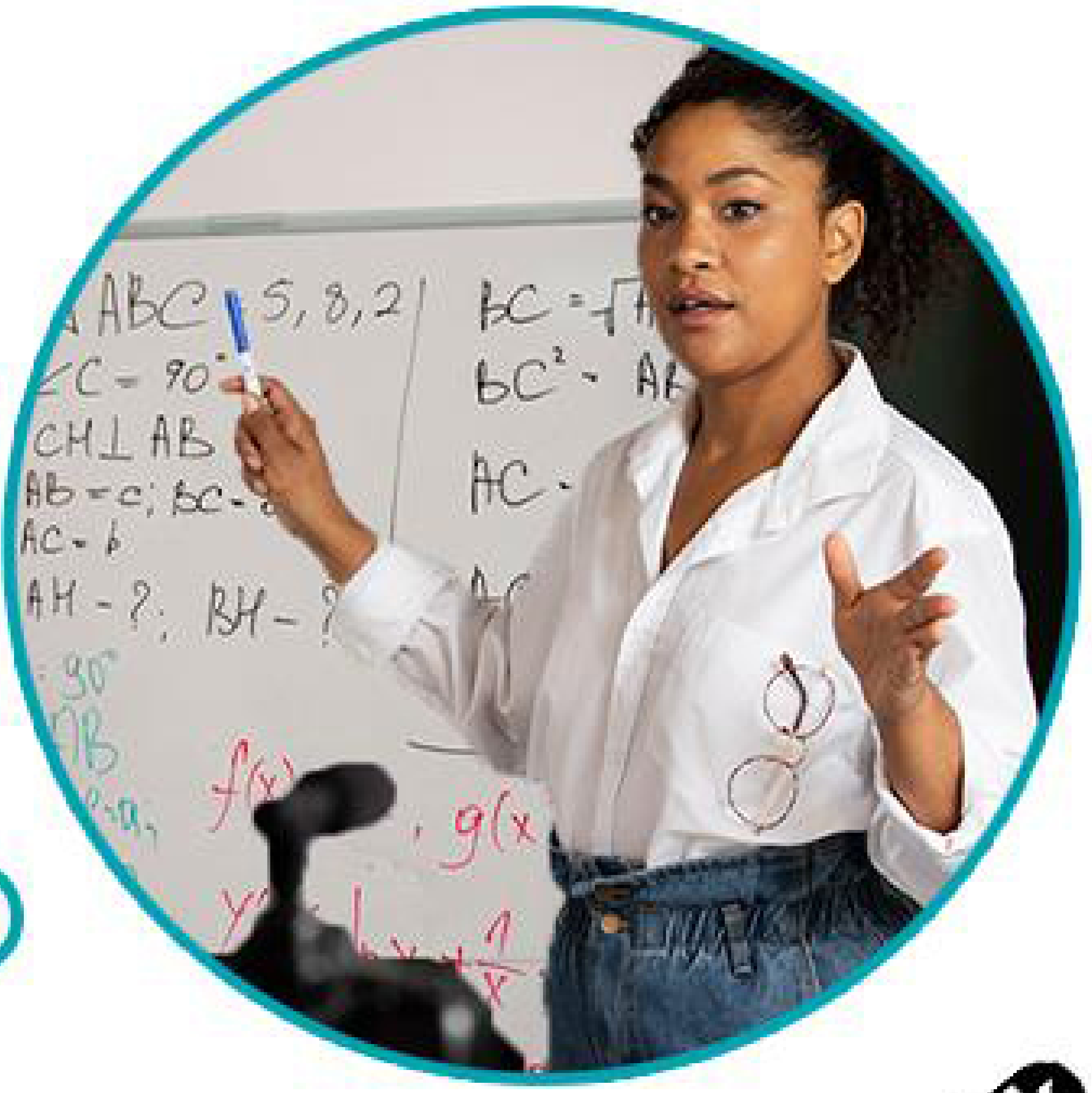

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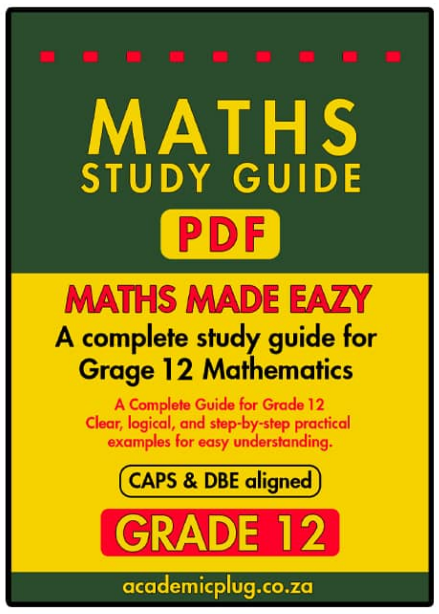
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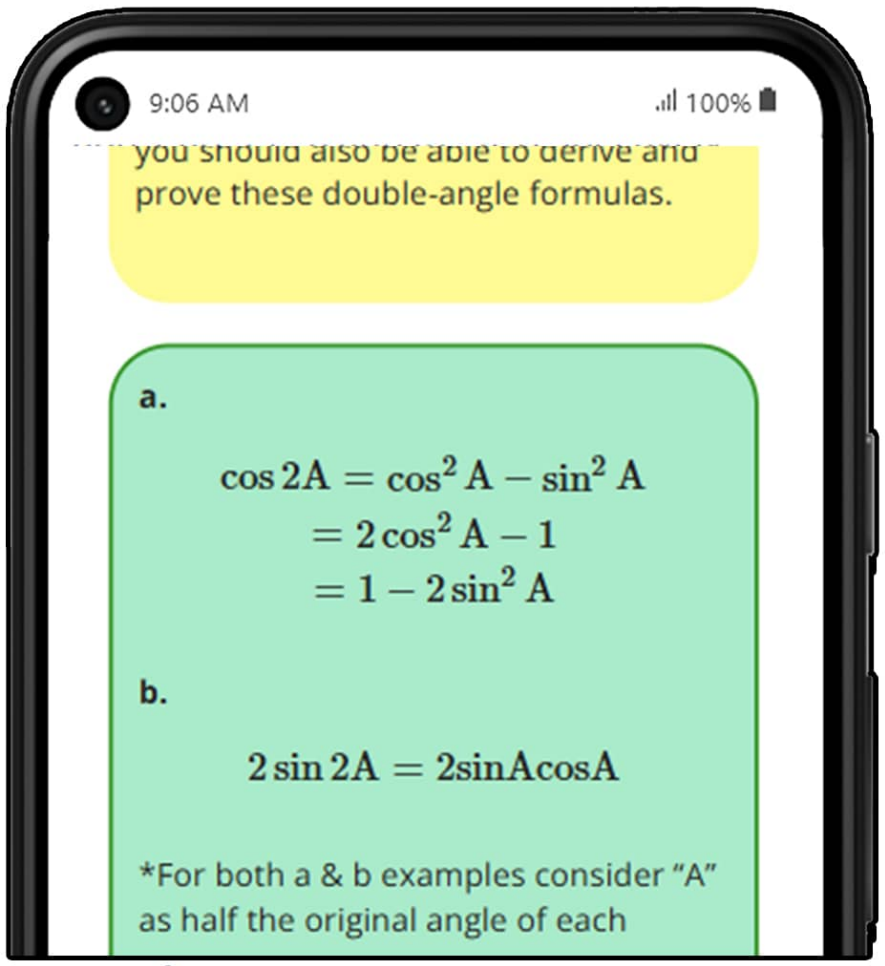

you should also be able to derive and prove these double-angle formulas.

a.

$$\cos 2A = \cos^2 A - \sin^2 A$$
$$= 2\cos^2 A - 1$$
$$= 1 - 2\sin^2 A$$

b.

$$2\sin 2A = 2\sin A\cos A$$

*For both a & b examples consider "A" as half the original angle of each

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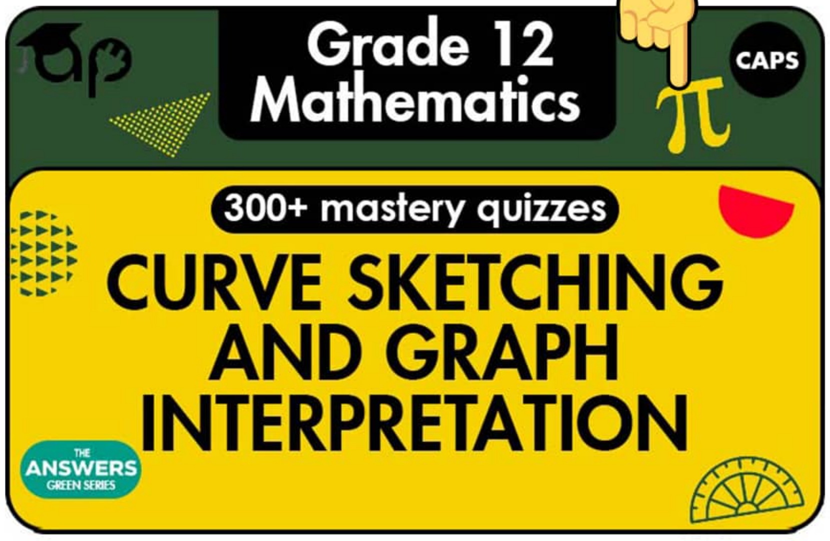
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CHAPTER 8
CURVE SKETCHING AND GRAPH INTERPRETATION

WHAT YOU WILL LEARN IN THIS CHAPTER

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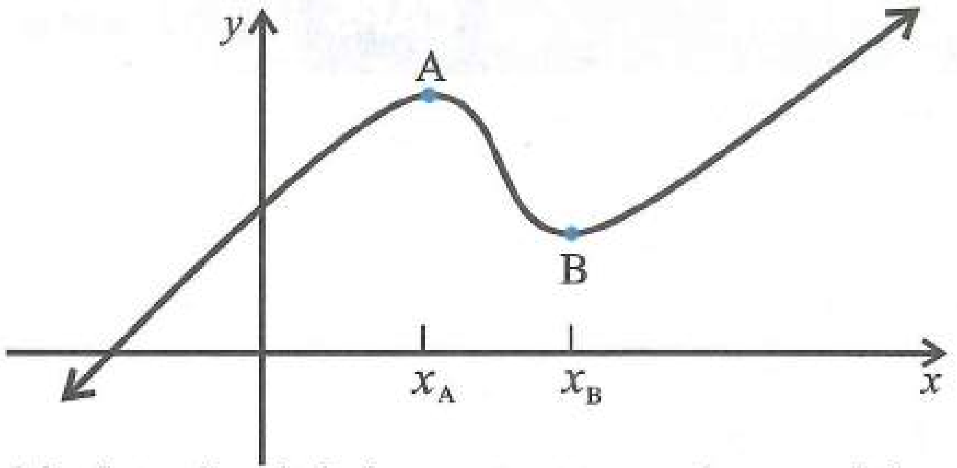
A. SKETCHING GRAPHS OF CUBIC FUNCTIONS

THE CUBIC FUNCTION

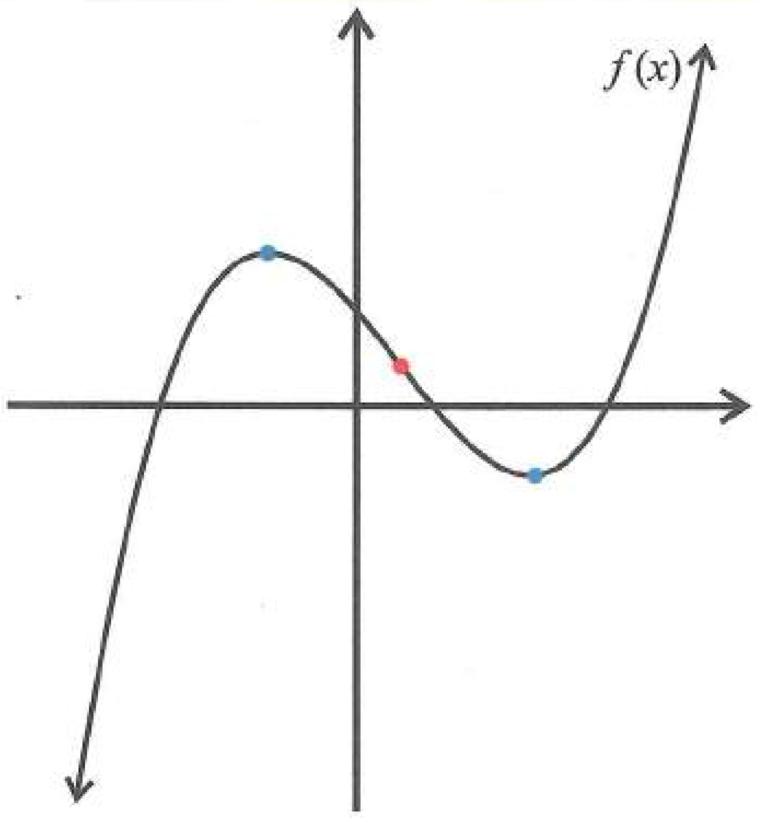
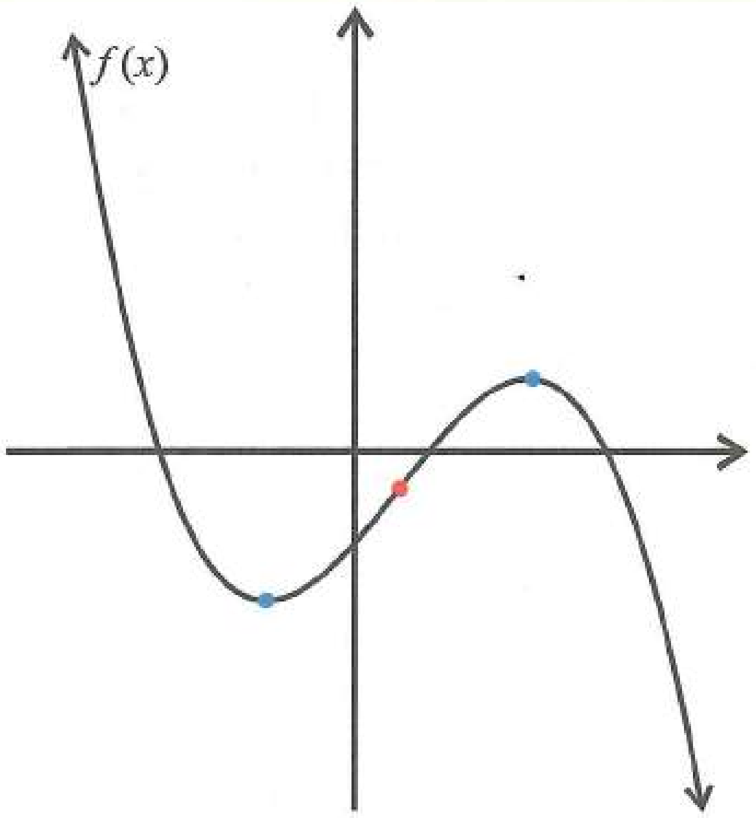
A cubic function has the form: $f(x) = ax^3 + bx^2 + cx + d$

In the graph below:
A and **B** are **stationary points** and have horizontal tangents.
A is called the local or relative **maximum turning point** (TP).
B is called the local or relative **minimum turning point**.

In the above equation $f(0) = d$. Therefore, in this form, d is the y-intercept.



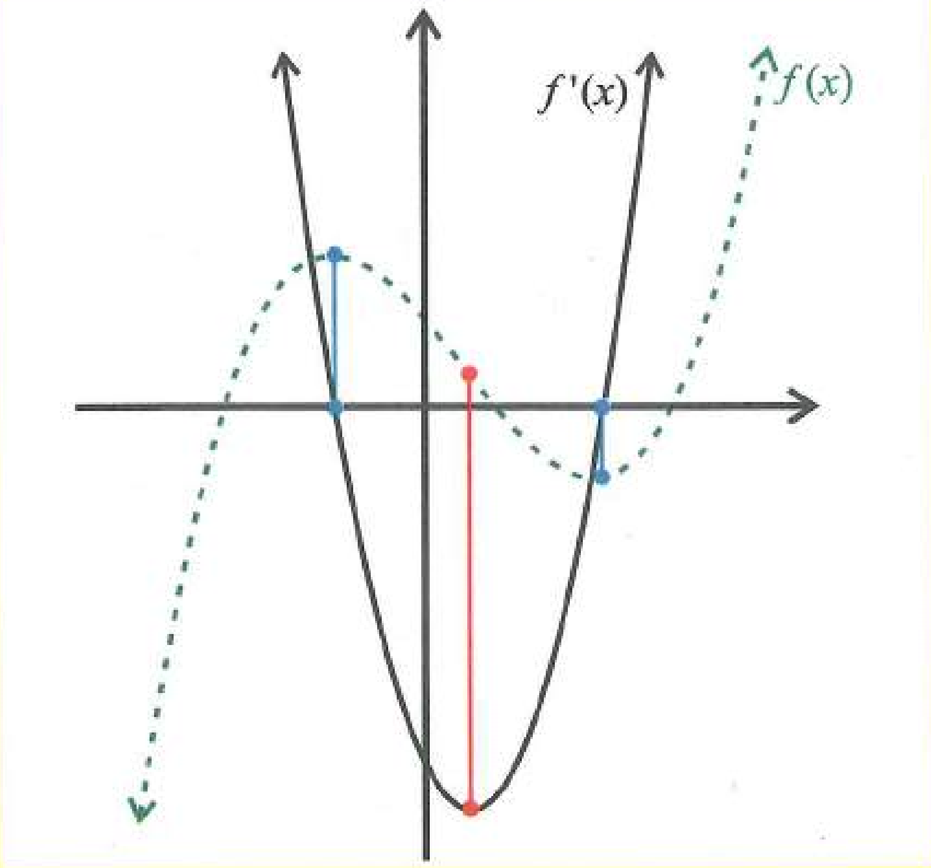
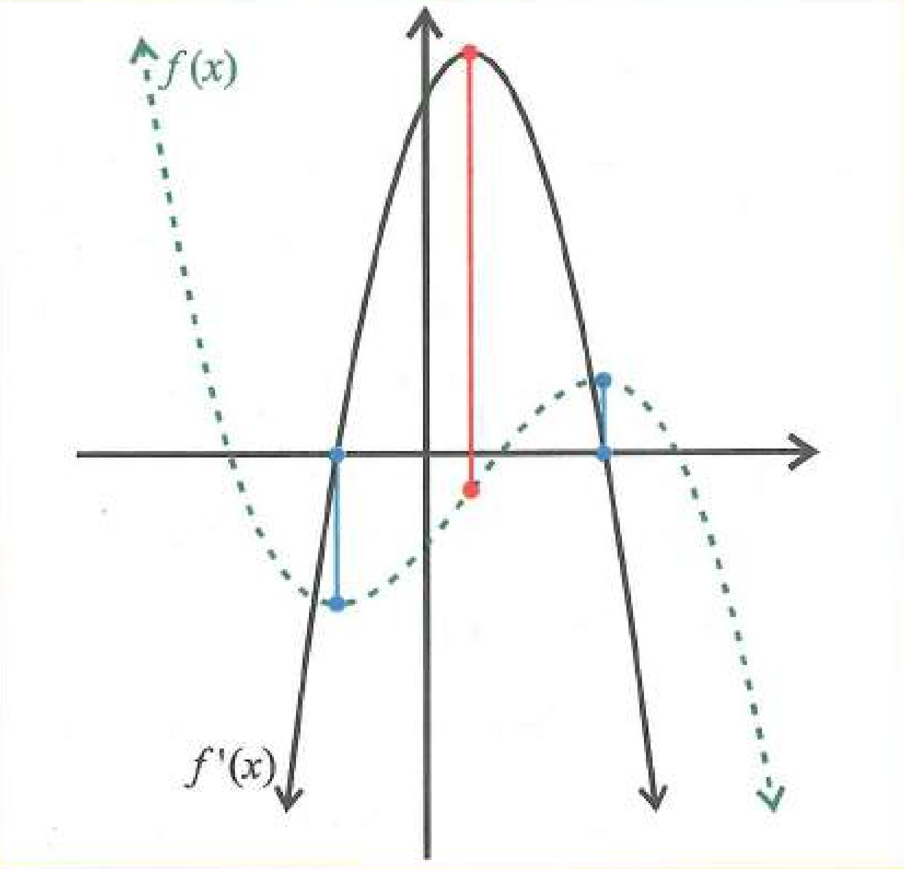
Before sketching the cubic function it is important to understand the shape and concavity of the cubic function. The diagrams below show the general shape of the cubic function for $a > 0$ and $a < 0$.

$f(x) = ax^3 + bx^2 + cx + d$ $a > 0$	$f(x) = ax^3 + bx^2 + cx + d$ $a < 0$
	

The derivative of a cubic function has the form $f'(x) = 3ax^2 + 2bx + c$. It can be seen that this represents a parabola. It is important to note that the y-values of the parabola are the values of the derivative for different x-values. This means that where the parabola is positive, $f(x)$ is increasing and where it is negative $f(x)$ is decreasing. The x-intercepts of $f'(x)$ (the parabola) represent the values where the derivative is equal to 0. Therefore, the x-intercepts represent the x-values of the turning points of the cubic function.

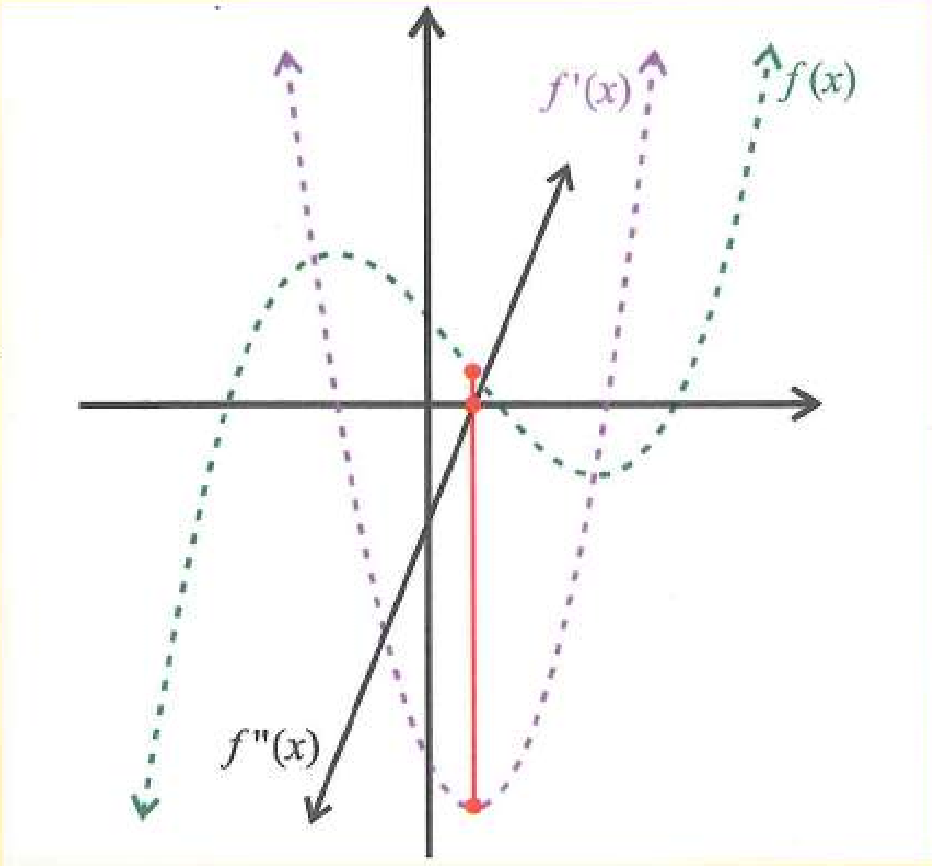
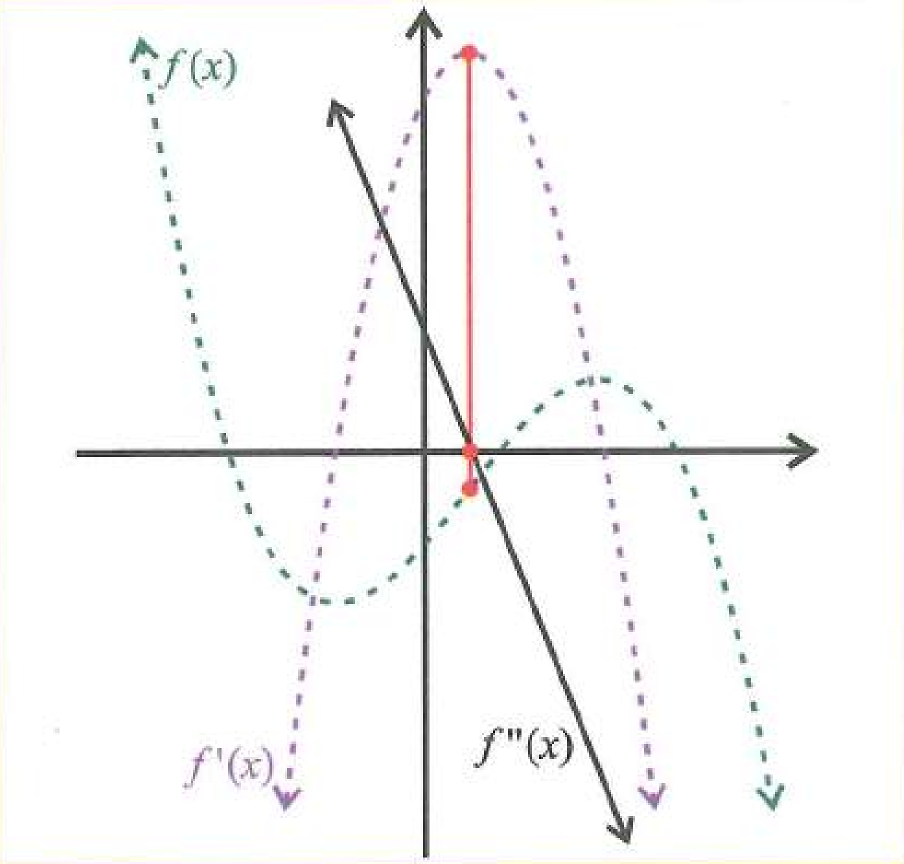
In the cubic functions above it can be seen that the graphs change from concave down to concave up if $a > 0$ and from concave up to concave down if $a < 0$. It should also be noted that the turning point of the parabola always lies exactly halfway between the x-intercepts. The x-value of the turning point of the parabola represents the x-value of the point of inflection of $f(x)$, as this where the second derivative is equal to 0 and the cubic function changes concavity.

The graphs below show the general shape of $f'(x)$ for $a > 0$ and for $a < 0$.

$f'(x) = 3ax^2 + 2bx + c$ $a > 0$	$f'(x) = 3ax^2 + 2bx + c$ $a < 0$
	

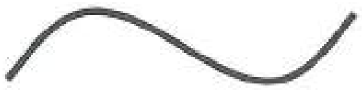
Similarly, it can be seen that the 2nd derivative of a cubic function is the derivative of a quadratic function and represents a linear function (straight line). The x-intercept of the 2nd derivative represents the x-value of the point of inflection of the original cubic function and the x-value of the turning point of $f'(x)$.

The graphs below show the general shape of $f''(x)$. for $a > 0$ and for $a < 0$.

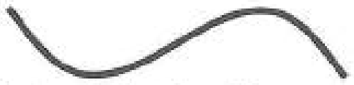
$f''(x) = 6ax + 2b$ $a > 0$	$f''(x) = 6ax + 2b$ $a < 0$
	

STEPS TO SKETCHING A CUBIC FUNCTION

1. **Determine the x-value/s of the stationary point/s by letting $f'(x)=0$.**
2. **Determine the y-value/s of the stationary point/s by substituting the x-values of the stationary point/s into $f(x)$, not $f'(x)$.**
3. **Determine the shape of the graph:**

   If $a > 0$ the shape of the graph is → 

   Note that in this case the stationary point with the lower x-value will be a local maximum and the stationary point with the larger x-value will be a local minimum.

   If $a < 0$ the shape of the graph is → 

   Note that in this case the stationary point with the lower x-value will be a local minimum and the stationary point with the larger x-value will be a local maximum.
4. **Determine the x-and y-intercepts by letting $y=0$ and $x=0$ respectively.**
5. **If asked, determine the x-value of the point of inflection by letting $f''(x)=0$.**
6. **Determine the y-value of the point of inflection by substituting the x-value of the point of inflection into $f(x)$, not $f'(x)$ or $f''(x)$.**

If the x-or y-value of a stationary point is equal to zero, then that point is also an intercept.

Example 1: **Sketch the graph of $f(x)=x^3-4x^2+4x$ showing all stationary points, intercepts with the axes and the point of inflection.**

Step 1: Determine the *x*-values of the stationary points

$f'(x)=3x^2-8x+4$

For stationary points: $f'(x)=0$

$\therefore 3x^2-8x+4=0$

$\therefore (3x-2)(x-2)=0$

$\therefore x=\frac{2}{3}$ or $x=2$

Step 2: Substitute the *x*-values into the equation to solve for the *y*-values

$$f\left(\frac{2}{3}\right)=\left(\frac{2}{3}\right)^3-4\left(\frac{2}{3}\right)^2+4\left(\frac{2}{3}\right)$$
$$=\frac{32}{27}$$

or

$$f(2)=(2)^3-4(2)^2+4(2)$$
$$=0$$

$\therefore$ The stationary points are $\left(\frac{2}{3};\frac{32}{27}\right)$ and $(2;0)$ → this is also an *x*-intercept

Step 3: Test for shape

$a>0$: → local max at $\left(\frac{2}{3};\frac{32}{27}\right)$ and local min at $(2;0)$

Step 4: Determine the intercepts

x-int: let $y=0$

$\therefore 0=x^3-4x^2+4x$

$\therefore 0=x\left(x^2-4x+4\right)$

$\therefore 0=x(x-2)^2$

$\therefore x=0$ or $x=2$

$\therefore$ *x*-int: $(0;0)$ and $(2;0)$

y-int: let $x=0$

$\therefore f(0)=(0)^3-4(0)^2+4(0)$

$\therefore f(0)=0$

$\therefore$ *y*-int: $(0;0)$

Step 5: Determine the *x*-value of the point of inflection

$f''(x)=6x-8$

For the point of inflection: $f''(x)=0$

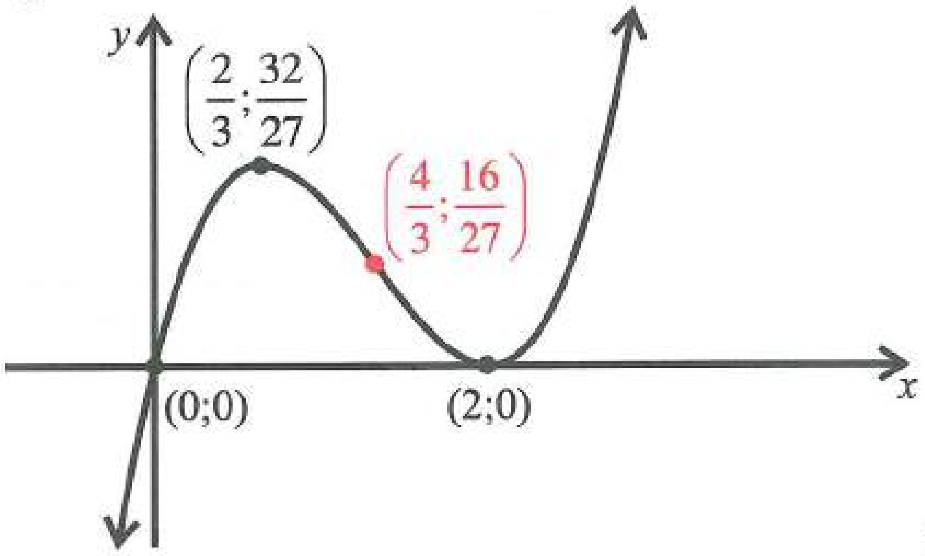
$\therefore 6x-8=0$

$\therefore x=\frac{4}{3}$

Step 6: Substitute the *x*-value into the equation to solve for the *y*-value

$$f\left(\frac{4}{3}\right)=\left(\frac{4}{3}\right)^3-4\left(\frac{4}{3}\right)^2+4\left(\frac{4}{3}\right)$$
$$=\frac{16}{27}$$

$\therefore$ The point of inflection is $\left(\frac{4}{3};\frac{16}{27}\right)$.



Example 2: **Sketch the graph of $f(x)=(2-x)(x+1)^2$ showing all stationary points, intercepts with the axes and the point of inflection.**

Step 1: Determine the x-values of the stationary points

$$
\begin{aligned}
f(x)&=(2-x)(x+1)^2\\
&=(2-x)\left(x^2+2x+1\right)\\
&=2x^2+4x+2-x^3-2x^2-x\\
&=-x^3+3x+2 \rightarrow \text{ write the equation in standard form so that you can differentiate}
\end{aligned}
$$

For stationary points: $f'(x)=0$

$\therefore -3x^2+3=0$

$\therefore -3\left(x^2-1\right)=0 \rightarrow$ divide both sides by -3 and factorise

$\therefore (x-1)(x+1)=0$

$\therefore x=1 \quad \text{or} \quad x=-1$

Step 2: Substitute the x-values into the equation to solve for the y-values

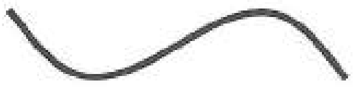
$f(1)=(2-1)(1+1)^2$
$=4$

or

$f(-1)=(2+1)(-1+1)^2$
$=0$

$\therefore$ The stationary points are $(1;4)$ and $(-1;0) \rightarrow$ this is also an x-intercept

Step 3: Test for shape

$a<0$:  $\rightarrow$ local min at $(-1;0)$ and local max at $(1;4)$

Step 4: Determine the intercepts

x-int: let $y=0$

$\therefore 0=(2-x)(x+1)^2$

$\therefore x=2 \quad \text{or} \quad x=-1$

$\therefore x$-int: $(-1;0)$ and $(2;0)$

y-int: let $x=0$

$\therefore f(0)=(2-0)(0+1)^2$

$\therefore f(0)=2$

$\therefore y$-int: $(0;2)$

Step 5: Determine the x-value of the point of inflection

$f''(x)=-6x$

For the point of inflection: $f''(x)=0$

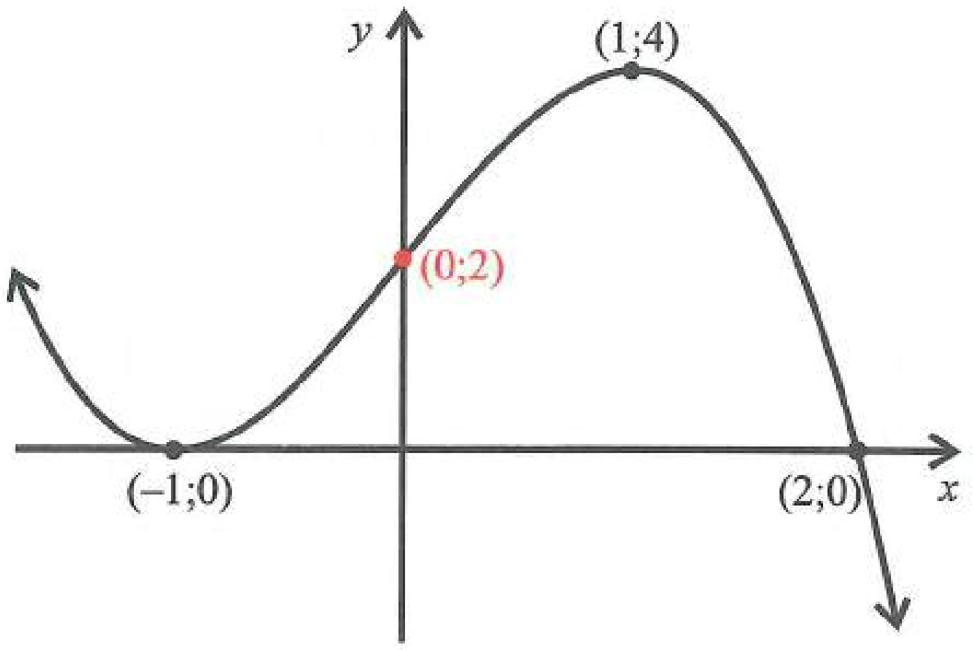
$\therefore -6x=0$

$\therefore x=0$

Step 6: Substitute the x-value into the equation to solve for the y-value

$f(0)=2$

$\therefore$ The point of inflection is $(0;2)$



Example 3: **Sketch the graph of $f(x)=-x^3+7x^2-15x+9$ showing all stationary points and intercepts with the axes.**

Step 1: Determine the x-values of the stationary points

$f'(x)=-3x^2+14x-15$

For stationary points: $f'(x)=0$

$\therefore -3x^2+14x-15=0$

$\therefore 3x^2-14x+15=0$

$\therefore (3x-5)(x-3)=0$

$\therefore x=\frac{5}{3}$ or $x=3$

Step 2: Substitute the x-values into the equation to solve for the y-values

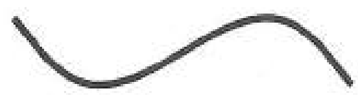
$$f\left(\frac{5}{3}\right)=-\left(\frac{5}{3}\right)^3+7\left(\frac{5}{3}\right)^2-15\left(\frac{5}{3}\right)+9 = -\frac{32}{27}$$

or

$$f(3)=-(3)^3+7(3)^2-15(3)+9 = 0$$

$\therefore$ The stationary points are $\left(\frac{5}{3};-\frac{32}{27}\right)$ and $(3;0)$ → this is also an x-intercept

Step 3: Test for shape

$a<0$:  → local min at $\left(\frac{5}{3};-\frac{32}{27}\right)$ and local max at $(3;0)$

Step 4: Determine the intercepts using the factor theorem and synthetic division

x-int: let $y=0$

$\therefore -x^3+7x^2-15x+9=0$

$\therefore x^3-7x^2+15x-9=0$

$f(3)=0$ → in step 2 it was shown that $(3;0)$ is an intercept

$\therefore x-3$ is a factor

$\therefore (x-3)(x^2-4x+3)=0$

$$\begin{array}{cccc|c} & 3 & -12 & 9 & \\ 1 & -7 & 15 & -9 & 3 \\ \hline 1 & -4 & 3 & 0 & \end{array}$$

$\therefore (x-3)(x-3)(x-1)=0$

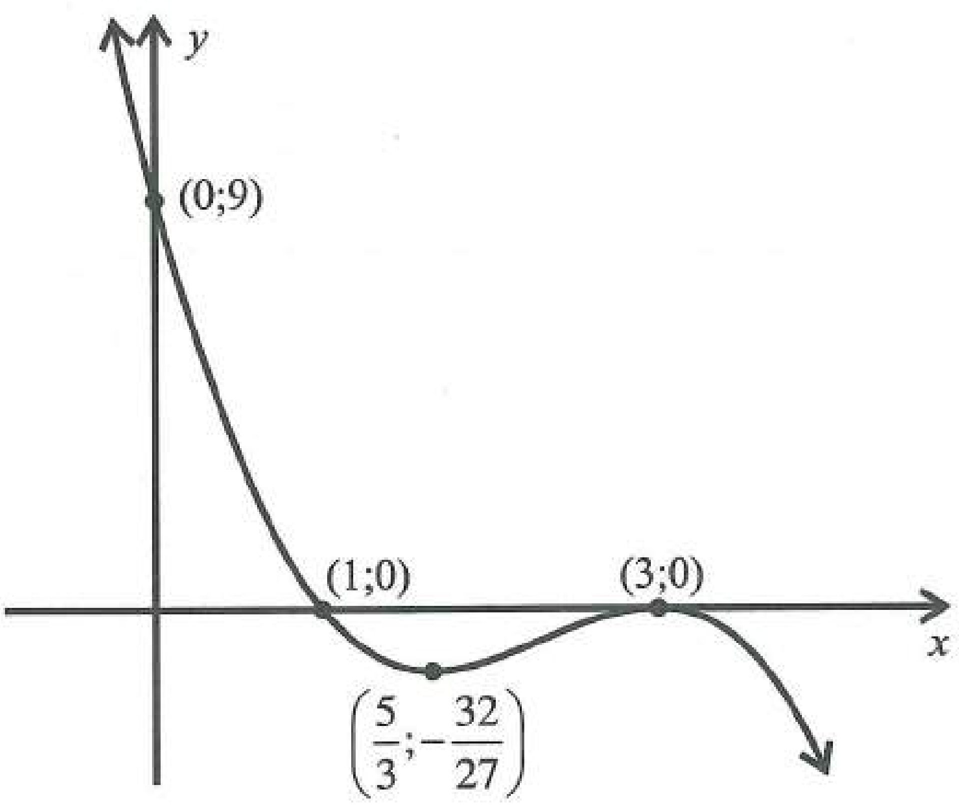
$\therefore x=3$ or $x=1$

$\therefore x$-int: $(1;0)$ and $(3;0)$

y-int: let $x=0$

$\therefore f(0)=9$

$\therefore y$-int: $(0;9)$



Example 4: Sketch the graph of $f(x)=x^3+x^2-x-10$ showing all stationary points and intercepts with the axes.

Step 1: Determine the x-values of the stationary points

$f'(x)=3x^2+2x-1$

For stationary points: $f'(x)=0$

$\therefore 3x^2+2x-1=0$

$\therefore (3x-1)(x+1)=0$

$\therefore x=\frac{1}{3}$ or $x=-1$

Step 2: Substitute the x-values into the equation to solve for the y-values

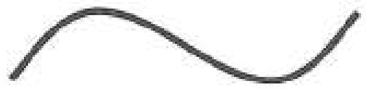
$$f\left(\frac{1}{3}\right)=\left(\frac{1}{3}\right)^3+\left(\frac{1}{3}\right)^2-\left(\frac{1}{3}\right)-10$$
$$=-\frac{275}{27}$$

or

$$f(-1)=(-1)^3+(-1)^2-(-1)-10$$
$$=-9$$

$\therefore$ The stationary points are $(-1;-9)$ and $\left(\frac{1}{3};-\frac{275}{27}\right)$

Step 3: Test for shape

$a>0$:  $\rightarrow$ local max at $(-1;-9)$ and local min at $\left(\frac{1}{3};-\frac{275}{27}\right)$

Step 4: Determine the intercepts using the factor theorem and synthetic division

x-int: let $y=0$

$\therefore x^3+x^2-x-10=0\rightarrow$ use the factor theorem

$f(2)=0$

$\therefore x-2$ is a factor

$\therefore (x-2)(x^2+3x+5)=0$

$$\begin{array}{cccc|c} & 2 & 6 & 10 & \\ 1 & 1 & -1 & -10 & 2 \\ \hline 1 & 3 & 5 & 0 & \end{array}$$

$\therefore x=2$ or $x^2+3x+5=0\rightarrow$ use the quadratic formula

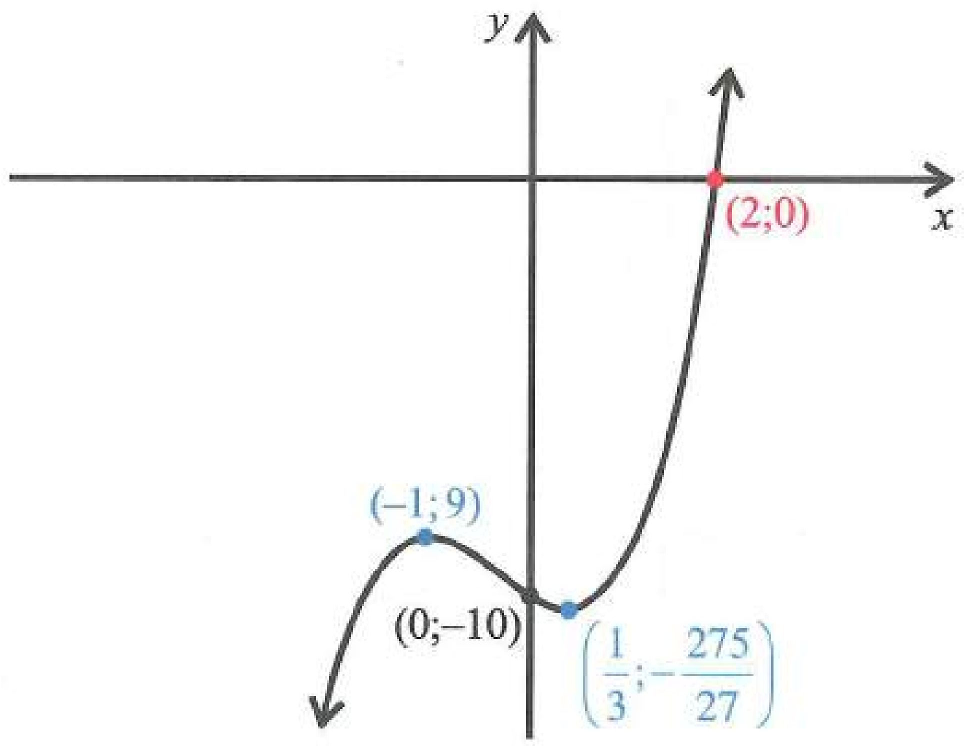
$\therefore x=2$ or $x=\frac{-3\pm\sqrt{-11}}{2}\rightarrow\frac{-b\pm\sqrt{b^2-4ac}}{2a}\rightarrow$ no solution

$\therefore$ only 1 x-int: $(2;0)$

y-int: let $x=0$

$\therefore f(0)=-10$

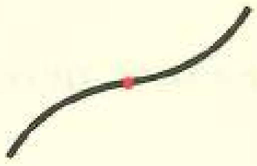
$\therefore$ y-int: $(0;-10)$



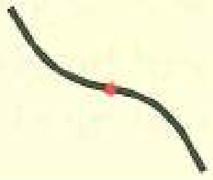
SKETCHING POINTS OF INFLECTIONS

When sketching a cubic function and there is a single stationary point, this point will be a point of inflection. The methods used to sketch cubic functions where there is only a point of inflection do not change, other than in the test for shape.

$a > 0 \rightarrow$ the shape is :



$a < 0 \rightarrow$ the shape is :



Example 5: **Sketch the graph of $g(x) = (x-1)^3 - 1$ showing all stationary points and intercepts with the axes.**

Step 1: Determine the x-values of the stationary points

$$\begin{aligned} g(x) &= (x-1)(x-1)^2 - 1 \\ &= (x-1)\left(x^2 - 2x + 1\right) - 1 \\ &= x^3 - 2x^2 + x - x^2 + 2x - 1 - 1 \\ &= x^3 - 3x^2 + 3x - 2 \rightarrow \text{ write the equation in standard form in order to differentiate} \end{aligned}$$

For stationary points: $g'(x) = 0$

$g'(x) = 3x^2 - 6x + 3$

$\therefore 3x^2 - 6x + 3 = 0$

$\therefore 3\left(x^2 - 2x + 1\right) = 0 \rightarrow$ divide both sides by -3 and factorise

$\therefore (x-1)(x-1) = 0$

$\therefore x = 1 \rightarrow$ only one stationary point, $\therefore$ this is a point of inflection

Step 2: Substitute the x-values into the equation to solve for the y-values

$g(1) = (1-1)^3 - 1 = -1$

$\therefore$ The point of inflection is $(1;-1)$

Step 3: Test for shape

$a > 0$:

Step 4: Determine the intercepts

x-int: let $y = 0$

$\therefore 0 = (x-1)^3 - 1$

$\therefore (x-1)^3 = 1 \rightarrow$ cube root both sides

$\therefore x - 1 = 1$

$\therefore x = 2$

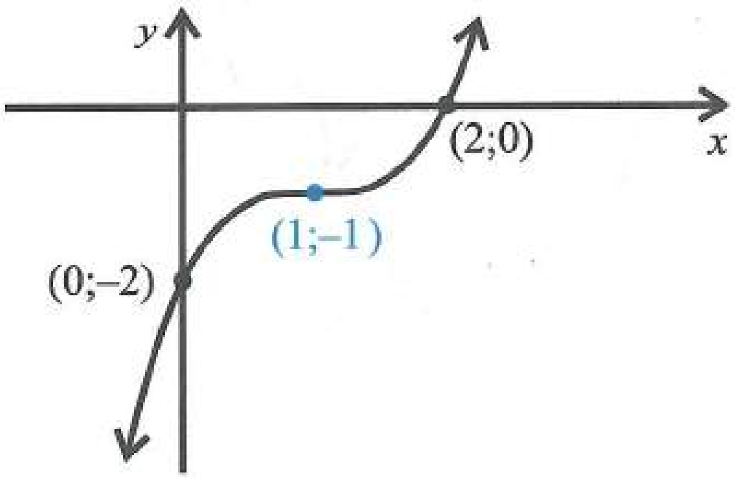
$\therefore x$-int: $(2;0)$

y-int: let $x = 0$

$\therefore g(0) = (0-1)^3 - 1$

$\therefore g(0) = -2$

$\therefore y$-int: $(0;-2)$



Complete Exercise 1 on page 183.

B. GRAPH INTERPRETATION

When determining the equation of a cubic function, you will usually be given one of the three cases below:

1. **You will be given the x-intercepts.**
2. **You will be given a stationary point and one other point.**
3. **You will be given both stationary points.**

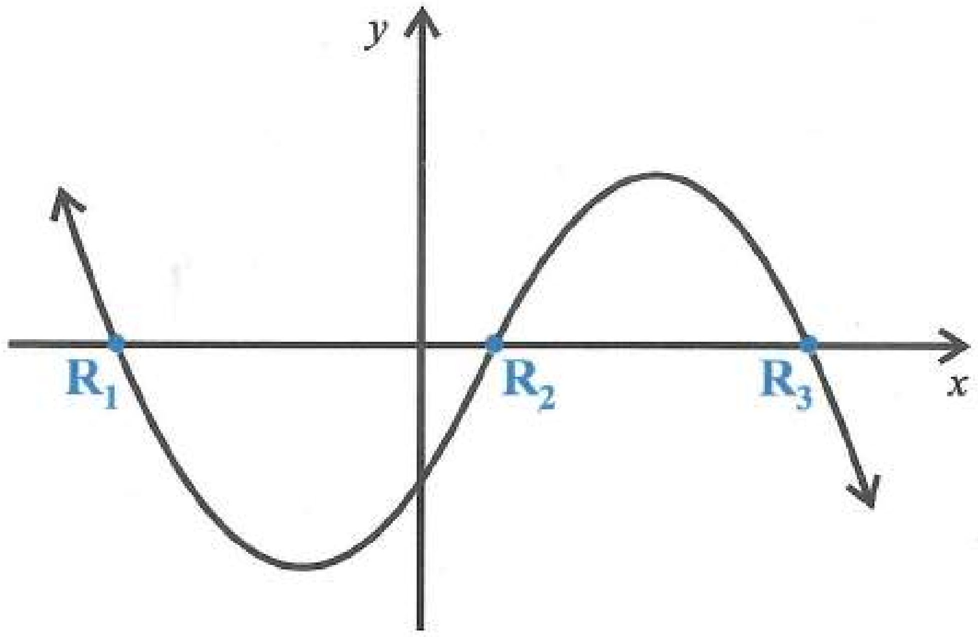
CASE 1: GIVEN THE x- INTERCEPTS

When given the x-intercepts, use the equation in the general form:

$$y=a(x-R_1)(x-R_2)(x-R_3)$$

Where R_1, R_2 and R_3 are the x-intercepts of the cubic functions.

The values of the x-intercepts can often be read from the graph.



You will be expected to determine the value of a in two ways:

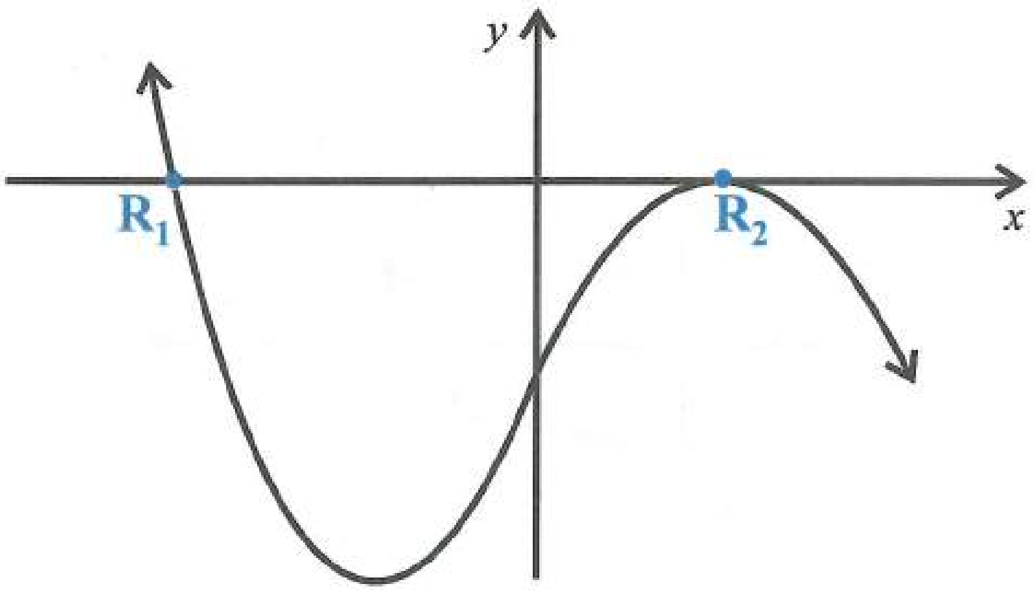
1. **It is given to you in the equation, a is the coefficient of x^3.**

   For example in the equation $y=2x^3+bx^2+cx+d$, $a=2$.
2. **A point on the curve other than the x-intercept is given.**
   You can then substitute the point into the equation and solve for a.

When an x-intercept is also a stationary point, use the equation in the general form:

$$y=a(x-R_1)(x-R_2)^2$$

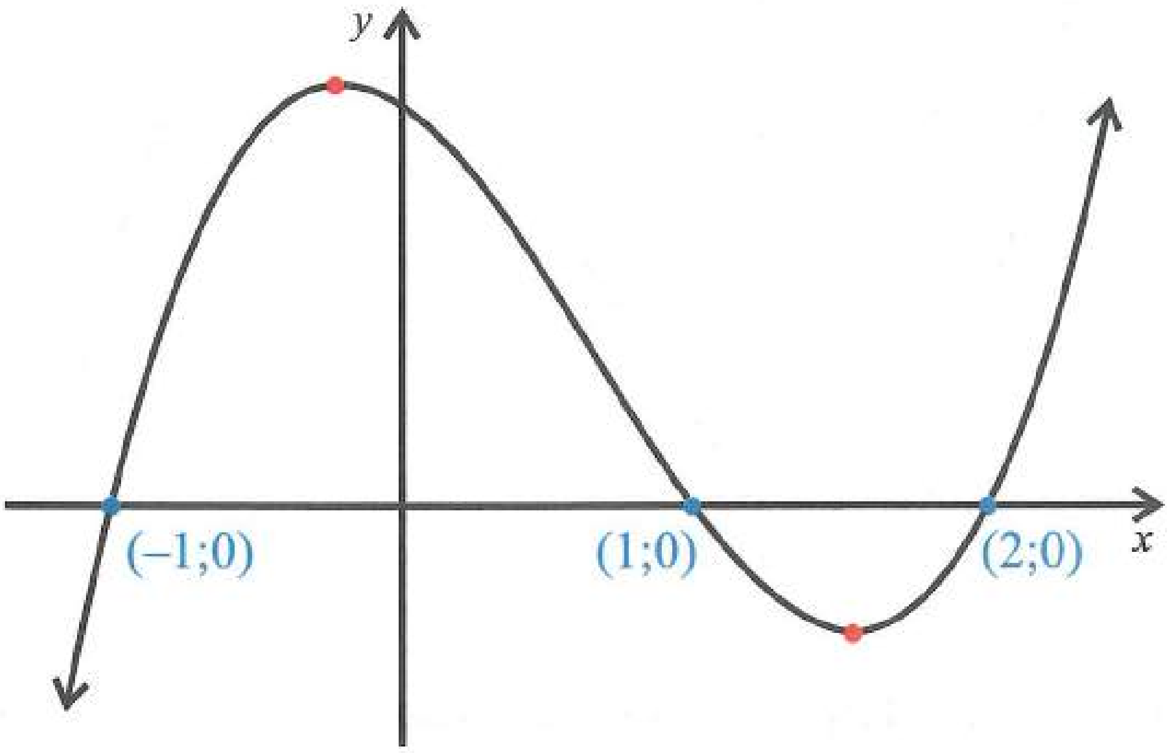
Where R_1 is an x-intercept and R_2 is an x-intercept and stationary point.



Example 1: **The graph shown below has the equation $y = 2x^3 + bx^2 + cx + d$.**

a) **Determine the values of b, c and d if the graph has x-intercepts of $(-1;0)$, $(1;0)$ and $(2;0)$.**

b) **Determine the x-values of the stationary points. Write your answer in simplest surd form.**



a) $y = a(x - R_1)(x - R_2)(x - R_3)$

$\therefore y = 2[x - (-1)](x - 1)(x - 2) \rightarrow$ a is the coefficient of x^3

$\therefore y = 2(x+1)(x-1)(x-2)$

$\therefore y = 2(x^2 - 1)(x - 2)$

$\therefore y = 2(x^3 - 2x^2 - x + 2)$

$\therefore y = 2x^3 - 4x^2 - 2x + 4$

$\therefore b = -4; c = -2$ and $d = 4$

b) **For stationary points $\frac{dy}{dx} = 0$:**

$\frac{dy}{dx} = 6x^2 - 8x - 2$

$\therefore 6x^2 - 8x - 2 = 0 \rightarrow$ divide both sides by 2

$\therefore 3x^2 - 4x - 1 = 0 \rightarrow$ use the quadratic formula

$\therefore x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(3)(-1)}}{2(3)} \rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

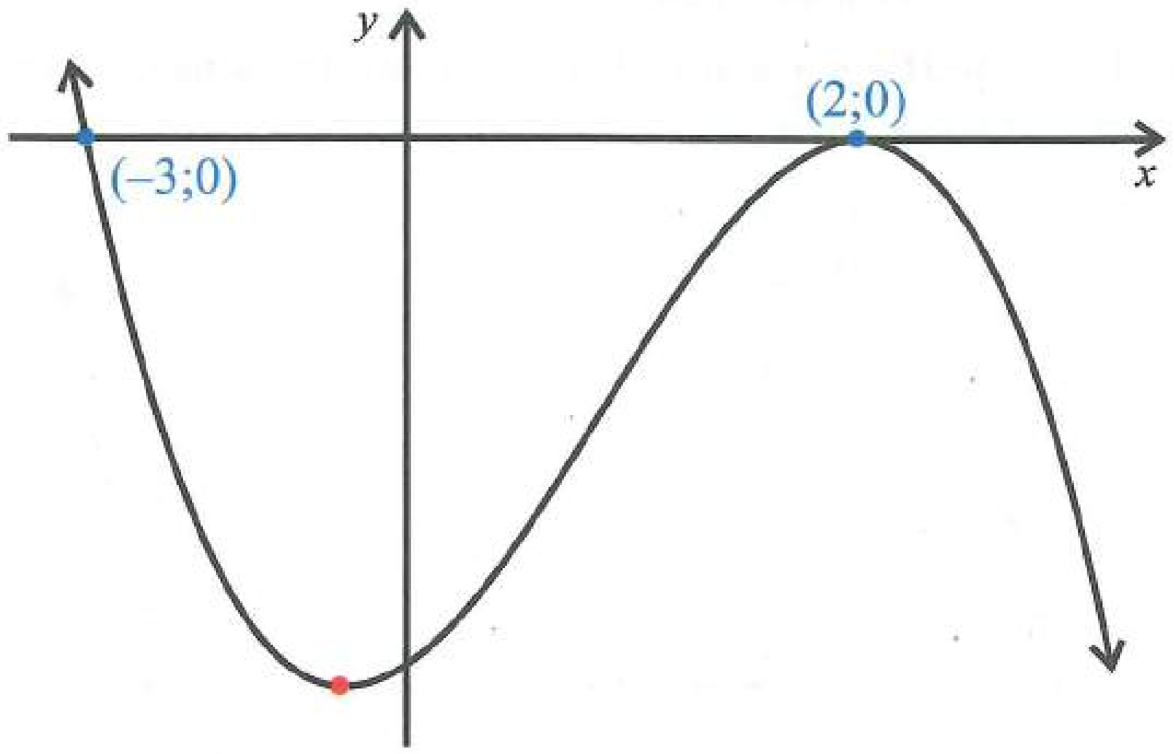
$\therefore x = \frac{4 \pm \sqrt{28}}{6} \rightarrow \sqrt{28} = \sqrt{4 \times 7} = 2\sqrt{7}$

$\therefore x = \frac{4 \pm 2\sqrt{7}}{6}$

$\therefore x = \frac{2(2 \pm \sqrt{7})}{6}$

$\therefore x = \frac{2 \pm \sqrt{7}}{3}$

Example 2: **Determine the equation of the graph shown below if it has x-intercepts of $(-3;0)$, $(2;0)$ and passes through the point $(3;-6)$.**



As one of the turning points is also a stationary point, use the equation in the form:

$y = a(x - R_1)(x - R_2)^2$

$\therefore y = a[x - (-3)](x - 2)^2$

$\therefore y = a(x + 3)(x - 2)^2$

Substitute the point $(3;-6)$ in order to solve for a

$\therefore -6 = a(3 + 3)(3 - 2)^2$

$\therefore -6 = 6a$

$\therefore a = -1$

$\therefore y = -1(x + 3)(x - 2)^2$

$\therefore y = -1(x + 3)(x^2 - 4x + 4) \rightarrow$ don't forget the middle term

$\therefore y = -1(x^3 - 4x^2 + 4x + 3x^2 - 12x + 12)$

$\therefore y = -1(x^3 - x^2 - 8x + 12)$

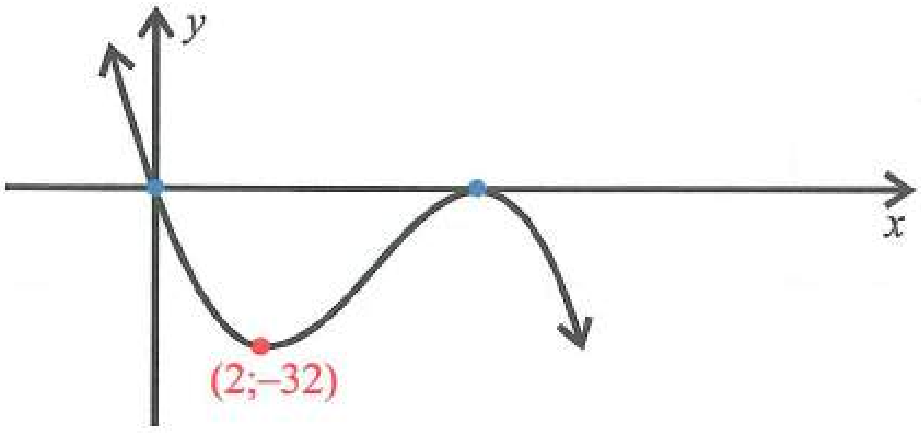
$\therefore y = -x^3 + x^2 + 8x - 12$

In most cases it is easier to substitute a point in to solve for a, before multypling out the equation.

CASE 2: GIVEN A STATIONARY POINT AND ONE OTHER POINT

In this case, you will be given the coordinates of a stationary point and one other point on the curve (usually the y-intercept). You will then be expected to solve for two unknown variables in the equation using simultaneous equations.

Example: The graph shown below has the equation $y=-x^3+bx^2+cx+d$. It has a stationary point at $A(2;-32)$ and passes through the origin. Determine the values of b, c and d. Hence, write down the equation of the function.



It is given that the function passes through the origin. This means that the y-intercept is the point $(0;0)$.

$\therefore d=0 \rightarrow d$ is the y-intercept

$\therefore y=-x^3+bx^2+cx+0$

Substitute $A(2;-32)$

$\therefore -32=-(2)^3+b(2)^2+c(2)+0$

$\therefore -32=-8+4b+2c$

$\therefore -24=4b+2c$

$\therefore -12=2b+c$

$\therefore c=-12-2b \rightarrow \boxed{1}$

$A(2;-32)$ is also a stationary point. This means that when $x=2$, the gradient of the function is equal to 0. In other words $\frac{dy}{dx}=0$ when $x=2$.

$\frac{dy}{dx}=-3x^2+2bx+c$

at $x=2$:

$-3(2)^2+2b(2)+c=0$

$\therefore c=12-4b \rightarrow \boxed{2}$

Substitute $\boxed{1}$ into $\boxed{2}$:

$\therefore -12-2b=12-4b$

$\therefore 2b=24$

$\therefore b=12 \rightarrow \boxed{3}$

Substitute $\boxed{3}$ into $\boxed{1}$:

$\therefore c=-36$

$\therefore b=12$ and $c=-36$

$\therefore$ The equation of the function is: $y=-x^3+12x^2-36x$

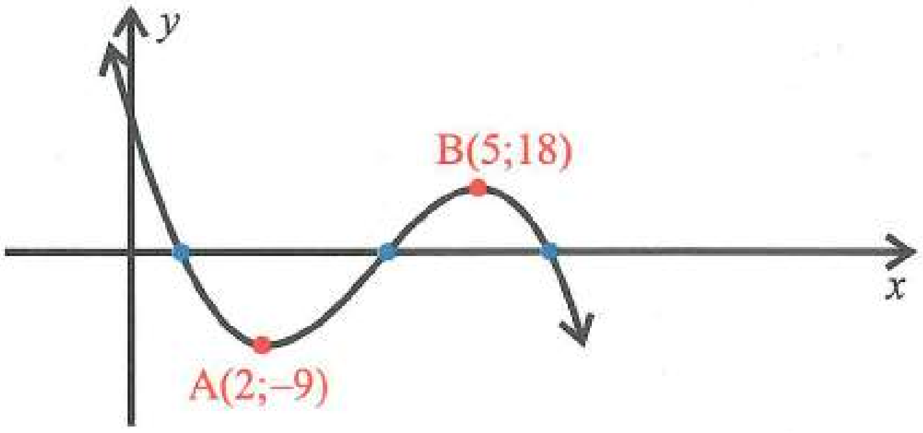
CASE 3: GIVEN BOTH STATIONARY POINTS

In this case, you will be given both stationary points. You will then have to use the fact that the derivative is equal to 0 at both points to form two equations and solve simultaneously. You may also have to use one of the points in the original equation to solve for a third variable.

Example: **The graph shown below has the equation $f(x) = -2x^3 + bx^2 + cx + d$.**

The graph has stationary points at $A(2;-9)$ and $B(5;18)$.

Determine the value of *b*, *c* and *d*.



As A and B are stationary points:

$f'(2) = 0$ and $f'(5) = 0$

$f'(x) = -6x^2 + 2bx + c$

$\therefore 0 = -6(2)^2 + 2b(2) + c$

$\therefore 0 = -24 + 4b + c$

$\therefore c = 24 - 4b \rightarrow \boxed{1}$

$\therefore 0 = -6(5)^2 + 2b(5) + c$

$\therefore 0 = -150 + 10b + c \rightarrow \boxed{2}$

Substitute $\boxed{1}$ into $\boxed{2}$:

$\therefore 0 = -150 + 10b + (24 - 4b)$

$\therefore 126 = 6b$

$\therefore b = 21 \rightarrow \boxed{3}$

Substitute $\boxed{3}$ into $\boxed{1}$:

$\therefore c = -60$

$\therefore f(x) = -2x^3 + 21x^2 - 60x + d$

In order to solve for *d* you can now substitute point A or B into *f*:

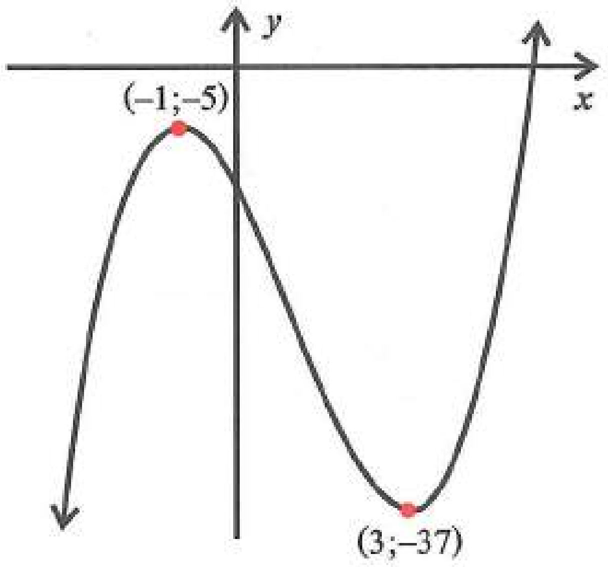
Substituting point $A(2;-9)$	**Substituting point $B(5;18)$**
$-9 = -2(2)^3 + 21(2)^2 - 60(2) + d$	$18 = -2(5)^3 + 21(5)^2 - 60(5) + d$
$\therefore -9 = -52 + d$	$\therefore 18 = -25 + d$
$\therefore d = 43$	$\therefore d = 43$

$\therefore b = 21$ and $c = -60$ and $d = 43$

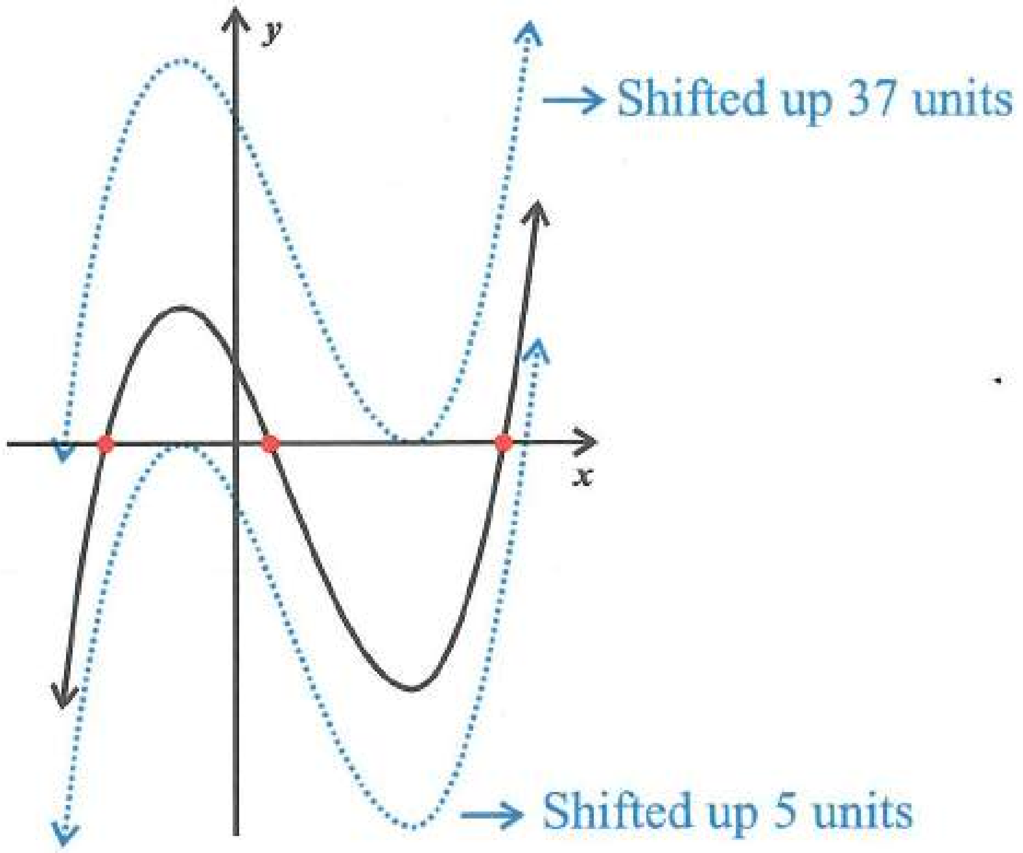
It is not necessary to substitute both A and B into *f*. Both methods have been shown above for the sake of clarity.

TRANSLATIONS OF CUBIC FUNCTIONS

Example 1: **The function $f(x)$ below has equation $f(x)=x^3-3x^2-9x-10$. Determine the value(s) of k for which $x^3-3x^2-9x-10+k=0$ will have 3 real roots.**



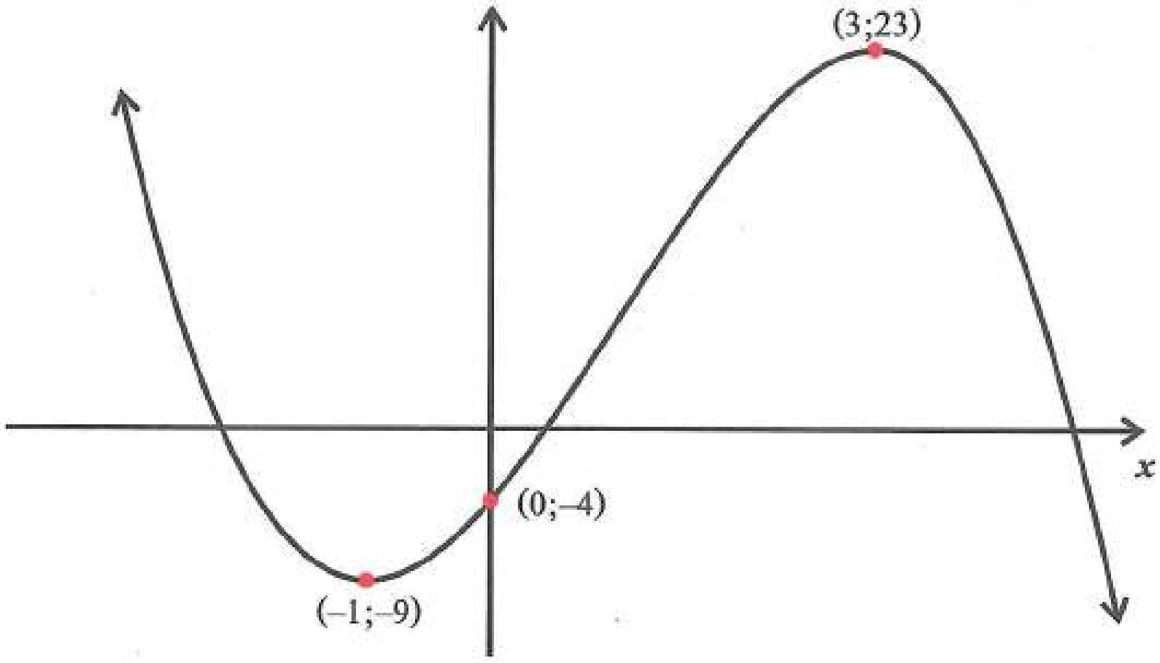
In this case, the value of k shifts the graph vertically up or down. To have three real roots, the graph has to have 3 x-intercepts. This means the graph has to be shifted up so that the local maximum turning point is **above** the x-axis and the local minimum turning point is **below** the x-axis. This is shown below where the graph has to lie between the two **blue** dotted graphs.



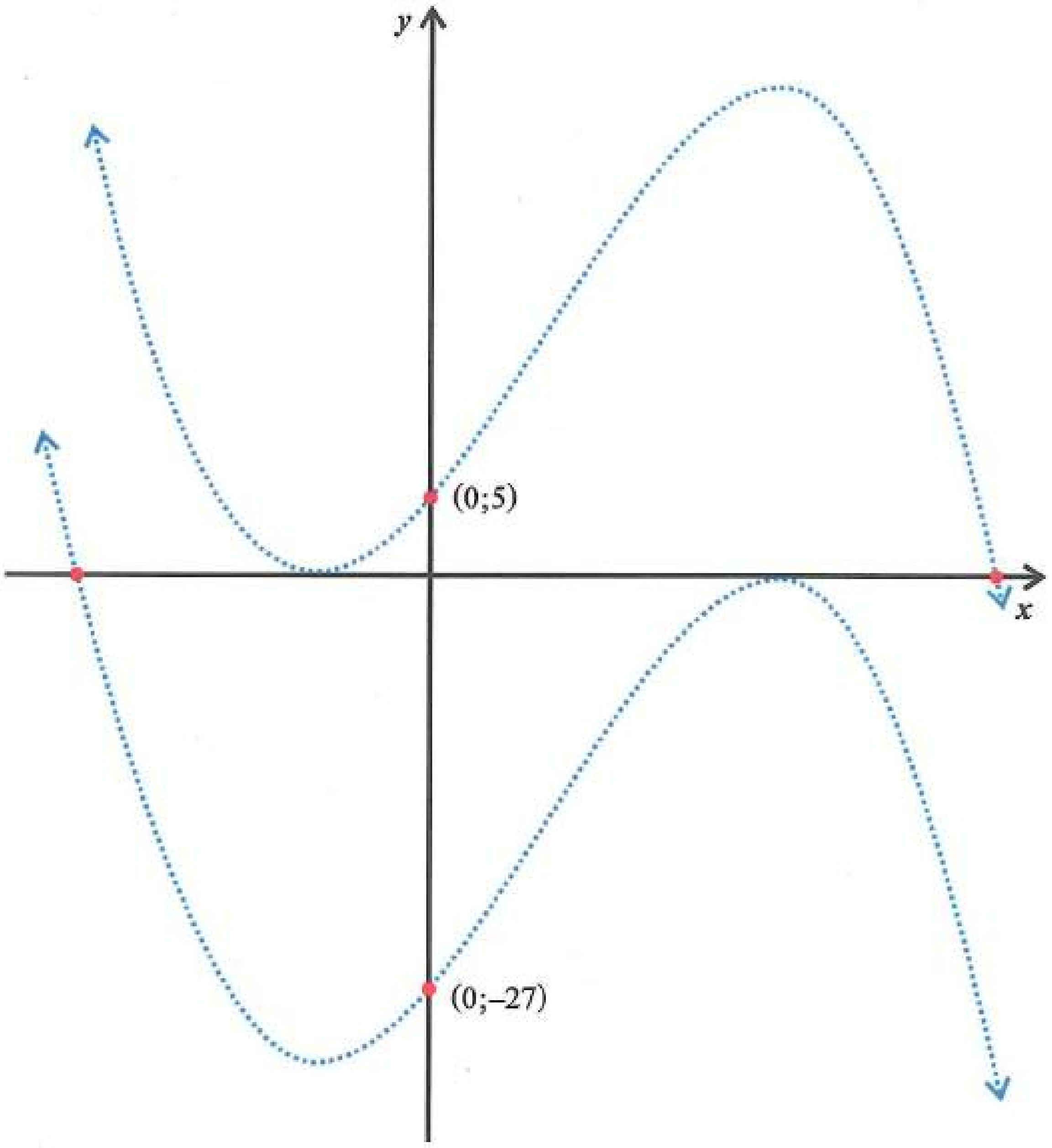
It should be clear that the graph has to be shifted vertically up by more than 5 units but less than 37 units.

$\therefore 5<k<37$

Example 2: **The function $f(x)$ below has equation $f(x)=-x^3+3x^2+9x-4$. Determine the value(s) of k for which $-x^3+3x^2+9x+k=0$ will have 1 real root.**



In this case, the value of k is the y-intercept. To have one real root, the graph has to have one x-intercept. This means the graph has to be shifted up so that the local minimum turning point is **above** the x-axis, or the graph has to shifted down so that the local maximum turning point is **below** the x-axis. This is shown below where the graph has to lie above or below the two **blue** dotted graphs.



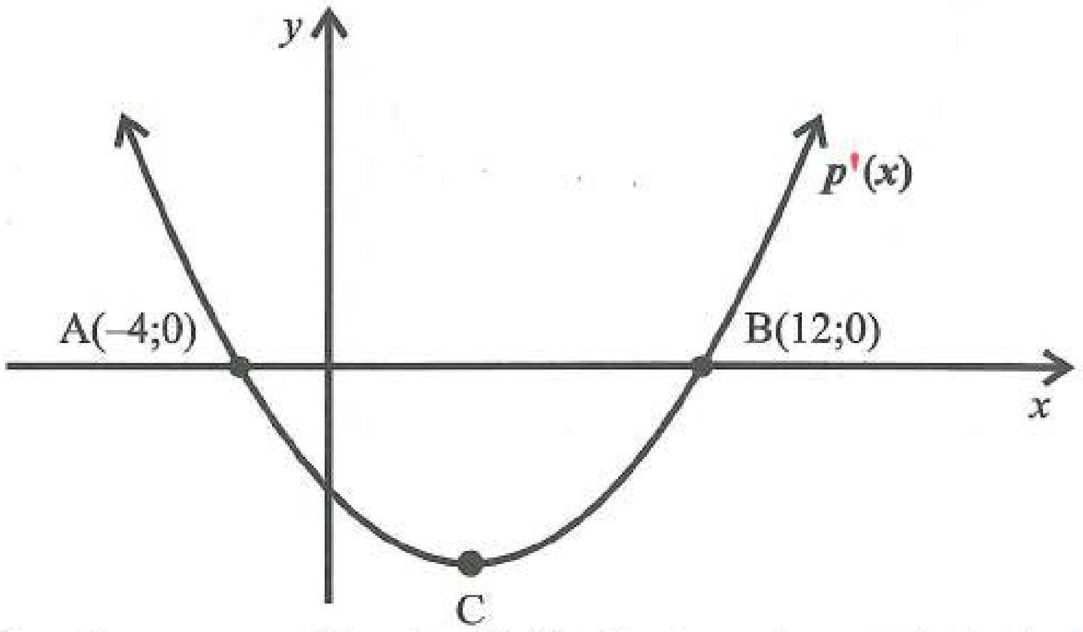
It should be clear that the graph has to be shifted vertically **up** by **more than 9** units, or shifted **down** by **more than 23 units**. The y-intercept of the original function is –4, if the graph is shifted up by more than 9 units the new y-intercept will be **greater than 5**. If the graph is shifted down by more than 23 units the new y-intercept will be **less than –27**.

$\therefore k>5$ or $k<-27$

Complete Exercise 2 on page 183.

MIXED EXAMPLES

Example 1: **The graph of $p'(x)=ax^2+bx+c$ is sketched below. $p'(x)$ is the derivative graph of a cubic function $p(x)$. Points $A(-4;0)$ and $B(12;0)$ are the x-intercepts of $p'(x)$.**



a) Determine the x-coordinate of C, the turning point of $p'(x)$.

b) Determine the x-coordinate of the point of inflection of $p(x)$.

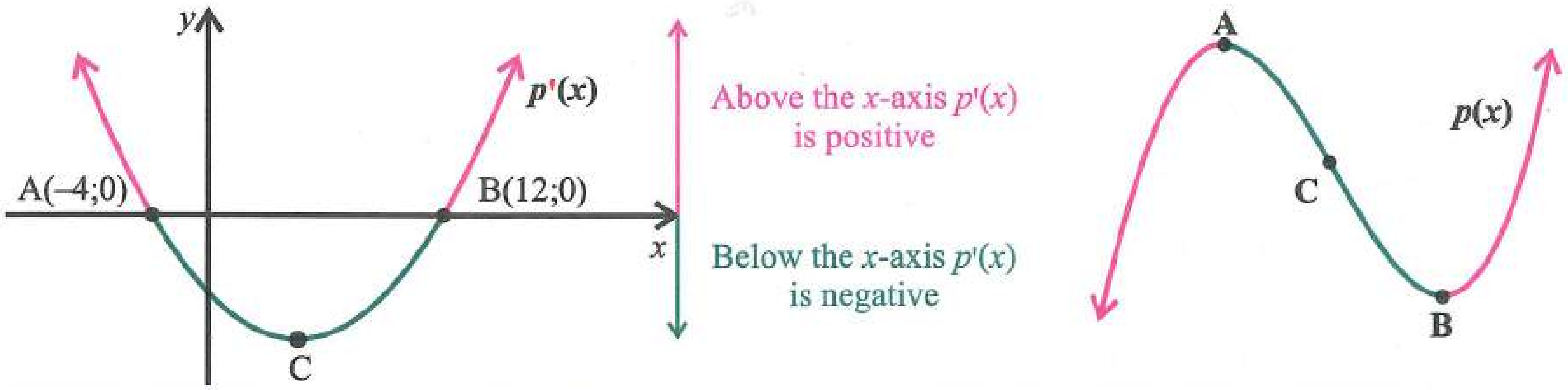
c) Explain why p has a local maximum at $x=-4$.

a) A parabola is symmetrical about its turning point. This means that the horizontal distance from point A to the turning point, is the same as the horizontal distance from the turning point to point B. In other words, the turning point lies halfway between A and B.

$$\begin{aligned}\therefore x_C &= \frac{x_A+x_B}{2}\\ &= \frac{-4+12}{2}\\ &= 4\end{aligned}$$

b) The point of inflection is where the $p''(x)=0$. This corresponds to the turning point of $p'(x)$. Therefore, the x-value of the point of inflection is **4**.

c) From the graph, it can be seen that $p'(x)>0$ where $x<-4$ and $p'(x)<0$ where $-4<x<12$. $p'(x)>0$ again, where $x>12$. This means that p is increasing on the interval where $x<-4$ and $x>12$. On the interval where $-4<x<12$ p is decreasing. This means that at point A, p has a local maximum and at point B, p has a local minimum.

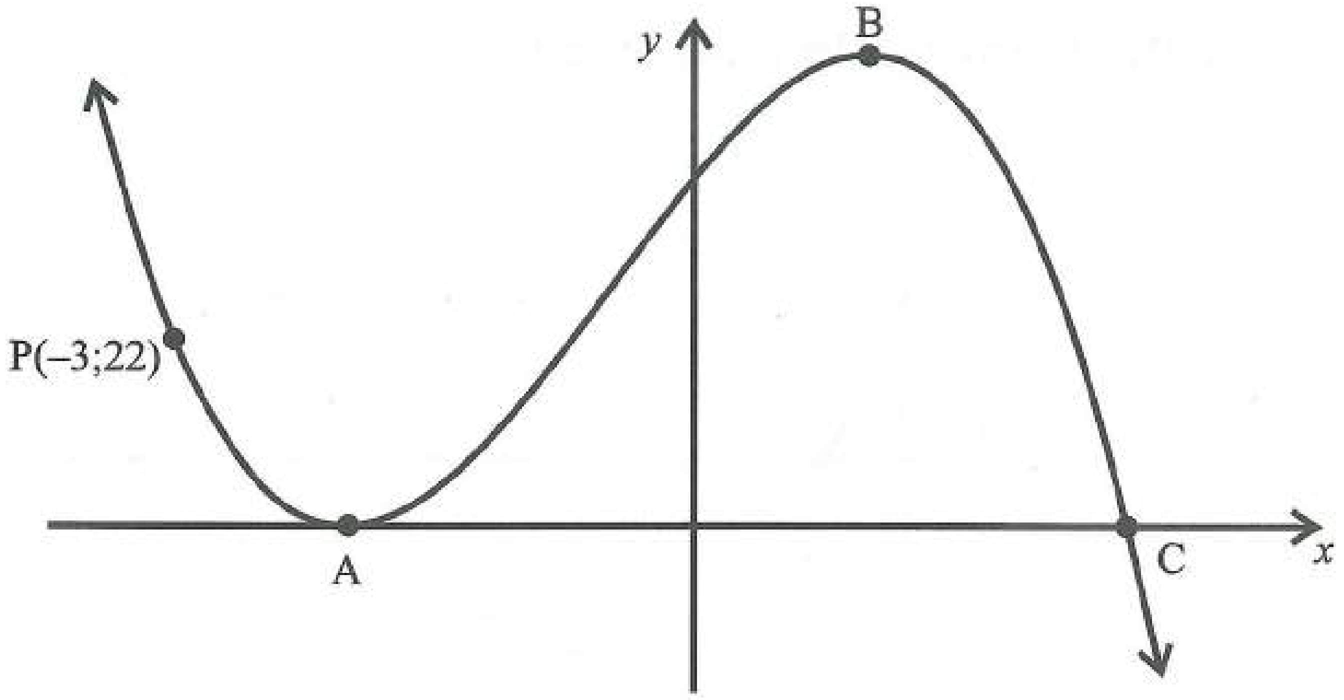

y
p'(x)
A(–4;0)
B(12;0)
x
C
Above the x-axis p'(x) is positive
Below the x-axis p'(x) is negative
A
p(x)
C
B

From the above example, it should be clear that the x-intercepts of the parabola represent the x-values of the turning points of a cubic function and that the turning point of the parabola represents the x-value of the point of inflection of a cubic function.

Example 2: **The graph of $f(x) = -4x^3 - 6x^2 + 24x + 40 = -2(2x-5)(x+2)^2$.**

A and B are turning points of f and A and C are the x-intercepts of f.

$P(-3;22)$ is a point on the graph.



a) Determine the length of AC.
b) Determine the coordinate of B.
c) Determine the equation of the tangent to f at P.
d) Determine the value(s) of k for which $-4x^3 - 6x^2 + 24x + 40 = k$ has 3 distinct roots.
e) Determine the x- coordinate of the point of inflection.

a) x-int : let $y = 0$

$\therefore -2(2x-5)(x+2)^2 = 0$

$\therefore x = \frac{5}{2}$ or $x = -2$

$\therefore$ A is the point $(-2;0)$ and C is the point $\left(\frac{5}{2};0\right)$

$AC = x_{\text{right}} - x_{\text{left}}$

$= \frac{5}{2} - (-2)$

$= 4{,}5 \text{ units}$

b) For stationary points: $f'(x) = 0$

$f'(x) = -12x^2 - 12x + 24$

$\therefore -12x^2 - 12x + 24 = 0 \rightarrow$ divide both sides by -12

$\therefore x^2 + x - 2 = 0$

$\therefore (x+2)(x-1) = 0$

$\therefore x = -2$ or $x = 1$

$\therefore$ At B: $x = 1 \rightarrow x$ is positive at B

$f(1) = -4(1)^3 - 6(1)^2 + 24(1) + 40 \rightarrow$ determine the y-value at B

$= 54$

$\therefore$ B is the point $(1;54)$

c) Equation of the tangent: $y = mx + c$ or $(y - y_1) = m(x - x_1)$

$f'(x) = -12x^2 - 12x + 24$

$f'(-3) = -12(-3)^2 - 12(-3) + 24$

$= -48$

$\therefore m_{\text{tangent}} = -48$

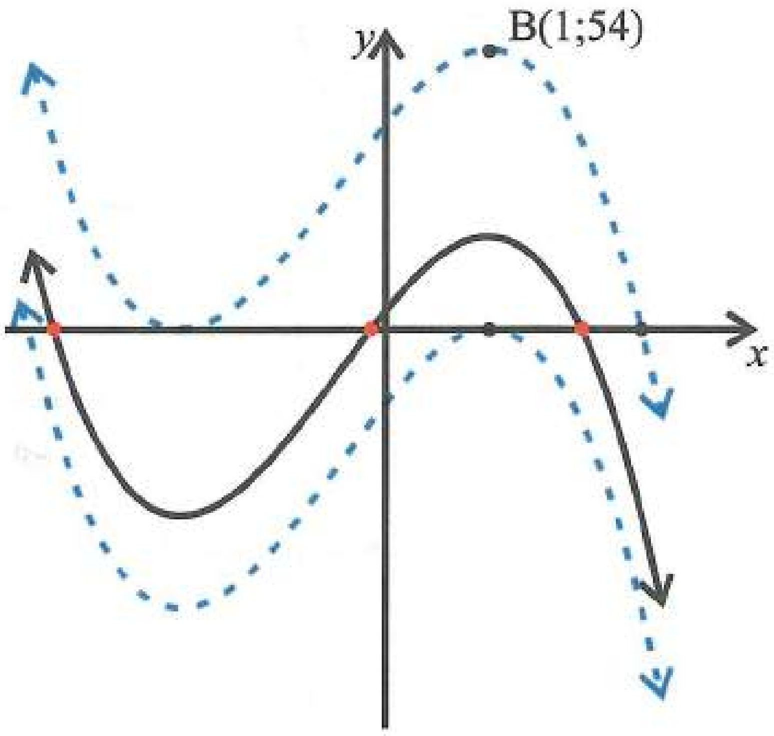
$\therefore y = -48x + c$

Method 1 using: $y = mx + c$	**Method 2 using:** $(y - y_1) = m(x - x_1)$
$y = -48x + c$ Substitute the point of contact $(-3;22)$: $\therefore 22 = -48(-3) + c$ $\therefore 22 = 144 + c$ $\therefore c = -122$ $\therefore y = -48x - 122$	$y - y_1 = -48(x - x_1)$ Substitute the point of contact $(-3;22)$: $\therefore y - 22 = -48[x - (-3)]$ $\therefore y - 22 = -48x - 144$ $\therefore y = -48x - 122$

d) $-4x^3 - 6x^2 + 24x + 40 - k = 0 \rightarrow k$ shifts the graph down when $k > 0$

In order to have 3 distinct roots the graph has to be shifted down by no more than 54 units.

$\therefore \mathbf{0 < k < 54}$



e) Point of Inflection: $f''(x) = 0$

$f''(x) = -24x - 12$

$\therefore -24x - 12 = 0$

$\therefore -24x = 12$

$\therefore x = -\frac{1}{2}$

Complete the Mixed Exercise on pages 184-186.

Chapter 8 Curve Sketching and Graph Interpretation (Exercises)

Sketching Cubic Functions

Exercise 1: (answers on page 356)

1) **Sketch the graphs of the following cubic functions, showing all stationary points and intercepts with the axes:**

   a) $f(x)=(x-2)^2(x+4)$

   b) $f(x)=x^3-4x^2-3x+18$

   c) $y=-x^3-x^2+8x+12$

   d) $g(x)=-x^3+6x^2-12x+16$

   e) $f(x)=-x^3+5x^2+8x-12$

   f) $f(x)=\frac{1}{900}x^3-\frac{1}{5}x^2+9x$

Graph Interpretation

Exercise 2: (answers on page 356 and 357)

1) **The curve $h(x)=2x^3+ax^2-4x+b$ has x-intercepts of 1 and –3.**

   a) Determine the values of a and b.

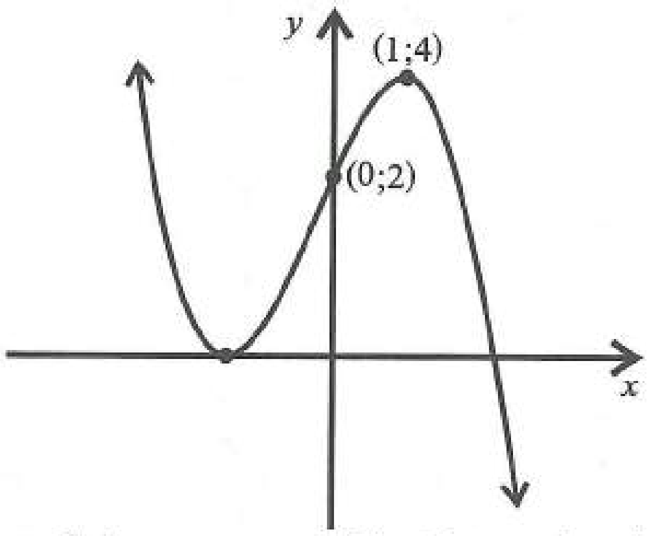
   b) Determine the stationary points of h.

2) **The curve $f(x)=4x^3+ax^2+bx+c$ has x-intercepts at $(-3;0)$ and $(4;0)$. If $(4;0)$ is also a stationary point:**

   a) Determine the values of a, b and c.

   b) Determine the stationary points of f.

3) **If polynomial $f(x)=-x^3+bx^2+cx+d$, whose graph is shown below, has a y-intercept of $(0;2)$ and a turning point at $(1;4)$:**



   a) Determine the values of b, c and d.

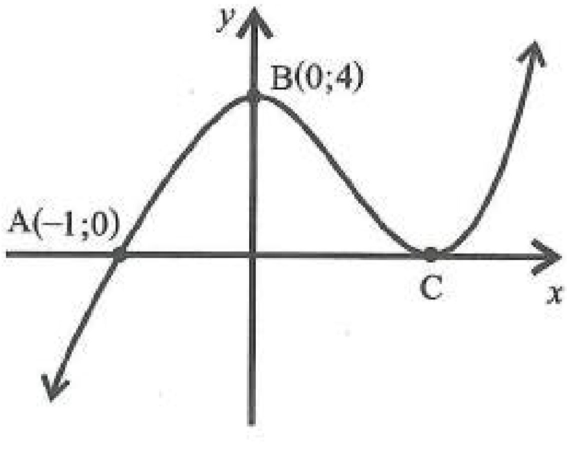
   b) Determine the x-intercepts of f.

4) **The graph of $h(x)=\frac{1}{3}x^3+2x^2-5x$ has a local minimum at M and a point of inflection at N.**

   a) Determine the coordinate of M.

   b) Determine the coordinate of N.

5) **The graph of $f(x)=x^3+ax^2+bx+4$ is sketched below. B and C are stationary points of f and A is an x-intercept.**



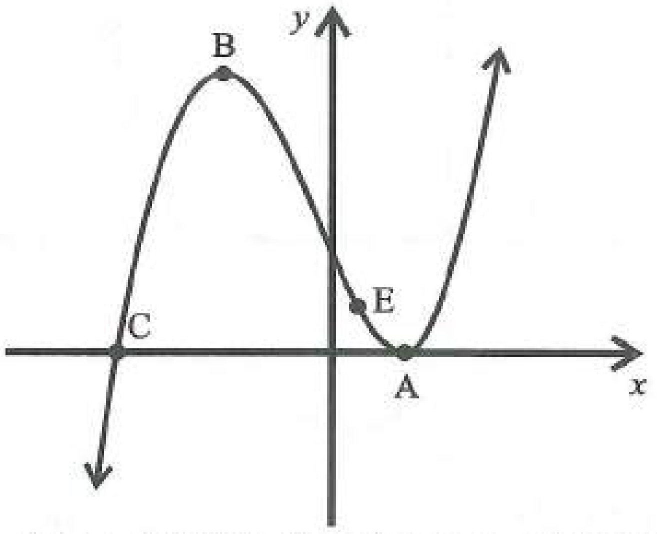
   a) Determine the values of a and b.

   b) Determine the coordinate of C.

   c) Determine the equation of the tangent at A.

   d) Determine the point of inflection of f.

6) The graph of $h(x)=x^3-3x^2-9x+27$ is sketched below. $B(-1;32)$ is a local maximum and A is local minimum. $E(2;5)$ is a point that lies on h.



a) Determine the coordinates of A and C if both points are x-intercepts of h.

b) A tangent to h has a gradient of 15. Determine the point(s) of contact.

c) Determine the coordinate of the y-intercept of f if $f(x)=h(x+2)$.

d) Determine the value(s) of k for which $-x^3+3x^2+9x-27=k$ has 3 unequal roots.

7) The graph of $p(x)=ax^3+bx^2-12x+4$ has a stationary point at $B(2;-16)$.

a) Determine the values of a and b.

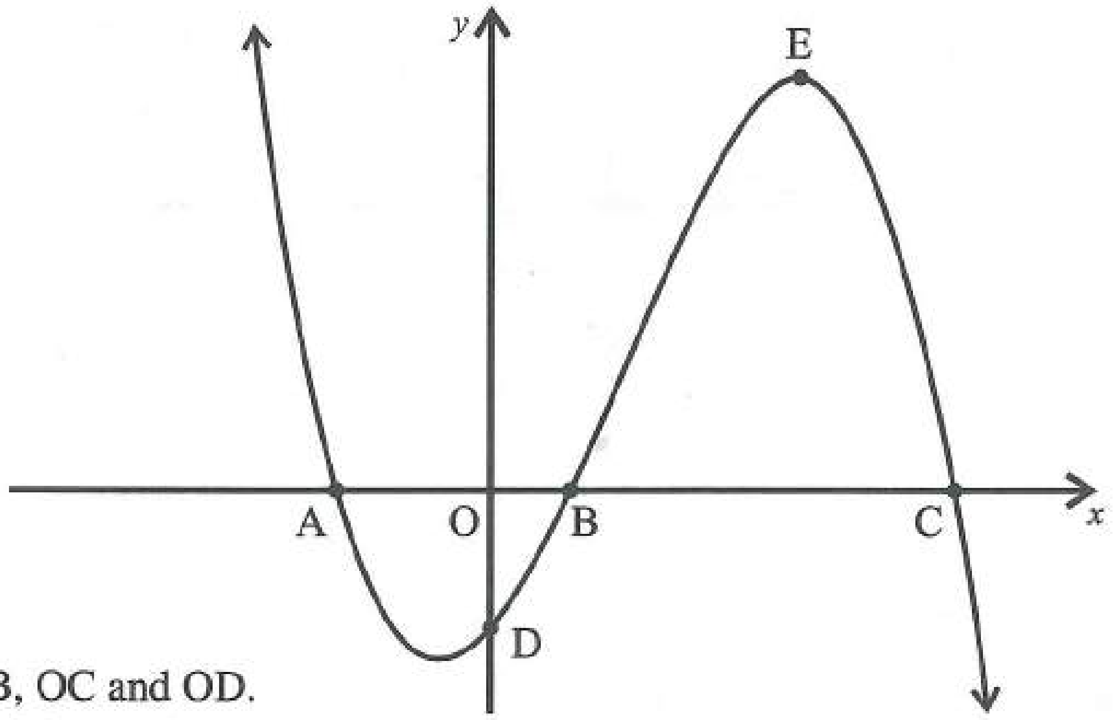
b) Determine the coordinates of R, another stationary point of p.

8) Use the information in the table below to draw a sketch graph of the cubic function $g(x)$, showing all stationary points and intercepts with the axes.

x	$g(x)$	$g'(x)$
$x<0$	No information given	Negative
-5	0	Negative
0	-5	0
$0<x<8$	No information given	Positive
3	0	Positive
8	10	0
12	0	Negative

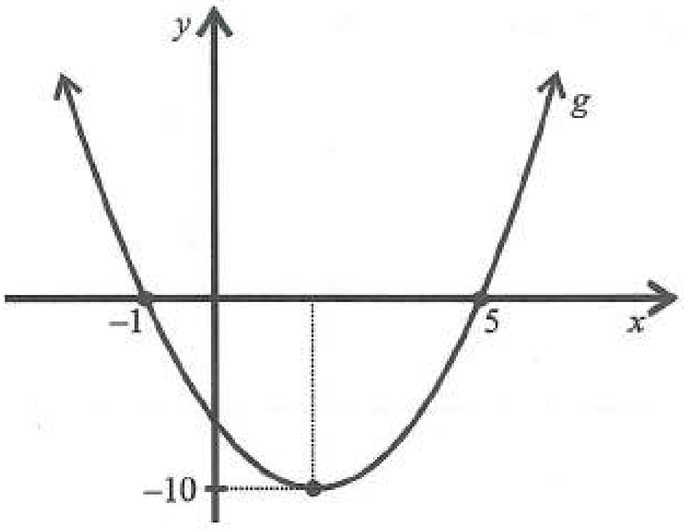
Mixed Exercise: (answers on page 357)

1) The sketch below represents the graph of $h(x)=-x^3+5x^2+8x-12$. Determine:



a) The lengths of AB, OC and OD.
b) The coordinates of the turning point E.
c) The equation of the tangent to f at D.

2) The graph of $g(x)$ below represents the derivative of a cubic function $f(x) = ax^3 + bx^2 + cx + d$.

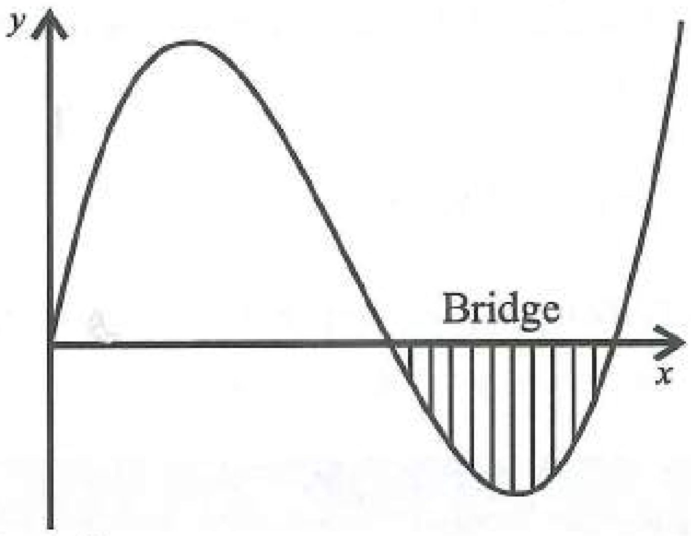


a) What are the x-values of the local minimum and maximum values of f? Indicate clearly which value is the local minimum and maximum respectively.

b) Give the coordinate on $g(x)$ which represents the point of inflection of f?

c) Is f increasing or decreasing on the interval $2 < x < 5$?

3) A road running over a mountain and then crossing a bridge can be modelled by the cubic function $f(x) = 0{,}5x^3 - 8x^2 + 30x$. If every unit on the graph below represents 10m, answer the following questions giving all answers correct to the nearest metre where applicable.



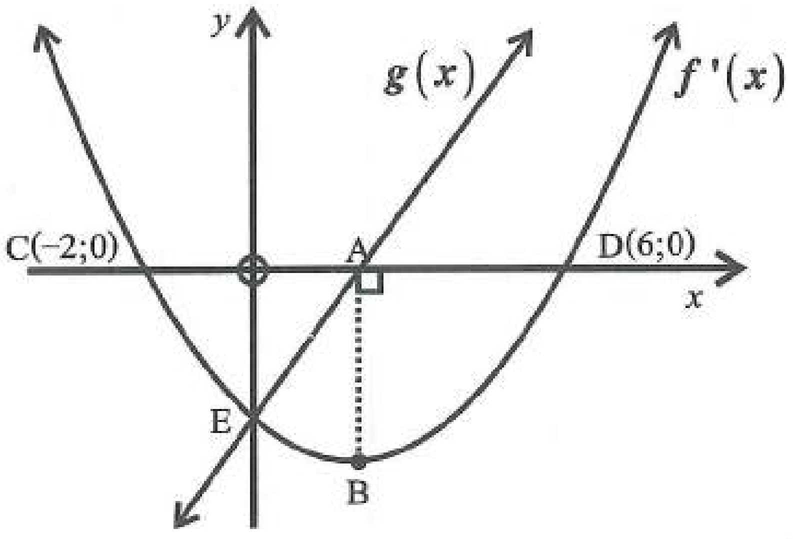
a) Determine the length of the bridge.

b) If a tunnel costs R50 000 per metre to construct, how much would it cost to construct a tunnel through the mountain?

c) Determine the height of the mountain correct to the nearest metre.

d) Determine the maximum height of the bridge above the valley correct to the nearest metre.

4) The graphs of $f'(x) = ax^2 + bx + c$ and $g(x) = 2x - 4$ are sketched below. E is the y-intercept of $f'(x)$ and B is the turning point of $f'(x)$.



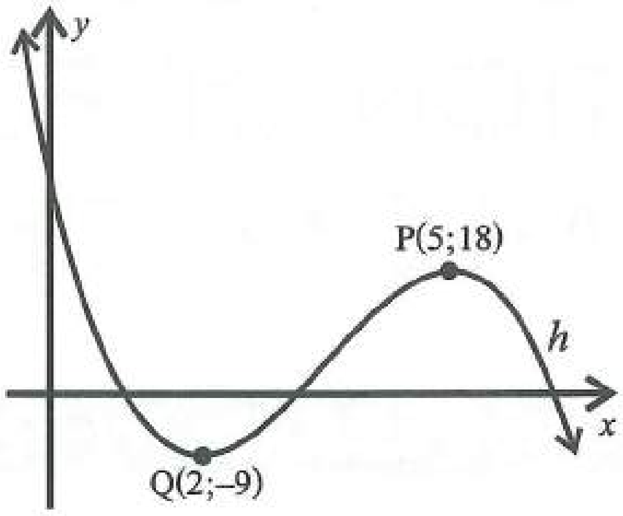
a) Determine the equation of the graph of $f'(x)$ in the form $f'(x) = ax^2 + bx + c$.

b) Determine the x-coordinates of the stationary points of $f(x)$.

c) What is the x-coordinate of the point of inflection of $f(x)$.

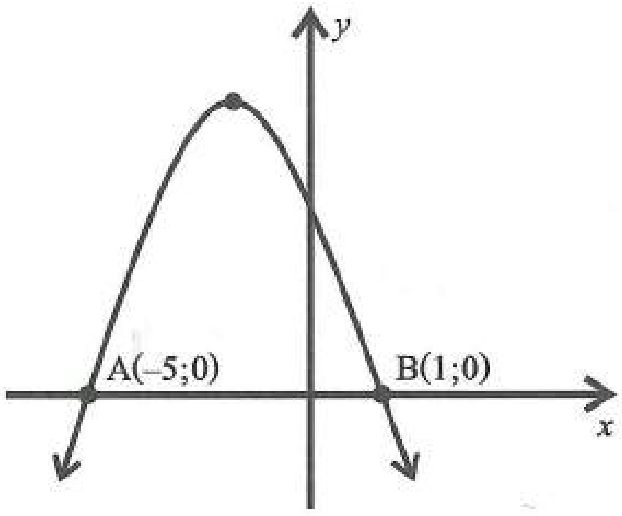
d) Does f have a local maximum or minimum where $x = -2$. Explain your answer.

5) **The function $h(x) = -2x^3 + ax^2 + bx + c$ is sketched below. If P and Q are stationary points of h:**



- **a)** **Determine the values of a, b and c.**
- **b)** **Determine the gradient of the tangent of h at the point where $x = 1$.**

6) **The graph of $y = f'(x)$, where f is a cubic function is sketched below:**



- **a)** **For which values of x is the graph of $f(x)$ increasing?**
- **b)** **For which values of x is the graph of $f'(x)$ increasing?**
- **c)** **What is the x-value of the point of inflection of f?**

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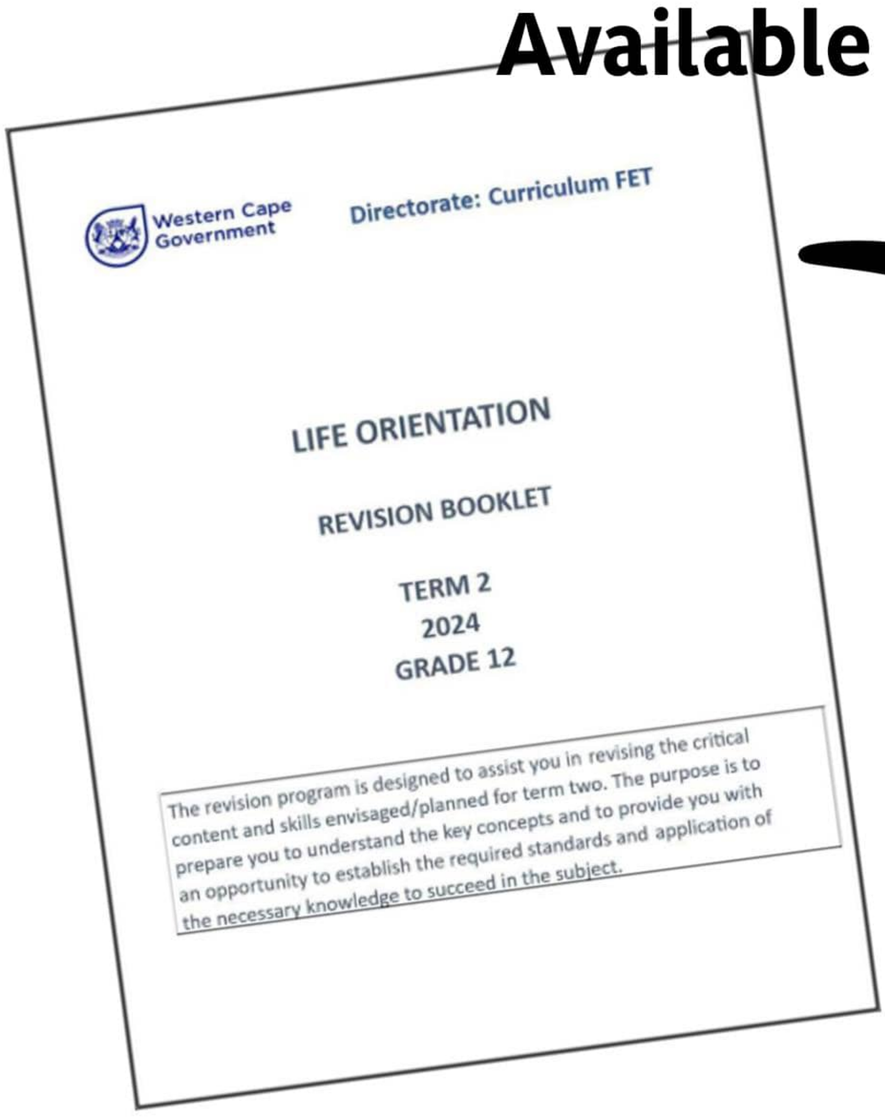

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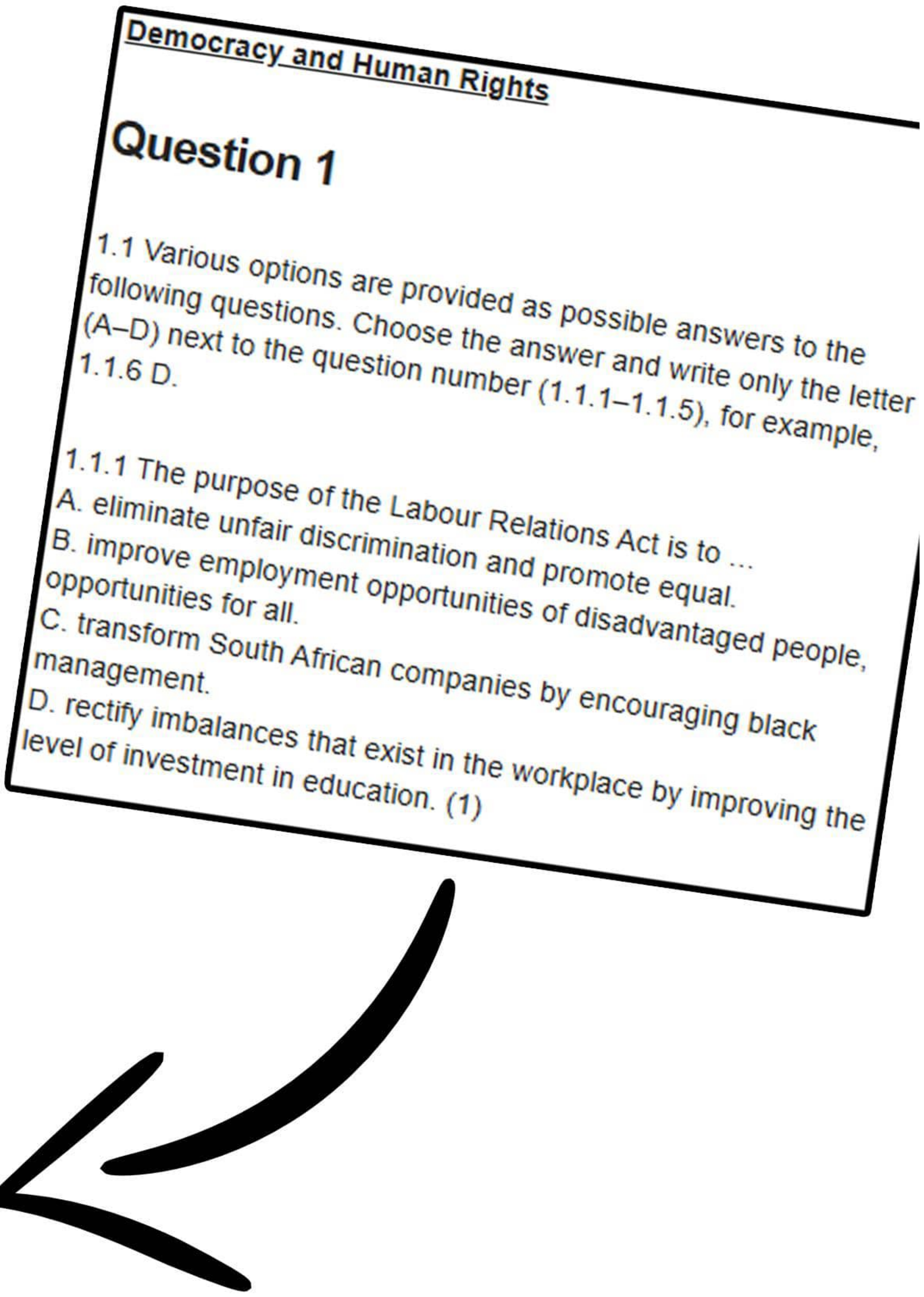
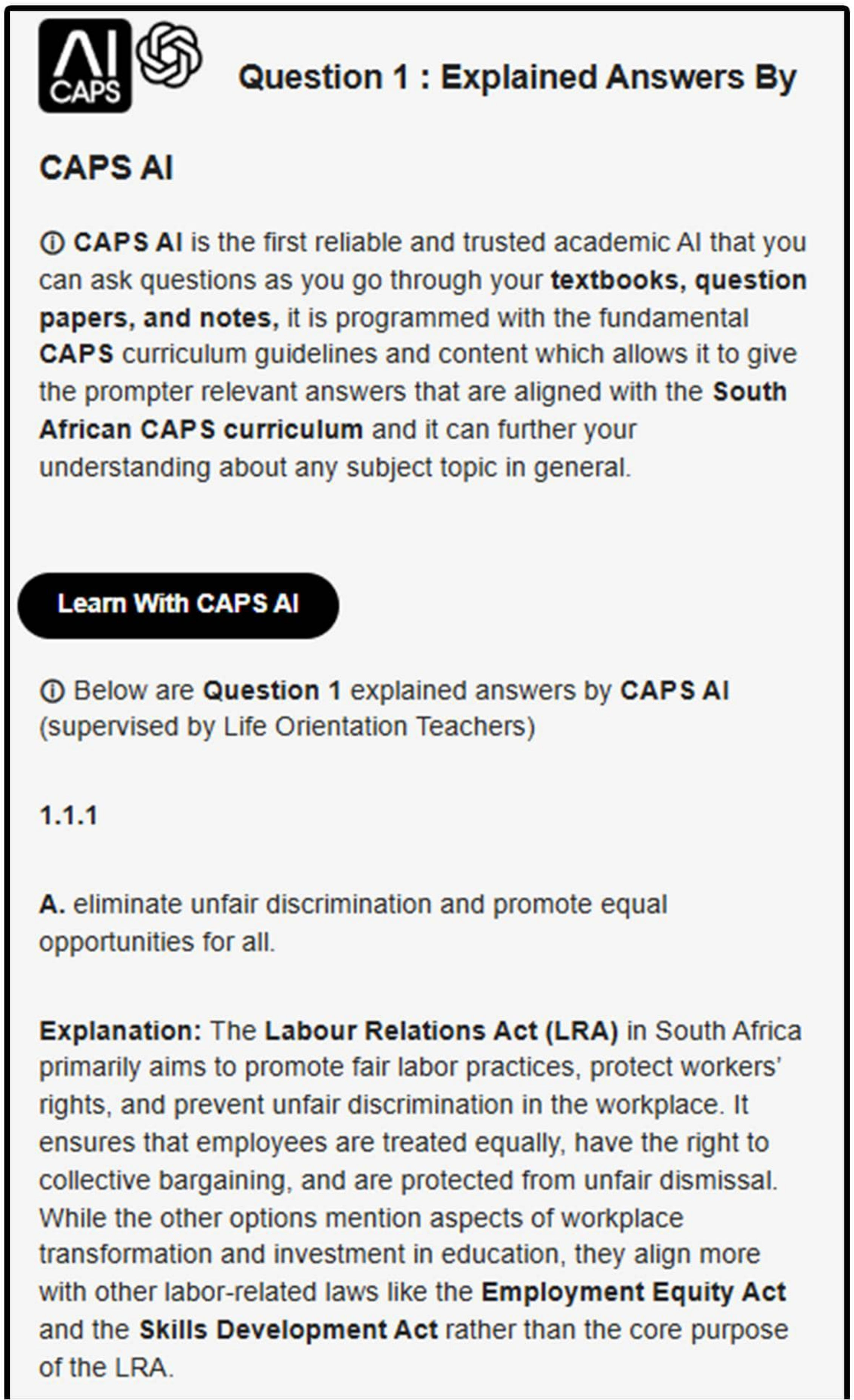


Democracy and Human Rights

Question 1

1.1 Various options are provided as possible answers to the following questions. Choose the answer and write only the letter (A–D) next to the question number (1.1.1–1.1.5), for example, 1.1.6 D.

1.1.1 The purpose of the Labour Relations Act is to …
A. eliminate unfair discrimination and promote equal opportunities for all.
B. improve employment opportunities of disadvantaged people.
C. transform South African companies by encouraging black management.
D. rectify imbalances that exist in the workplace by improving the level of investment in education. (1)



Question 1 : Explained Answers By CAPS AI

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ⓘ Below are **Question 1** explained answers by **CAPS AI** (supervised by Life Orientation Teachers)

1.1.1

A. eliminate unfair discrimination and promote equal opportunities for all.

Explanation: The **Labour Relations Act (LRA)** in South Africa primarily aims to promote fair labor practices, protect workers' rights, and prevent unfair discrimination in the workplace. It ensures that employees are treated equally, have the right to collective bargaining, and are protected from unfair dismissal. While the other options mention aspects of workplace transformation and investment in education, they align more with other labor-related laws like the **Employment Equity Act** and the **Skills Development Act** rather than the core purpose of the LRA.

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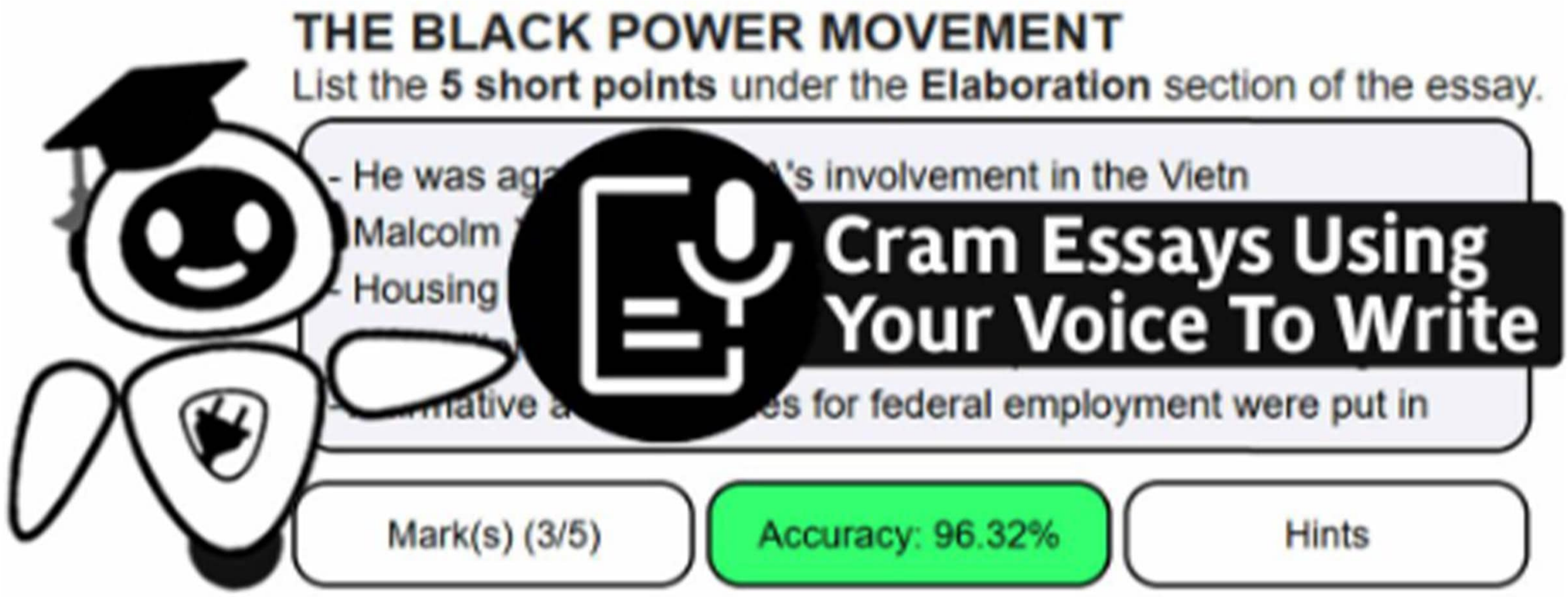

THE BLACK POWER MOVEMENT

List the **5 short points** under the **Elaboration** section of the essay.

- He was ag A's involvement in the Vietn
Malcolm
- Housing
mative a es for federal employment were put in

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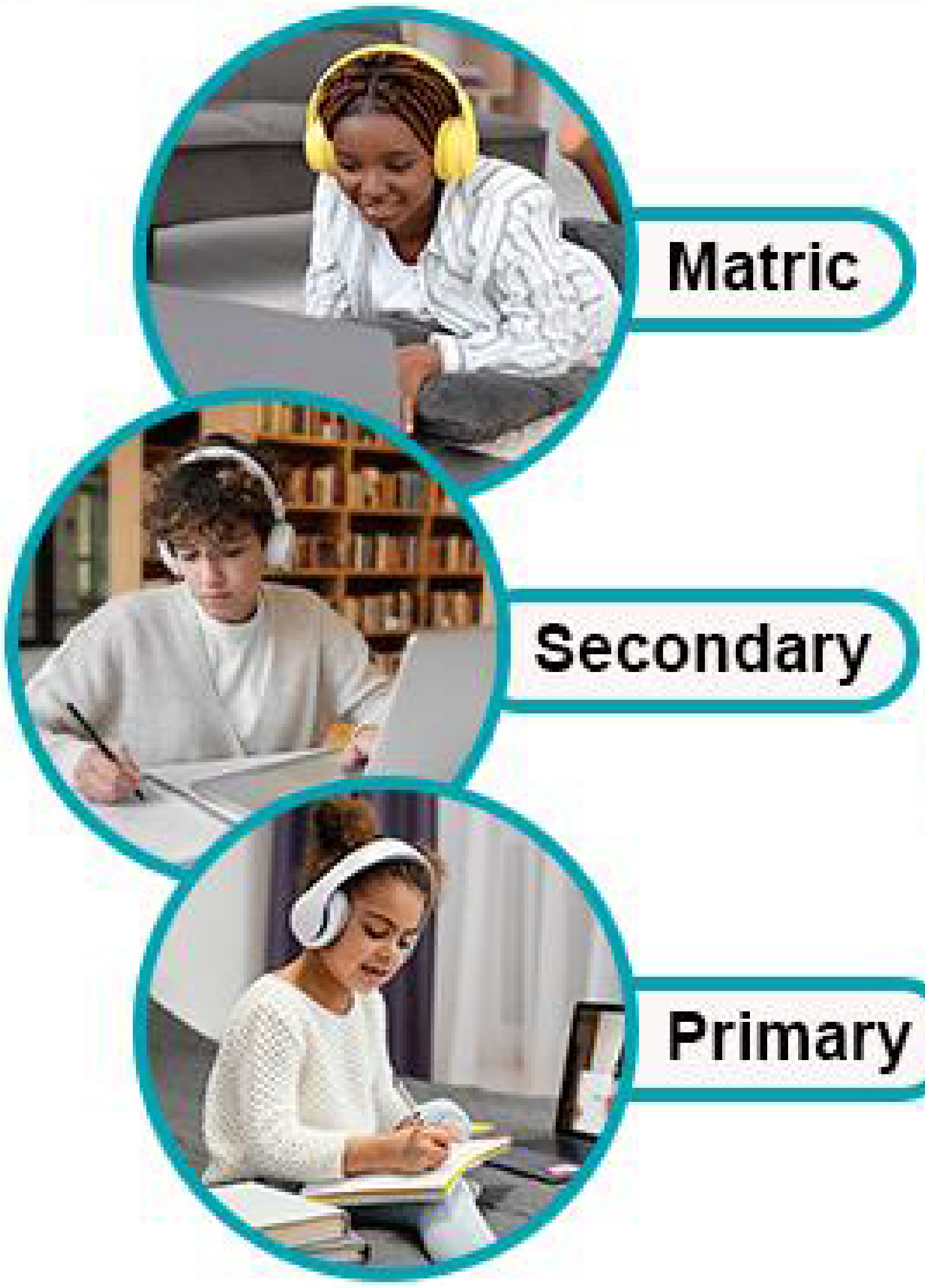

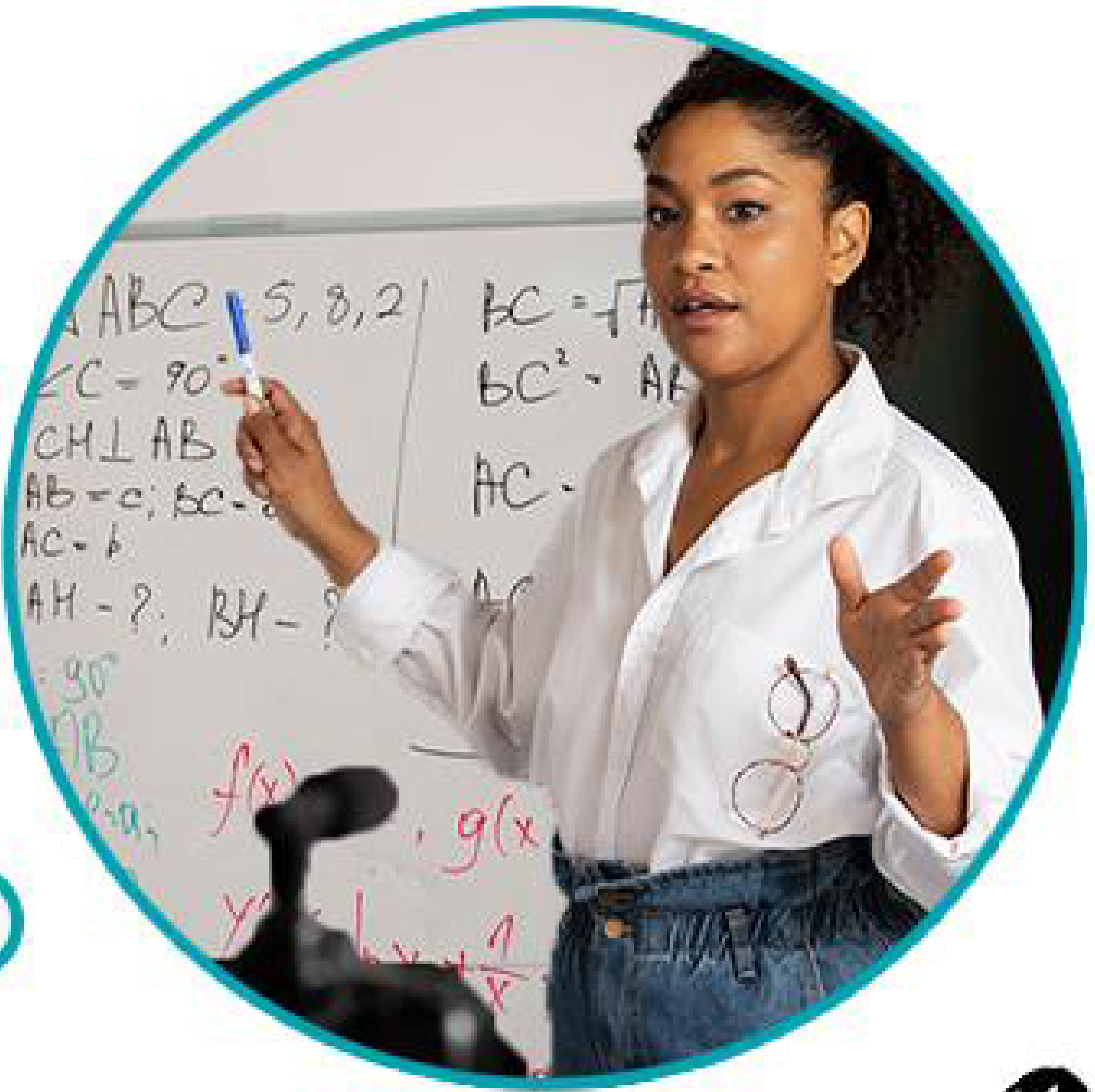

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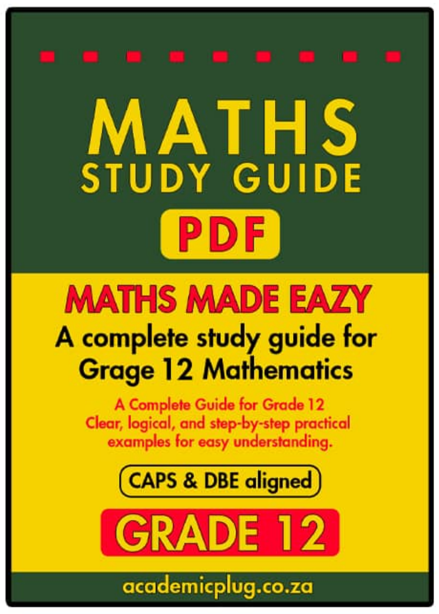

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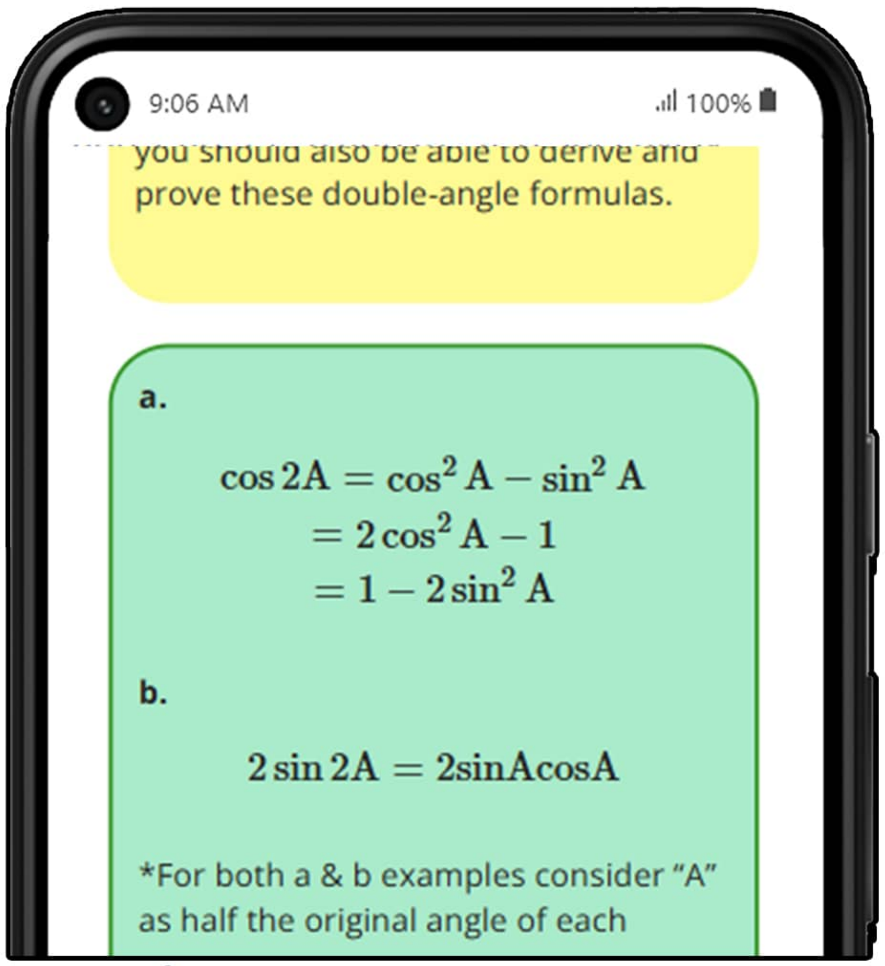

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you should also be able to derive and prove these double-angle formulas.

a.

$$\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A \\ &= 2\cos^2 A - 1 \\ &= 1 - 2\sin^2 A\end{aligned}$$

b.

$$2\sin 2A = 2\sin A\cos A$$

*For both a & b examples consider "A" as half the original angle of each

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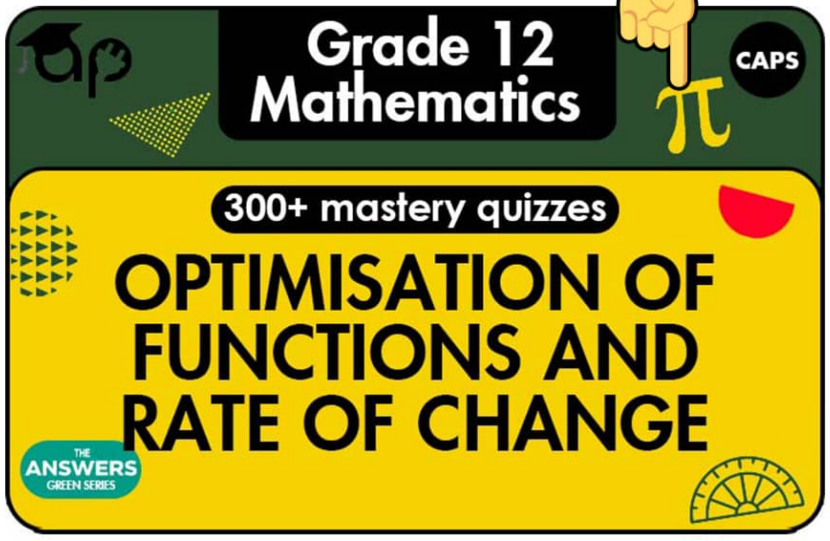

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CHAPTER 9

OPTIMISATION OF FUNCTIONS AND RATES OF CHANGE

WHAT YOU WILL LEARN IN THIS CHAPTER

PAGE	CONTENT	CAPS	ASSESSMENTS YOU WILL BE ABLE TO:
188	**A. Maxima and Minima Problems**	5	• Solve practical problems involving optimisation, rates of change and calculus of motion.
188	• Steps to Solving Maxima and Minima Problems		
195	**B. Rates of Change**		
199	**C. Calculus of Motion**		
200	• Velocity and Acceleration (Average and Instantaneous)		

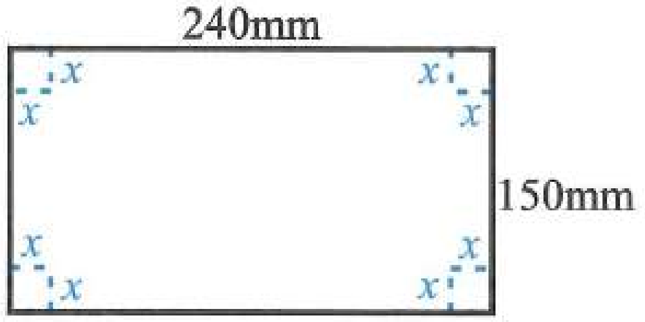
A. MAXIMA AND MINIMA PROBLEMS

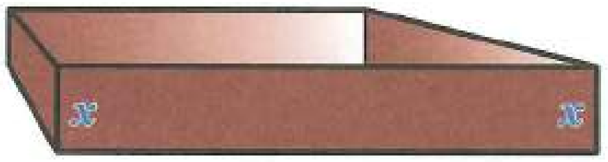
In all maxima and minima problems you will be given a function or have to formulate your own function. You will then have to determine the value of a variable in order to optimise the function. In order to find the maximum or minimum, you have to determine the derivative and set it equal to zero.

STEPS TO SOLVING MAXIMA AND MINIMA PROBLEMS

1. **Highlight what the question asks and make an expression for what has to be maximised or minimised.**
2. **Draw a diagram if possible.**
3. **If there are 2 variables, eliminate one using the given information and simultaneous equations.**
4. **Write down $f(x)$ and $f'(x)$**
5. **Set the derivative equal to 0 for maximum or minimum.**
6. **Answer the question.**

Example 1: A sheet of cardboard 240mm by 150mm has square edges of side x millimetres cut out. The edges are then folded up so as to form a box x millimetres high without a lid. Find x so that the volume of the box is a maximum, and hence find the maximum volume.





Volume $= l \times b \times h$

$\therefore V = (240-2x)(150-2x)(x)$

$= x\left(36000-780x+4x^2\right)$

$= 4x^3-780x^2+36000x$

$V'(x) = 12x^2-1560x+36000 \rightarrow$ derivative of $V(x)$

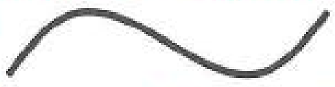
For max or min $V'(x)=0$

$\therefore 12x^2-1560x+36000=0 \rightarrow$ divide both sides by 12

$\therefore x^2-130x+3000=0$

$\therefore (x-30)(x-100)=0$

$\therefore x=30$ or $x=100$

You now need to determine which x value will result in a maximum. From the equation for volume, it can be seen that the coefficient of x^3 is positive. This means the graph of volume for this container will have the shape: 

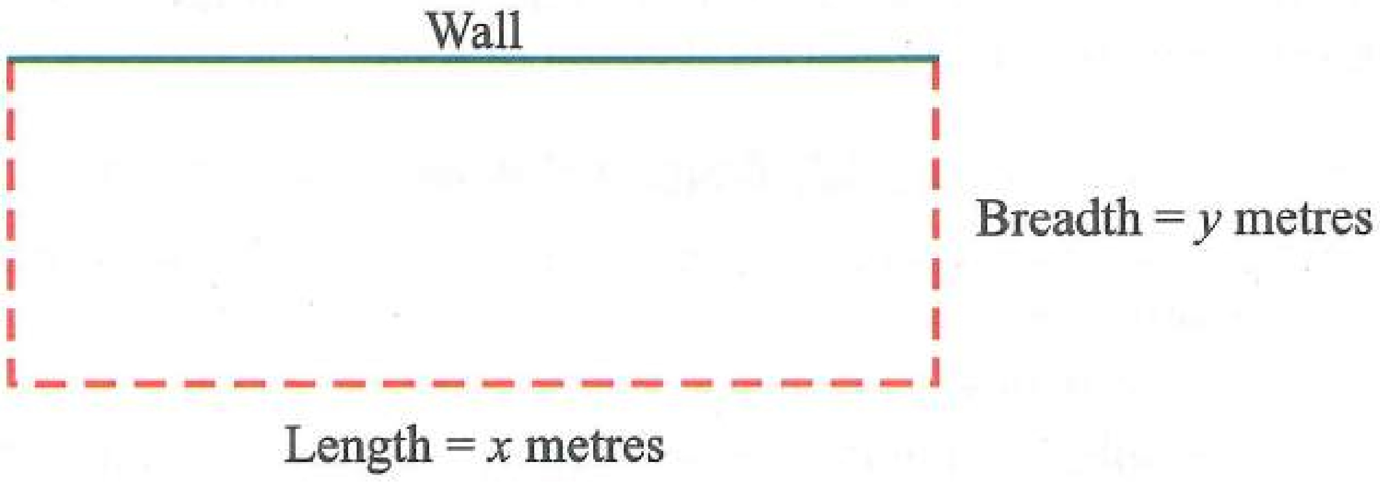
This means that the lowest x-value of your solution will be a maximum.

$\therefore$ maximum at $x=30$ mm

In order to determine the maximum volume, you now substitute $x=30$ into the equation for volume, $V(x)$ not $V'(x)$.

$\therefore$ Max Volume $= [240-2(30)][150-2(30)](30) = 486000\text{mm}^3$

Example 2: **A farmer has 200m of wire fencing from which to build a rectangular enclosure. He intends to use an existing wall for one of the sides. Find the dimensions that will result in a maximum enclosed area.**



Area $= l \times b$ and Perimeter $= 2y + x$

$\therefore A(y) = xy \rightarrow \boxed{1}$ and $\therefore 2y + x = 200 \rightarrow$ perimeter is equal to the length of the wire

$\therefore x = 200 - 2y \rightarrow \boxed{2}$

Substitute $\boxed{2}$ into $\boxed{1}$:

$$\begin{aligned}\therefore A(y) &= xy \\ &= (200 - 2y)\,y \\ &= 200y - 2y^2\end{aligned}$$

$A'(y) = 200 - 4y$

For max or min, $A'(y) = 0$:

$\therefore 200 - 4y = 0$

$\therefore 4y = 200$

$\therefore y = 50 \rightarrow \boxed{3}$

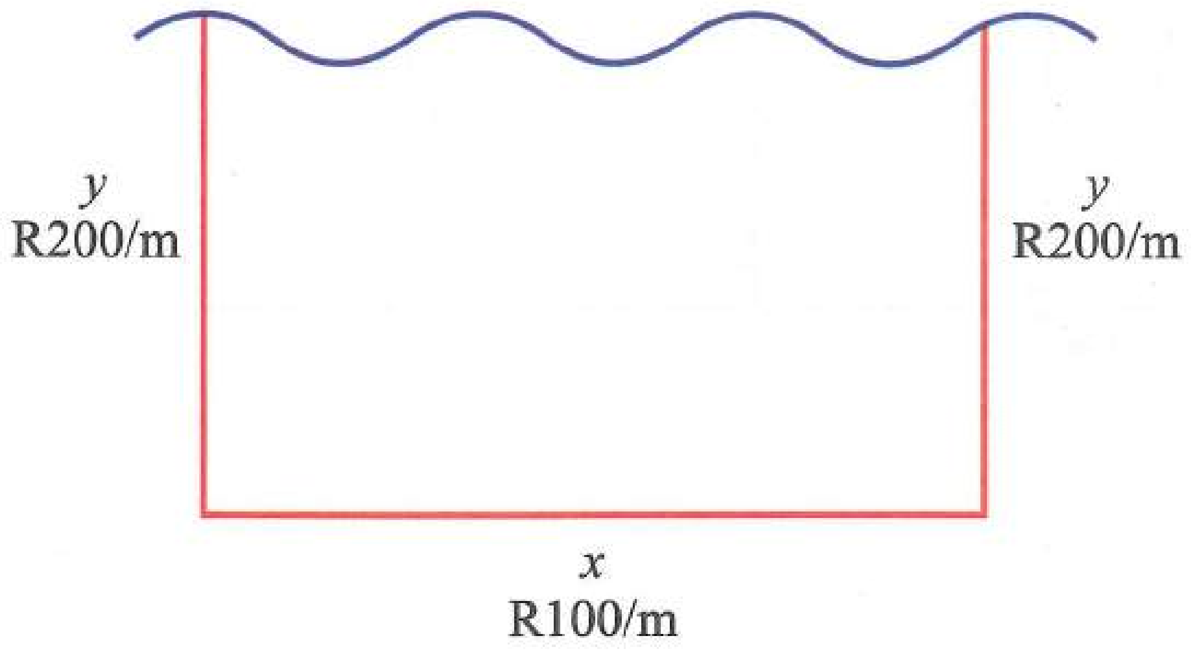
Substitute $\boxed{3}$ into $\boxed{2}$:

$\therefore x = 200 - 2(50)$

$\therefore x = 100$

$\therefore$ The length must be 100m and the breadth must be 50m for a maximum enclosed area.

Example 3: **A rectangular enclosure is to be fenced in on three sides, with the fourth side being formed by a river. The cost of fencing at right angles to the river is R200/m while that parallel to the river is R100/m. If the enclosed area is to be 100m^2, find the dimensions so that the cost of fencing is a minimum. Verify your answer is a minimum by performing the 2nd derivative test.**



$Area = l \times b$

$\therefore xy = 100$

$\therefore y = \dfrac{100}{x} \rightarrow \boxed{1}$

$\text{Cost} = C = 200y + 100x + 200y$

$\therefore C = 400y + 100x \rightarrow \boxed{2}$

Substitute $\boxed{1}$ into $\boxed{2}$:

$$\therefore C(x) = 400\left(\frac{100}{x}\right) + 100x$$

$$= 40000x^{-1} + 100x$$

$C'(x) = 100 - 40000x^{-2}$

For max or min, $C'(x) = 0$:

$$\therefore 100 - \frac{40000}{x^2} = 0$$

$$\therefore 100x^2 - 40000 = 0$$

$$\therefore x^2 - 400 = 0$$

$$\therefore (x-20)(x+20) = 0$$

$\therefore x = 20$ or $x = -20 \rightarrow$ N/A as length cannot be negative

Use the 2nd derivative test to check if the answer is a minimum:

$$\therefore C''(x) = \frac{80000}{x^3}$$

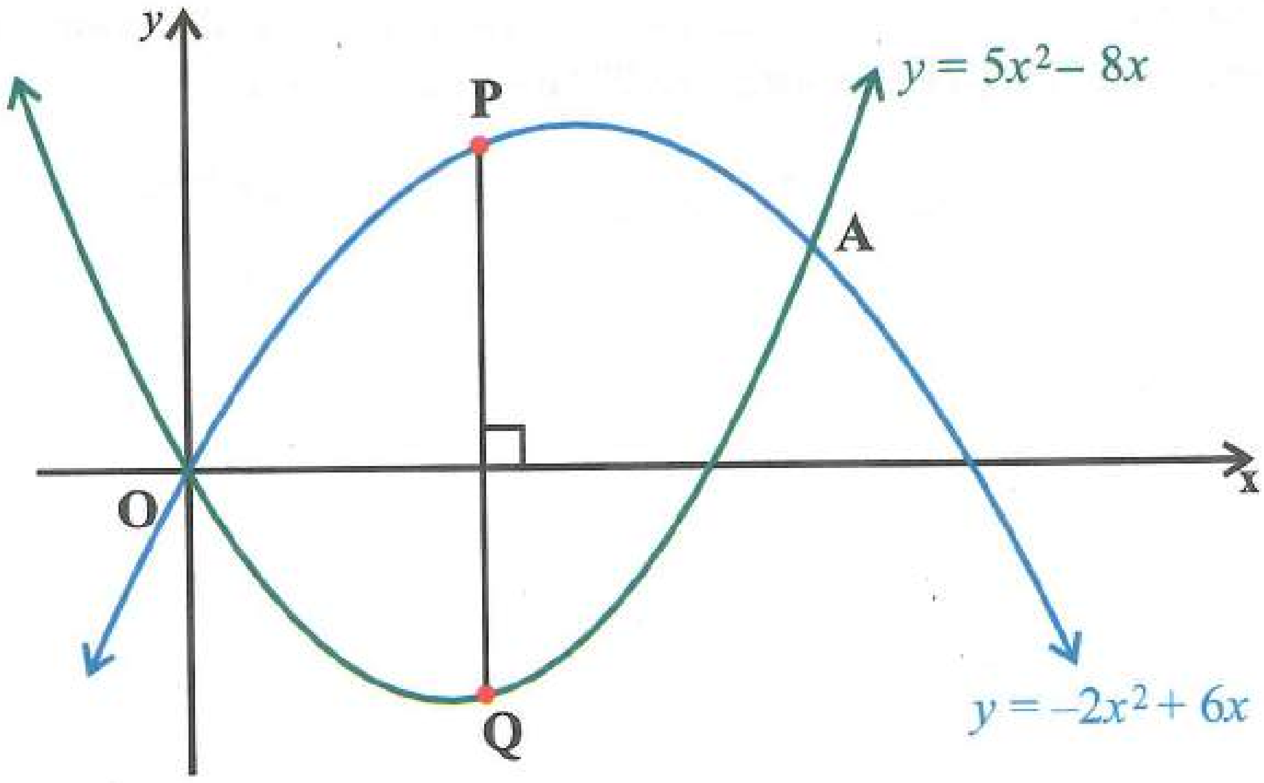
$$\therefore C''(20) = 10$$

$$\therefore C''(20) > 0$$

$\therefore$ Minimum cost at $x = 20$ and $y = 5 \rightarrow$ found by substituting $x = 20$ into equation $\boxed{1}$

$\therefore$ You need 20m of fencing parallel to the river and 5m at right angles to the river.

Example 4: **The line PQ lies anywhere between the origin and A as shown below. Determine the maximum length that PQ can have.**



$PQ = y_{top} - y_{bottom}$

$\therefore PQ(x) = -2x^2 + 6x - (5x^2 - 8x)$

$= -2x^2 + 6x - 5x^2 + 8x$

$= -7x^2 + 14x$ → this function represents the length of PQ at different x-values

$PQ'(x) = -14x + 14$

For max or min, $PQ'(x) = 0$:

$\therefore -14x + 14 = 0$

$\therefore x = 1$

$\therefore$ Maximum at $x = 1$

In order to calculate the maximum length of PQ, substitute $x = 1$ into the original equation, not the derivative.

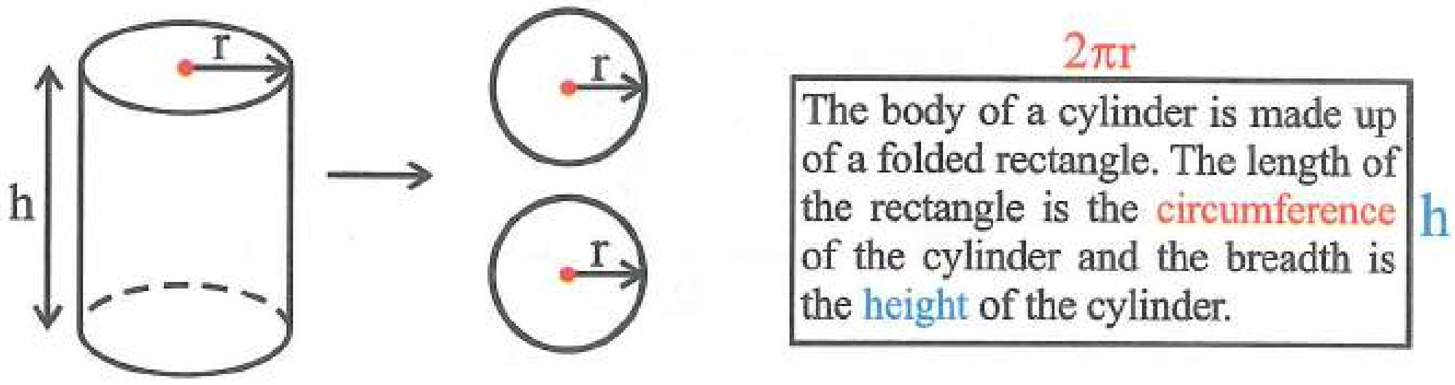
$\therefore PQ(x) = -7x^2 + 14x$

$\therefore PQ(1) = -7(1)^2 + 14(1)$

$= 7$

$\therefore$ Maximum length of PQ is equal to 7 units → always put in units

Example 5: **A cylindrical container has a capacity of 64cm^3. Find the dimensions so that the amount of metal needed to make the closed cylinder is a minimum. Use the 2nd derivative test to verify that your answer is minimum.**



The body of a cylinder is made up of a folded rectangle. The length of the rectangle is the circumference of the cylinder and the breadth is the height of the cylinder.

The amount of metal is the surface area of the cylinder.

$\text{Surface Area} = 2\pi r^2 + 2\pi rh \rightarrow \boxed{1}$ and $V = \pi r^2 h$

$\therefore 64 = \pi r^2 h$

$\therefore h = \dfrac{64}{\pi r^2} \rightarrow \boxed{2}$

Substitute $\boxed{2}$ into $\boxed{1}$:

$\therefore A(r) = 2\pi r^2 + 2\pi r\left(\dfrac{64}{\pi r^2}\right)$

$= 2\pi r^2 + \dfrac{128}{r}$

$= 2\pi r^2 + 128r^{-1}$

$A'(r) = 4\pi r - 128r^{-2}$

For max or min, $A'(r) = 0$:

$\therefore 4\pi r - \dfrac{128}{r^2} = 0$

$\therefore 4\pi r^3 - 128 = 0$

$\therefore r^3 = \dfrac{128}{4\pi}$

$\therefore r = \sqrt[3]{\dfrac{128}{4\pi}} = 2{,}17$

Use the 2nd derivative to check that the answer is a minimum:

$\therefore A''(r) = 4\pi + \dfrac{256}{r^3}$

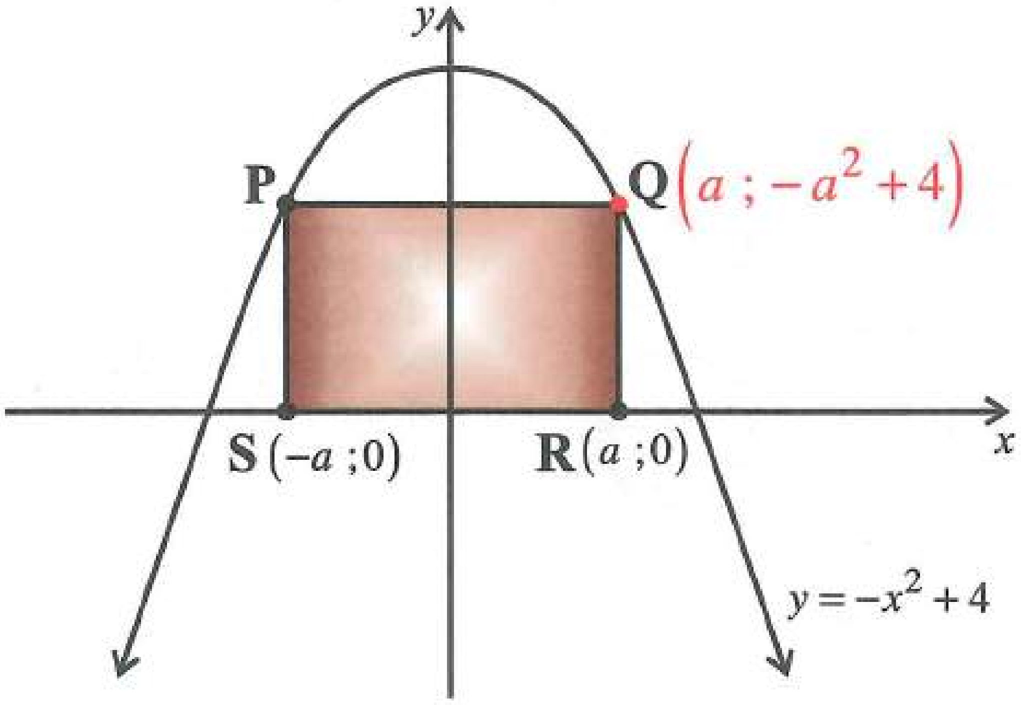
$\therefore A''(2{,}17) = 37{,}62$

$\therefore A''(2{,}17) > 0$

$\therefore$ Minimum surface area at $r = 2{,}17$ and $h = \dfrac{64}{\pi(2{,}17)^2} = 4{,}33$

$\therefore$ A radius of $2{,}17\,\text{cm}$ and a height of $4{,}33\,\text{cm}$ is required

Example 6: **The equation of the parabola shown below is $y = -x^2 + 4$. Determine the dimensions of PQRS that will result in a maximum area.**



$A_{PQRS} = l \times b = SR \times QR$

Let the length of $SR = 2a$ *units*

$\therefore$ The coordinates are:

$S:(-a;0)$ $\qquad$ $R:(a;0)$

$SR = 2a$

and

$QR = y_{top} - y_{bottom} = y_Q - y_R$

The y-value at Q is found by substituting the x-value at Q into $y = -x^2 + 4$

$\therefore y_Q = -a^2 + 4$

$\therefore QR = -a^2 + 4 - (0)$

$\therefore A(a) = 2a\left(-a^2 + 4\right)$

$= -2a^3 + 8a$

$A'(a) = -6a^2 + 8$

For max or min, $A'(a) = 0$:

$\therefore -6a^2 + 8 = 0$

$\therefore a^2 = \frac{8}{6}$

$\therefore a^2 = \frac{4}{3}$

$\therefore a = \pm\sqrt{\frac{4}{3}}$ → the negative answer is not applicable as a length cannot be negative

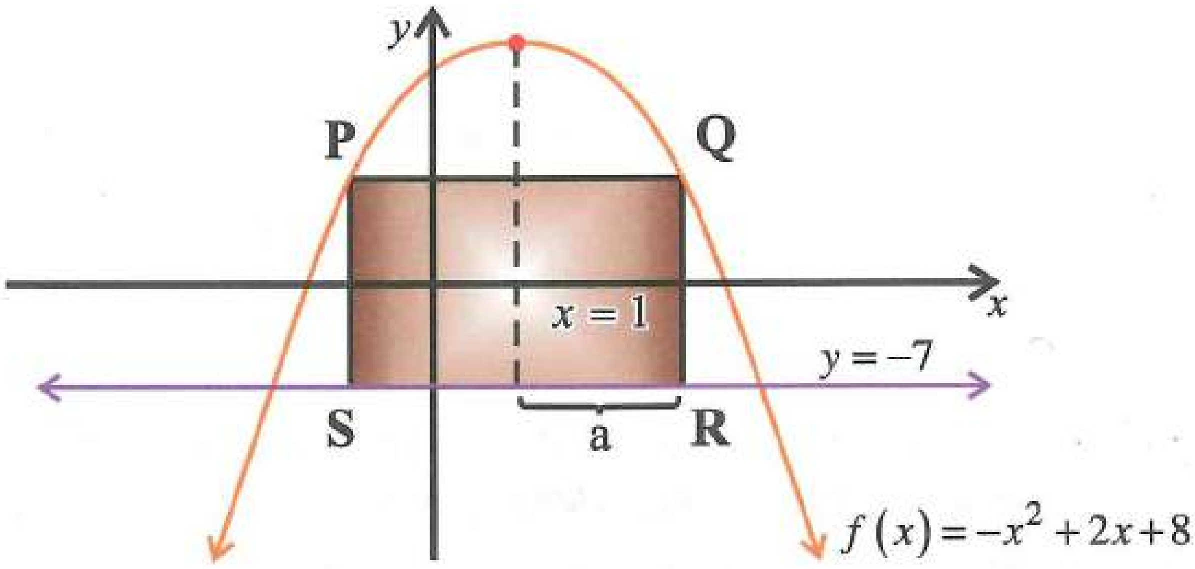
$= \frac{2}{\sqrt{3}}$

$\therefore$ The dimensions are:

$SR = PQ = 2 \times \frac{2}{\sqrt{3}} = \frac{4}{\sqrt{3}}$ units and $PS = QR = -\left(\frac{2}{\sqrt{3}}\right)^2 + 4 = \frac{8}{3}$ units → always put in units

Example 7: The equation of the parabola shown below is $f(x)=-x^2+2x+8$ and the equation of the straight line is $y=-7$.

Determine the length of QR and SR that will result in a maximum area, given that the x-value of the turning point of f is equal to 1 and SR is $2a$ units in length.



$\text{Area}_{PQRS}=\text{SR}\times\text{QR}$

$\text{QR}=y_{top}-y_{bottom}=y_Q-y_R$

The x-value at the turning point is equal to 1. This means that the x-value at R and Q is equal to: $1+a$

$$\therefore y_Q=f(1+a)=-(1+a)^2+2(1+a)+8$$
$$=-\left(1+2a+a^2\right)+2+2a+8$$
$$=-1-2a-a^2+2+2a+8=-a^2+9$$

R is on the line $y=-7$. This means the y-value at R is equal to -7.

$$\therefore \text{QR}=-a^2+9-(-7)\rightarrow y_Q-y_R$$
$$=-a^2+16$$

$$\therefore \text{A}(a)=2a\times\left(-a^2+16\right)\rightarrow \text{Area in terms of } a=\text{SR}\times\text{QR}$$
$$=-2a^3+32a$$

$$\therefore \text{A}'(a)=-6a^2+32$$

For max or min, $\text{A}'(a)=0$:

$$\therefore -6a^2+32=0$$

$$\therefore a^2=\frac{16}{3}$$

$$\therefore a=\pm\frac{4}{\sqrt{3}}\rightarrow \text{the negative is not applicable as a length can't be negative}$$

$$\therefore a=\frac{4}{\sqrt{3}}$$

$$\therefore \text{QR}=-\left(\frac{4}{\sqrt{3}}\right)^2+16 \quad\text{and}\quad \text{SR}=2\left(\frac{4}{\sqrt{3}}\right)$$

$$=\frac{32}{3}\text{ units} \qquad\qquad =\frac{8}{\sqrt{3}}\text{ units}$$

Complete Exercise 1 on pages 202 and 203.

B. RATES OF CHANGE

The derivative $f'(x)$ is the instantaneous rate of change of y with respect to x.
Whenever you are dealing with a problem where one variable is changing with respect to another you will have to use differentiation.

Recap: Average Gradient (average rate of change)

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Example 1: If the number of cells (n) dividing in a human hair is governed by the equation $n=12t^2+350t+5000$, where t is time in minutes, at what rate are the cells dividing after 3 minutes?

$n(t) = 12t^2 + 350t + 5000$

$\therefore n'(t) = 24t + 350$

$\therefore n'(3) = 24(3) + 350 = 422$ cells / min → always put in units

Example 2: The volume of acid in a tank is governed by the equation $V = 12 + 10t - t^2$ (where V is in m^3 and t is in minutes). Determine:

a) The initial volume.
b) The rate at which the volume is increasing when $t = 2$ min.
c) The volume at $t = 2$ min.
d) The rate at which the volume is increasing at $t = 3$ min.
e) The maximum volume in the tank.
f) At what time the volume starts to decrease?
g) At what time the tank is empty?
h) The rate at which the volume is changing at $t = 7$ min.
i) The time at which volume is decreasing at a rate of 2m^3/min?

Before starting to answer the questions, determine the derivative of V:

$V = 12 + 10t - t^2$ → volume

$\therefore V'(t) = 10 - 2t$ → $V'(t)$ = rate of change of volume with respect to time

a) $V(0) = 12 + 10(0) - (0)^2$

$= 12\text{m}^3$

b) $V'(2) = 10 - 2(2)$

$= 6\text{m}^3/\text{min}$

c) $V(2) = 12 + 10(2) - (2)^2$

$= 28\text{m}^3$

d) $V'(3)=10-2(3)$

$=4\text{ m}^3/\text{min}$

e) For max. or min. $V'(t)=0$

$\therefore 10-2t=0$

$\therefore -2t=-10$

$\therefore t=5 \rightarrow$ this is the time when maximum volume occurs

$\therefore V_{\max}=12+10(5)-(5)^2$

$=37\text{ m}^3$

f) Volume is decreasing when $V'(t)<0$:

$\therefore 10-2t<0$

$\therefore 10<2t$

$\therefore 5<t$

$\therefore$ The volume will start to decrease after 5 minutes.

g) The tank will be empty when $V(t)=0$:

$\therefore -t^2+10t+12=0$

$\therefore t^2-10t-12=0 \rightarrow$ use the quadratic formula

$$\therefore t=\frac{-(-10)\pm\sqrt{(-10)^2-4(1)(-12)}}{2(1)} \rightarrow t=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\therefore t=\frac{10\pm2\sqrt{37}}{2}$$

$$\therefore t=\frac{\cancel{2}\left(5\pm\sqrt{37}\right)}{\cancel{2}}$$

$\therefore t=5+\sqrt{37} \rightarrow$ only '+' as time cannot be negative

$\therefore t=11{,}08$ minutes

h) $V'(7)=10-2(7)$

$=-4\text{ m}^3/\text{min} \rightarrow$ the '$-$' means that volume is decreasing when $t=7$

i) $V'(t)=-2 \rightarrow$ volume is decreasing, $\therefore$ the derivative is negative

$\therefore 10-2t=-2$

$\therefore -2t=-12$

$\therefore t=6$ minutes

Example 3: During an experiment, the concentration of glucose $M(t)$ in the human body varied according to the formula:

$M(t) = 4,5 - 0,03t^2$ where t is measured in hours.

- **a) Determine the average rate of change of glucose concentration in the body over the interval $t = 0$ hours to $t = 3$ hours.**
- **b) Determine the instantaneous rate of change of glucose concentration when $t = 1$.**
- **c) Did the concentration of glucose increase or decrease during the first three hours?**

a) average rate of change $= \dfrac{M(3) - M(0)}{3 - 0}$

$$= \frac{\left[4,5 - 0,03(3)^2\right] - \left[4,5 - 0,03(0)^2\right]}{3}$$

$= -0,09$ units/hour

b) $M'(t) = -0,06t$

$M'(1) = -0,06(1)$

$\therefore M'(1) = -0,06$ glucose units/hour

c) It was shown in (*a*) that the average rate of change over the first 3 hours was $-0,09$ units per hour. Therefore, the concentration decreased.

Example 4: The volume, V, of water in a kettle t minutes after it starts to boil, is given by: $V(t) = 86 - \frac{1}{8}t - \frac{1}{4}t^3$ where V is measured in ml.

- **a) How many millilitres of water are there in the kettle just before it starts to boil?**
- **b) As the water boils, the volume in the kettle decreases. Determine the rate at which the volume is decreasing when $t = 2$ minutes.**

a) $V(0) = 86 - \frac{1}{8}(0) - \frac{1}{4}(0)^3$ → the initial volume is when $t = 0$

$= 86\,\text{ml}$

b) $V'(t) = -\frac{1}{8} - \frac{3}{4}t^2$ → the derivative is the rate of change

$\therefore V'(2) = -\frac{1}{8} - \frac{3}{4}(2)^2$

$= -\frac{25}{8}$ ml/min

Example 5: The profit yielded by a flight charter company is dependent on the average speed at which its plane is flown. The profit (P) in Rands/hour is calculated using the formula:

$$P(x) = -\frac{1}{25}x^2 + 52x - 16000$$

Where x is the average speed in km/h and $x \geq 600$.

a) What is the speed at which no profit is yielded?

b) Find the most economical speed and the corresponding profit per hour.

a) No Profit is made when $P(x) = 0$

$\therefore -\frac{1}{25}x^2 + 52x - 16000 = 0 \rightarrow$ mulitply both sides by $-\frac{1}{25}$

$\therefore x^2 - 1300x + 400000 = 0$

$\therefore (x - 800)(x - 500) = 0$

$\therefore x = 800$ or $x = 500 \rightarrow$ N/A as $x \geq 600$

$\therefore$ No profit is made at 800 km/h

b) $P(x) = -\frac{1}{25}x^2 + 52x - 16000$

$\therefore P'(x) = -\frac{2}{25}x + 52$

For max or min $P'(x) = 0$:

$\therefore -\frac{2}{25}x + 52 = 0 \rightarrow$ multiply both sides by 25

$\therefore -2x + 1300 = 0$

$\therefore -2x = -1300$

$\therefore x = 650 \rightarrow$ max profit is at 650 km/h

In order to determine the maximum profit substitute $x = 650$ into $P(x)$.

$$\therefore P(650) = -\frac{1}{25}(650)^2 + 52(650) - 16000$$
$$= 900$$

$\therefore$ Maximum profit is R900 per hour.

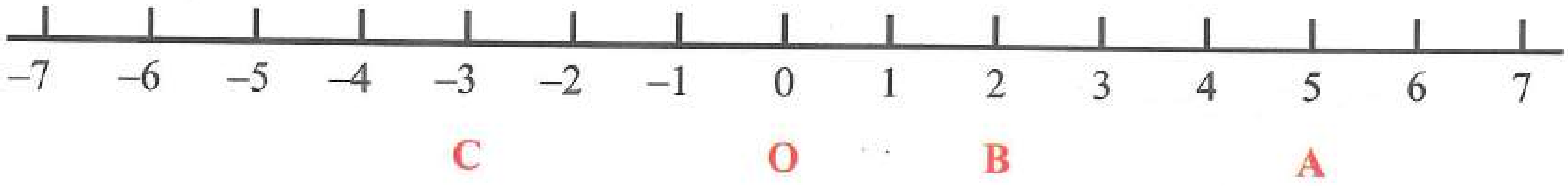
The quadratic equation in (a) could also have been solved using the quadratic formula if you were unable to factorise the expression.

Complete Exercise 2 on page 203.

C. CALCULUS OF MOTION

Calculus of motion is an extension of the rates of change section. In this section, we deal with rates of change with regard to displacement, velocity and acceleration.

- **Displacement refers to the distance covered in a certain direction.**
- **Velocity refers to the speed of an object in a specific direction and is the rate of change (derivative) of displacement.**
- **Acceleration is the rate of change (derivative) of velocity.**



On the above number line, O is the origin and the lengths are measured in metres.
If a particle moves from O to A then back to B, then:
Distance = 8 metres
Displacement = 2 metres

For a particle that moved from O to C:
Displacement = -3m → the negative indicates direction

Displacement → $s(t)=0$: at the origin, where $t=$ time
$s(t)>0$: movement to the right of the origin
$s(t)<0$: movement to the left of the origin

Velocity is a speed in a certain direction. If a particle moves to the right, its velocity is positive.

For example, if a particle moves from C to B, the velocity is positive and the displacement is increasing. If a particle's velocity is negative, it does not necessarily mean that it is decelerating but rather that it is moving towards the left and its displacement is decreasing.

Velocity → $v(t)>0$: Particle moves right: $s(t)$ is increasing
$v(t)<0$: Particle moves left: $s(t)$ is decreasing

VELOCITY AND ACCELERATION (AVERAGE AND INSTANTANEOUS)

Avg. velocity $= \frac{\Delta s}{\Delta t} \rightarrow$ change $= \frac{s_2 - s_1}{t_2 - t_1}$ (m/s or km/h)	Instantaneous velocity $= v(t)$ Instantaneous velocity $= s'(t)$
Avg. acceleration $= \frac{\Delta v}{\Delta t} \rightarrow$ change $= \frac{v_2 - v_1}{t_2 - t_1}$ (m/s^2 or km/h^2)	Instantaneous acceleration $= a(t)$ Instantaneous acceleration $= v'(t)$ Instantaneous acceleration $= s''(t)$

Example 1: The displacement of a particle is given by:
$s(t) = 120 - 10t$ where s is in metres and t is in seconds.

a) Find the displacement when
  i) $t = 0$
  ii) $t = 1$
  iii) $t = 2$
  iv) Explain what these values tell you about the motion of the particle?

b) Determine the velocity, $v(t)$ of the particle.

c) What is the speed of the particle?

Displacement	*Velocity*	*Acceleration*
$s(t) = 120 - 10t$	$s'(t) = v(t) = -10$	$s''(t) = v'(t) = a(t) = 0$

a)

i) $s(0) = 120 - 10(0)$
$= 120\text{ m}$

ii) $s(1) = 120 - 10(1)$
$= 110\text{ m}$

iii) $s(2) = 120 - 10(2)$
$= 100\text{ m}$

iv) It can be seen from the above answers that the particle is moving at a constant velocity as the change in displacement is constant every second. As the displacement is decreasing it can be seen that the particle is moving in the negative direction.

b) $v(t) = s'(t) = -10\text{ m/s}$

c) It is important to remember that the difference between speed and velocity is that speed does not have a direction.
$\therefore$ The speed of the particle $= 10\text{ m/s}$

Example 2: **A stone (A) is thrown vertically upwards and it reaches a height of S_A meters above the ground after t seconds according to:**

$$S_A(t) = 30t - 5t^2$$

At the same instant, another stone (B) is dropped from the roof of a building 120m above the ground. It falls a distance of S_B meters in t seconds according to:

$$S_B(t) = 5t^2$$

a) At what time(s) is A 25m above the ground?
b) How long will it take A to reach its maximum height and what is this maximum height?
c) Calculate the height of B above the ground after 2 seconds.
d) After how many seconds will A and B be at the same height above the ground for the first time?

Displacment $S_A(t) = 30t - 5t^2$	*Velocity* $S_A'(t) = v(t) = 30 - 10t$
Displacment $S_B(t) = 5t^2$	*Velocity* $S_B'(t) = v(t) = 10t$

a) $S_A(t) = 25$

$\therefore 30t - 5t^2 = 25$

$\therefore 5t^2 - 30t + 25 = 0$

$\therefore t^2 - 6t + 5 = 0$

$\therefore (t-1)(t-5) = 0$

$\therefore t = 1$ or $t = 5$

$\therefore$ A is 25m above the ground after 1 second and then again after 5 seconds

b) $S_A'(t) = 0 \rightarrow$ maximum height when velocity $= 0$

$\therefore 30 - 10t = 0$

$\therefore t = 3\text{s}$

$$\therefore S_A(3) = 30(3) - 5(3)^2$$
$$= 45\text{ m}$$

c) $S_B(2) = 5(2)^2 = 20$

$\therefore$ Height above the ground of $B = 120 - 20 \rightarrow$ B started 120 m above the ground

$$= 100\text{ m}$$

d) $S_A = 120 - S_B \rightarrow$ B started 120 m above the ground

$\therefore 30\text{t} - 5t^2 = 120 - 5t^2$

$\therefore 30t = 120$

$\therefore t = 4s$

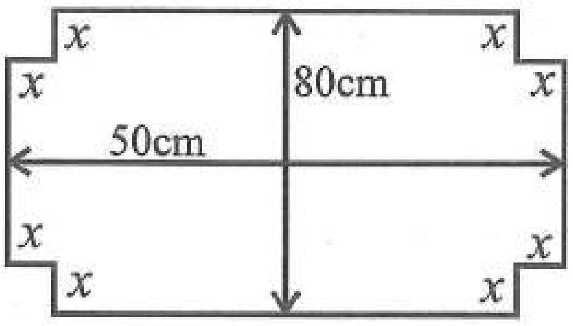
Complete Exercise 3 on page 204 and the Mixed Exercise on pages 204 and 205.

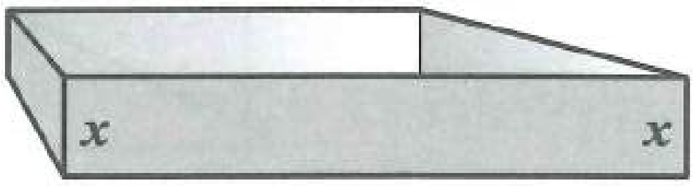
Chapter 9 Optimisation of Functions and Rates of Change (Exercises)

Maxima and Minima Problems

Exercise 1: (answers on page 358)

1) **A piece of aluminium 50cm by 80cm has square corners of length x cm cut out. The edges are then folded up and welded together so as to form a box x centimetres high, as shown below.**

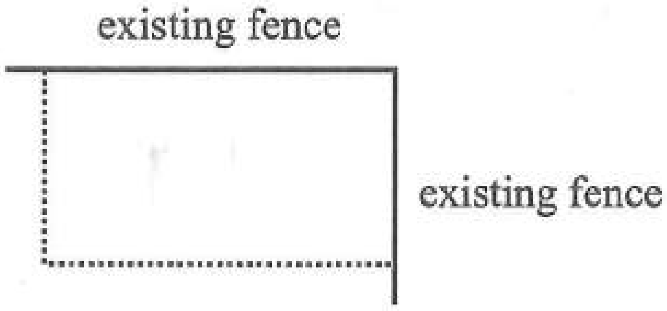




   a) Show that the volume of the box can be expressed as $V(x) = 4000x - 260x^2 + 4x^3$.

   b) Determine the value of x that will maximise the volume of the box.

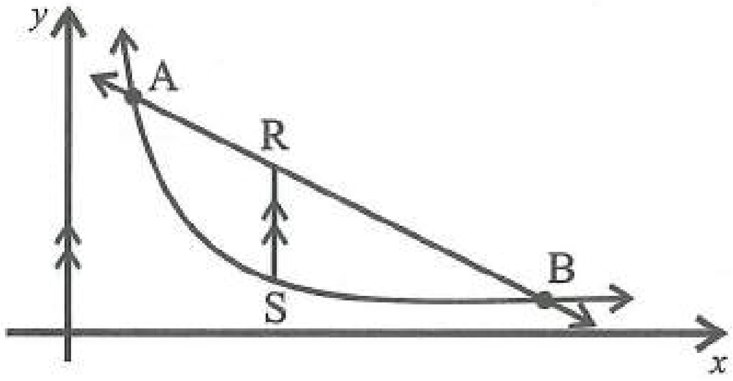
2) **A prison needs to enlarge its outdoor exercise yard. There is 100m of razor wire fencing available and the prison plans to use two sides of existing fences for the enclosure.**



   a) Show that the area of the prison yard can be given as $A(x) = 100x - x^2$.

   b) Determine the dimension which will provide a maximum enclosed area.

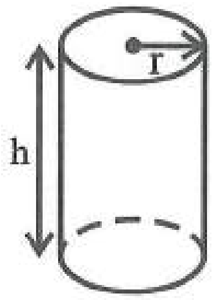
3) **A portion of the straight line $f(x) = -x + 3$ and the hyperbola with equation $xy = 1$ are sketched below. RS is a line parallel to the y-axis that lies between points A and B. R is a point on the straight line and S is a point on the hyperbola.**



   a) Determine an expression for the length of RS in terms of x.

   b) Determine the maximum length of RS.

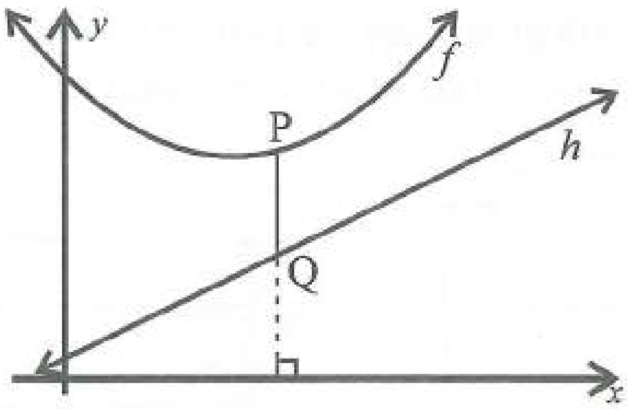
4) **A closed cylindrical can has a volume of 400cm^3. If the height of the can is h cm and the radius of the base is r cm:**



   a) Find an expression for the total surface area of the can in terms of r.

   b) Determine the dimensions of the can which will minimise the surface area of the can correct to 1 decimal place.

5) **The graph of f has the equation $y = x^2 - 4x + 15$ and the graph of h has the equation $y = 2x + 1$. The line PQ is perpendicular to the x-axis. P is a point on f and Q is a point on h.**



a) Determine an expression for the length of line PQ in terms of x.
b) Determine the minimum length of line PQ.

6) **A theatre owner realises that by cooling down his theatre he can increase the profits at his snack bar due to people buying more hot food and beverages. With the theatre at temperature T, the profit of the snack bar in Rands can be represented by the function $S(T) = 200 - \frac{361}{T}$. Determine the temperature that the theatre should be kept at in order to maximise profit if the coldest temperature setting on the cooling system is 15° C and the cost of running the cooling system for the duration of a performance can be given by $C(T) = 20 + T$.**

Rates of Change

Exercise 2: (answers on page 358)

1) **Live cultures known as Probiotics are bacteria which are consumed by humans and are beneficial to the human body. These bacteria are often cultivated in laboratories. In a laboratory the mass in micrograms (μg) of a bacteria, after t minutes is given by the equation $m(t) = 800 + 160t - 10t^2$.**

a) What was the mass of the bacteria at the start of cultivation?
b) What is the mass of the bacteria 5 after minutes?
c) At what rate is the mass of the bacteria increasing after 5 minutes?
d) What is the maximum mass the bacteria can reach?

2) **According to an estimate from a market survey, the number of a certain magazine sold per month, t months from now can be given by $m(t) = 60t^2 - 100t + 8000$.**

a) How many magazines are sold at the time of the initial publication?
b) How many magazines are sold in the 6th month?
c) At what rate are the sales of magazines changing 4 months from the initial publication?

3) **The volume of water in a rainwater collection tank t minutes after it starts to empty is given by the equation $V(t) = 4(5-t)^2$ where volume is measured in litres (l).**

a) Determine the initial volume of water in the tank.
b) At what rate is the water in the tank changing after 180 seconds?
c) How long will it take for the tank to empty?

4) **When a storage room reaches a certain temperature a cooling system turns on and operates for 10 minutes. The room temperature (T) in degrees Celsius, t minutes after the cooling system turns on is given by the expression: $T(t) = 28 - 0{,}008t^3 + 0{,}16t$ where $t \in [0;10]$**

a) At what temperature does the cooling system turn on?
b) At what rate is the temperature changing 3 minutes after the cooling system turns on?
c) What is the minimum temperature the room will reach?

Exercise 3: (answers on page 358)

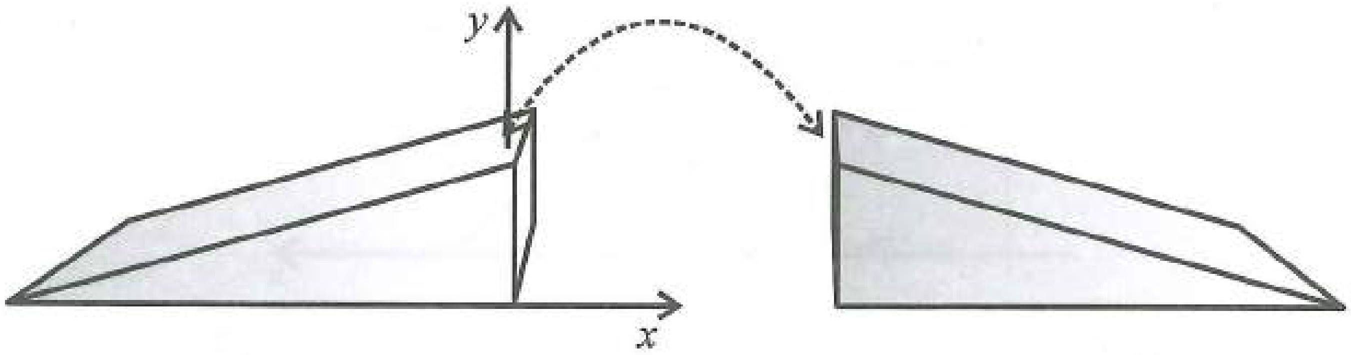
1) **The displacement (in meters) of a car, t seconds after going through a traffic light, is given by the formula $s(t)=10t-0{,}5t^2$.**
   - a) What is the car's initial velocity?
   - b) What is the car's maximum displacement?
   - c) What is the car's acceleration after 10s?

2) **The gravitational acceleration on a planet is approximately 60% of that on earth. If an object is dropped on the planet, the distance in meters that it falls in t seconds is given by $s(t)=2{,}94t^2$.**
   - a) Determine an expression for the velocity of the object at any time above the planet?
   - b) What is the value of gravitational acceleration on earth? Show all of your workings.
   - c) If a ball is dropped from a height of 8 meters on the planet, how long will it take to hit the ground?
   - d) At what velocity will the ball hit the ground if it is dropped from a height of 8m on the planet?

3) **A builder throws a brick to his friend standing 4m above him on scaffolding. The displacement in meters of the brick from the builder is given by $s(t)=8t-5t^2$, where t is in seconds.**
   - a) At what speed does the brick leave the builder's hands?
   - b) Does the brick reach the person standing on the scaffolding?

Mixed Exercise: (answers on page 359)

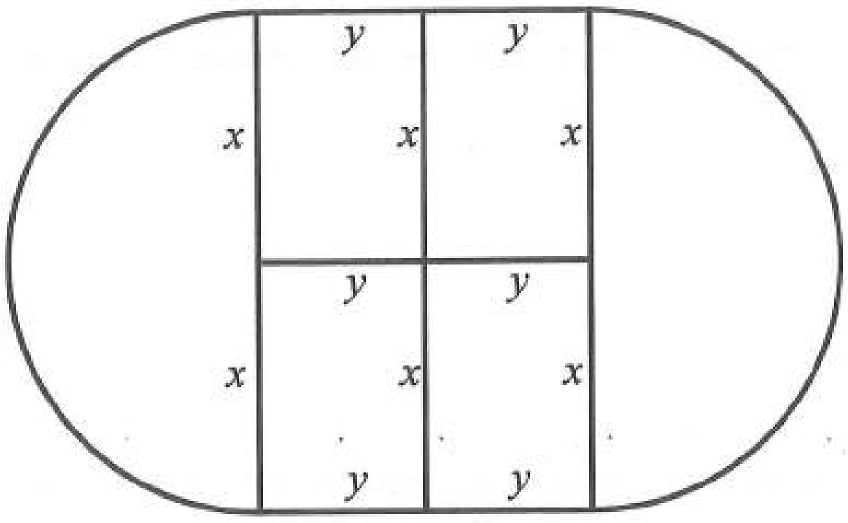
1) **Daredevil Freddy Reddy is planning to jump over two buses, each 7m high, piled on top of each other. He launches his rally car off a ramp that is 10m high. During practice runs it was determined that Freddy's height in meters above the ground could be represented by the equation: $h(x)=-\frac{3}{800}(x-40)^2+16$ where x is the horizontal distance in meters from the landing ramp.**



   - a) If Freddy reaches his maximum height directly above the buses, will he make the jump?
   - b) What is the maximum distance the landing ramp can be placed from the first ramp for Freddy still to land on the landing ramp?
   - c) What horizontal distance has Freddy travelled when he is 13 meters off the ground for the first time?
   - d) At what rate is Freddy ascending when he has travelled a horizontal distance of 30m?

2) **As a safety measure all dams are built with sluice gates. These are opened during times of heavy rainfall to release excess water and stop the dam from bursting. After the sluice gates are opened the level of water in a dam can be given by the equation: $L(t)=64-\frac{1}{8}t-\frac{1}{4}t^3$, where t is the time in hours after the sluice gates are opened and L is measured in meters.**
   - a) Determine the level of the water after the sluice gates have been open for 5 hours.
   - b) At what rate is the water decreasing 3 hours after the sluice gates are opened?
   - c) If a water level of 1 meter in the dam equates to 10 million litres of water, how many litres of water can the dam hold before the sluice gates have to be opened?

3) A farmer plans to divide a portion of his land according to the figure below, by using 18 kilometres of fencing. The rectangular divisions have a length of x kilometres and a width of y kilometres.



a) Show that the area of the fenced in portion of land can be given as $A(x) = 12x - x^2\left(4 + \frac{\pi}{3}\right)$.

b) Determine the values of x and y that will maximise the area the farmer can create with 18km of fencing and hence determine this maximum area.

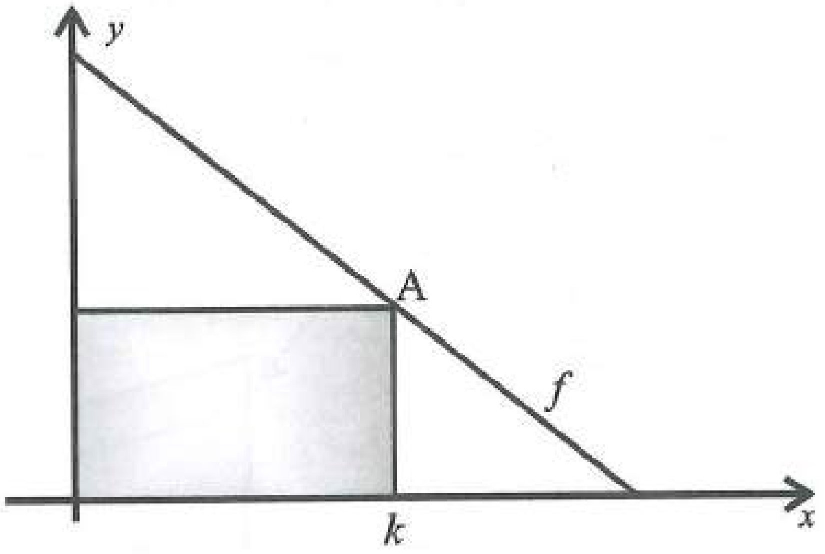
4) The height of a soccer ball t seconds after a penalty shot has been kicked can be given by the equation: $s(t) = 5t - 2t^2$, where s is measured in meters.

a) Determine how long it takes for the ball to reach a height of 2 meters above the ground for the first time.

b) Determine the velocity of the ball at this instant.

c) If the height of the goal posts is 2,4 meters and the ball reaches its maximum height at the goal posts, is it possible to score a goal?

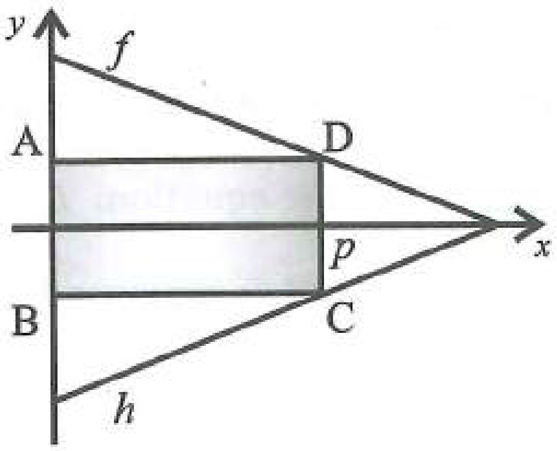
5) The rectangle below is k units in length and touches the graph of $f(x) = -2x + 12$ at point A.



a) Show that the area of the shaded region can be expressed as $A(x) = -2k^2 + 12k$.

b) Determine the maximum area of the shaded region.

6) The rectangle below is p units in length and touches the graphs of $f(x) = -4x + 8$ at point D and $h(x) = 4x - 8$ at point C.



a) Show that the area of shaded region can be expressed as $A(p) = -8p^2 + 16p$.

b) Determine the maximum area of the shaded region.

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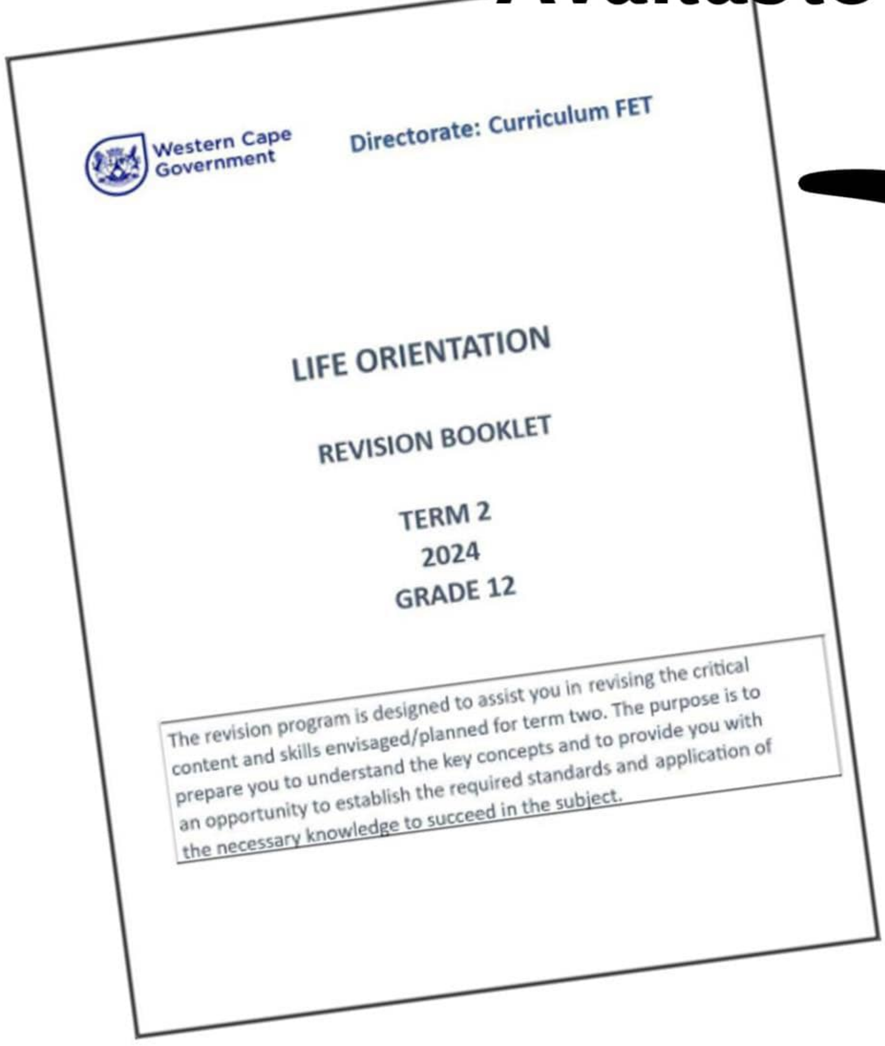

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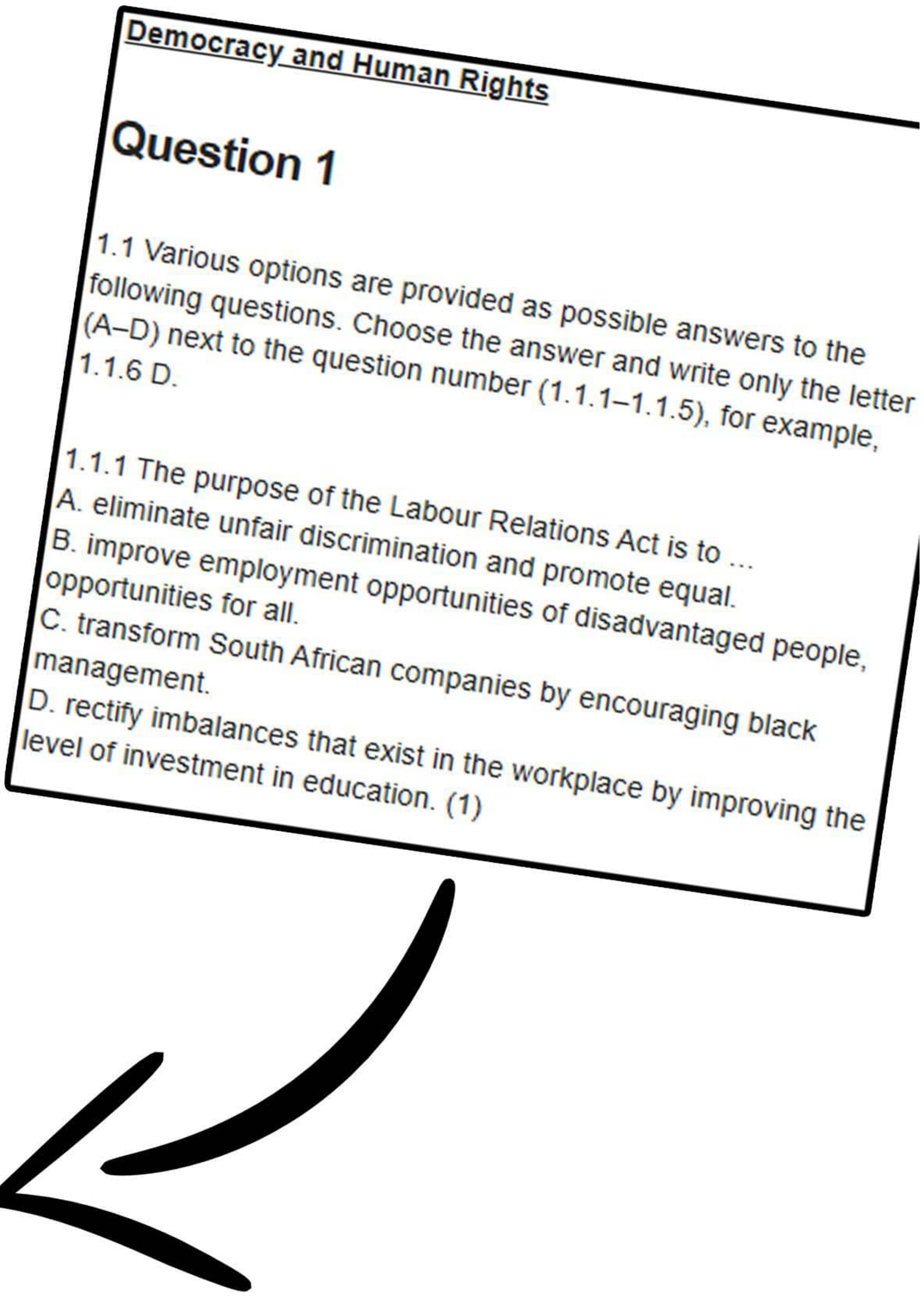
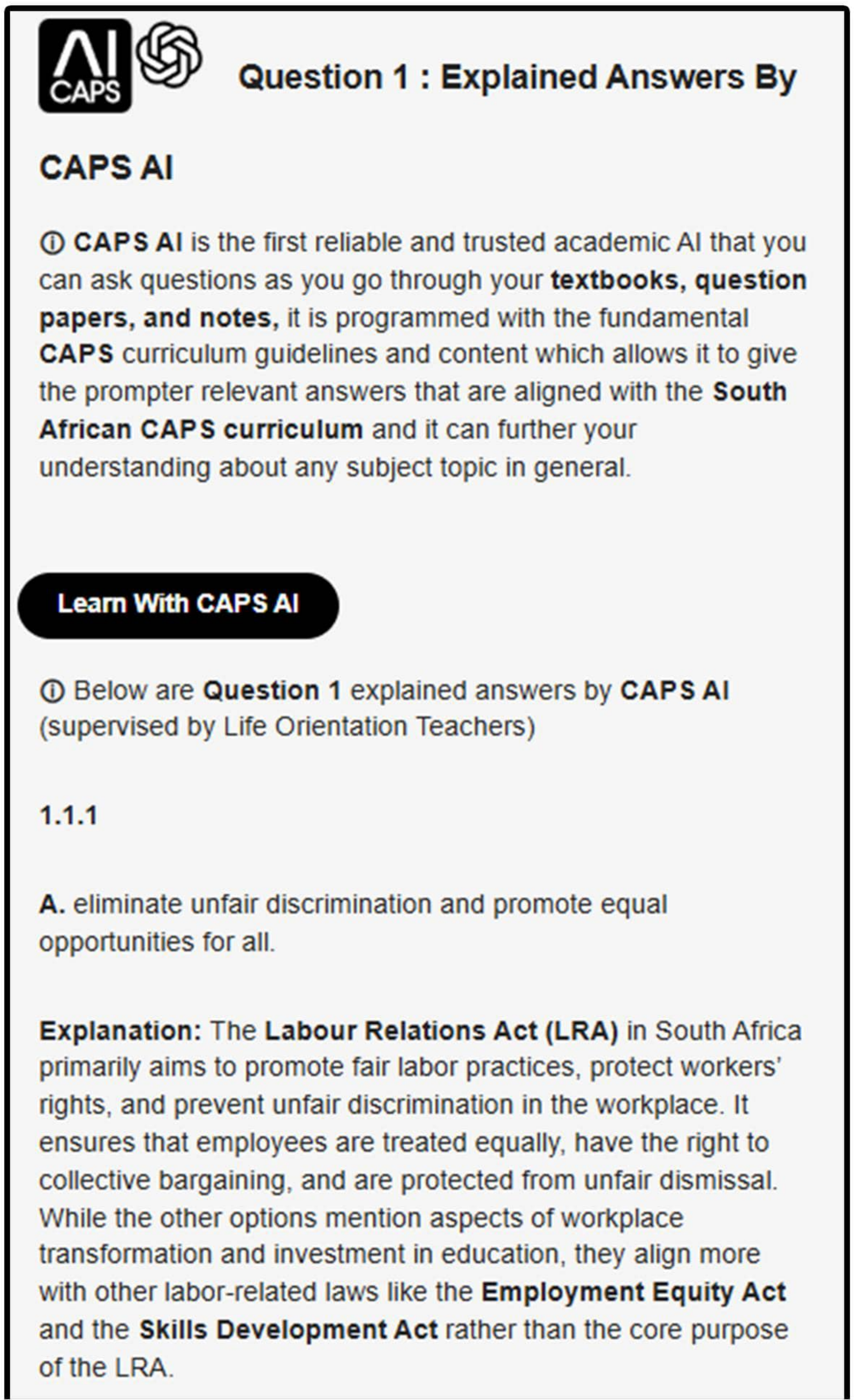


Democracy and Human Rights

Question 1

1.1 Various options are provided as possible answers to the following questions. Choose the answer and write only the letter (A–D) next to the question number (1.1.1–1.1.5), for example, 1.1.6 D.

1.1.1 The purpose of the Labour Relations Act is to …
A. eliminate unfair discrimination and promote equal opportunities for all.
B. improve employment opportunities of disadvantaged people,
C. transform South African companies by encouraging black management.
D. rectify imbalances that exist in the workplace by improving the level of investment in education. (1)



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ⓘ Below are **Question 1** explained answers by **CAPS AI** (supervised by Life Orientation Teachers)

1.1.1

A. eliminate unfair discrimination and promote equal opportunities for all.

Explanation: The **Labour Relations Act (LRA)** in South Africa primarily aims to promote fair labor practices, protect workers' rights, and prevent unfair discrimination in the workplace. It ensures that employees are treated equally, have the right to collective bargaining, and are protected from unfair dismissal. While the other options mention aspects of workplace transformation and investment in education, they align more with other labor-related laws like the **Employment Equity Act** and the **Skills Development Act** rather than the core purpose of the LRA.

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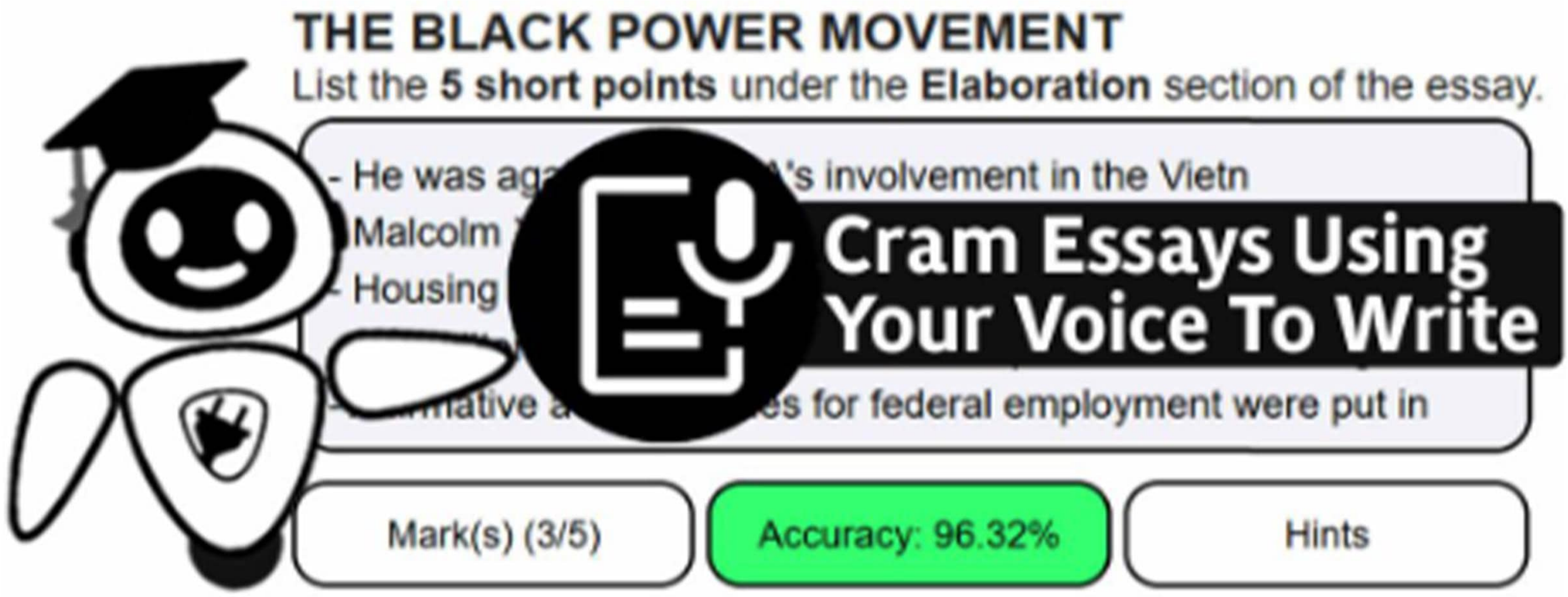

THE BLACK POWER MOVEMENT

List the **5 short points** under the **Elaboration** section of the essay.

- He was ag A's involvement in the Vietn
- Malcolm
- Housing
- mative a es for federal employment were put in

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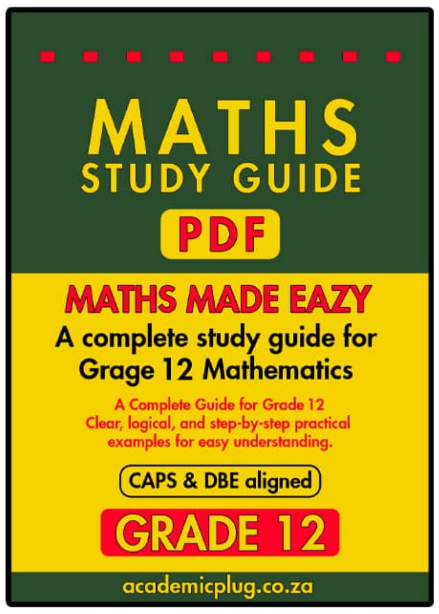
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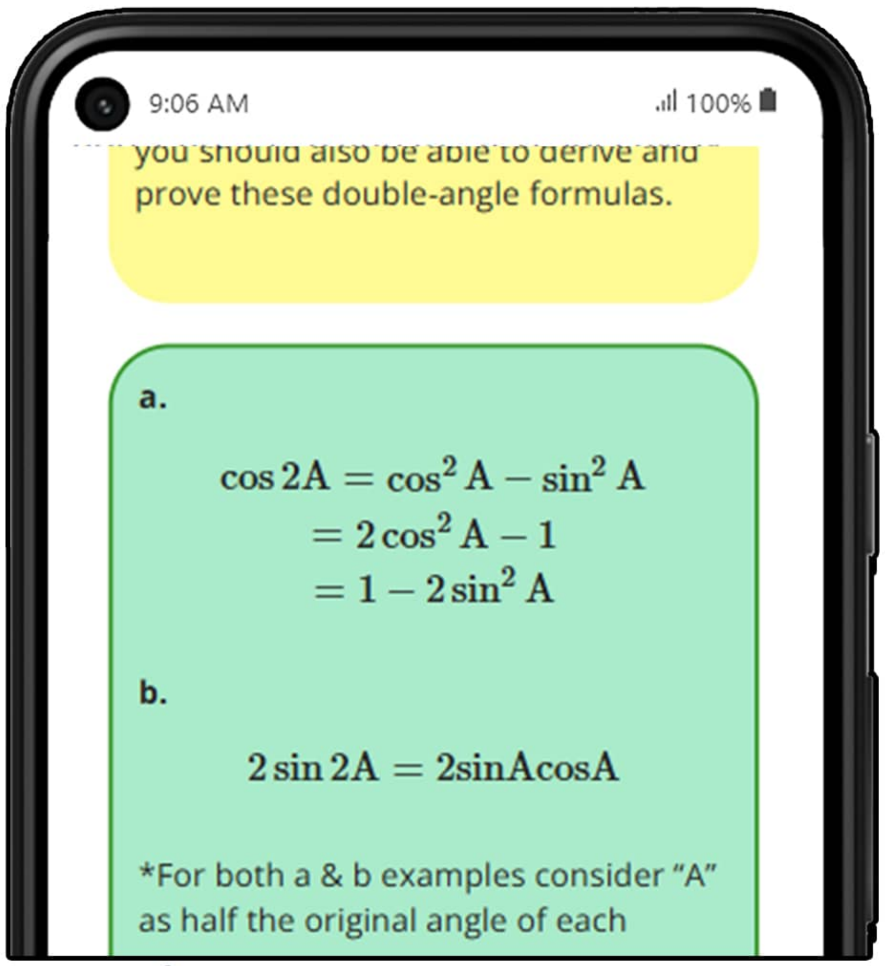

you should also be able to derive and prove these double-angle formulas.

a.

$$\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A \\ &= 2\cos^2 A - 1 \\ &= 1 - 2\sin^2 A\end{aligned}$$

b.

$$2 \sin 2A = 2\sin A\cos A$$

*For both a & b examples consider "A" as half the original angle of each

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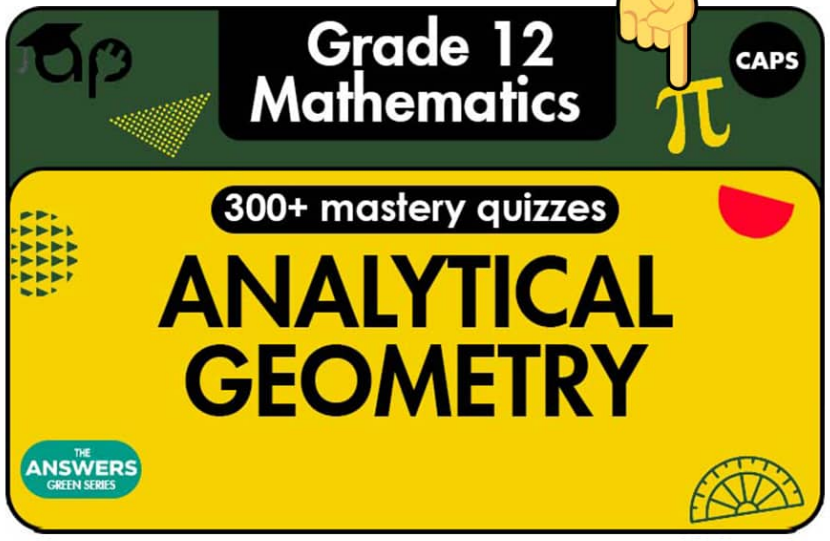
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CHAPTER 10
ANALYTICAL GEOMETRY

WHAT YOU WILL LEARN IN THIS CHAPTER

PAGE	CONTENT	CAPS	ASSESSMENTS YOU WILL BE ABLE TO:
207	**A. Summary of Formulas**	9	- Use a two-dimensional coordinate system to derive and apply the equation of a circle with any centre and determine the equation of a tangent to a circle at a given point.
208	**B. Grade 11 Revision**		
211	**C. The Circle**		
211	- Equation of a Circle		
213	- Intersection of Circles		
215	- Straight Lines and Tangents to Circles		

A. SUMMARY OF FORMULAS

Average Gradient:

The average gradient passing through points A and B is: $m_{AB} = \dfrac{y_b - y_a}{x_b - x_a}$

Distance formula:
The distance between any two points A and B is:

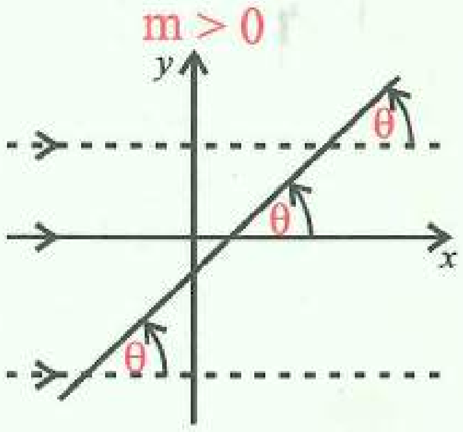
$AB = \sqrt{(y_b - y_a)^2 + (x_b - x_a)^2}$ → this is also written as $AB^2 = (y_b - y_a)^2 + (x_b - x_a)^2$

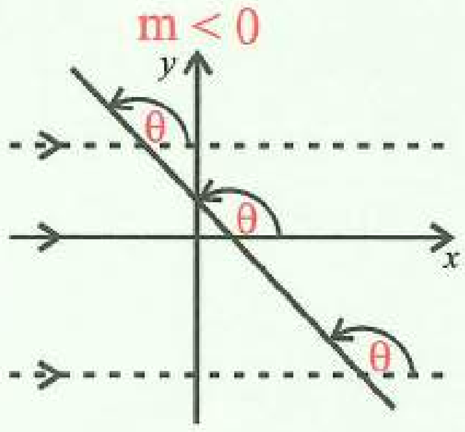
Midpoint formula:

The coordinates of the midpoint of any line segment AB are: $\left(\dfrac{x_a + x_b}{2}; \dfrac{y_a + y_b}{2}\right)$

Angle of inclination and gradient:
The angle of inclination is often denoted as θ.
The gradient of a line is equal to the tangent of the angle of inclination (θ):

$\therefore$ Gradient $= m = \tan\theta = \dfrac{y_2 - y_1}{x_2 - x_1}$ where $\theta \in [0°; 180°]$.





Equation of a circle: $(x-a)^2 + (y-b)^2 = r^2$

Where a and b are the coordinates of the centre of the circle and r is the length of its radius.
In the equation of a circle, x and y are coordinates of points which lie on the circumference of the circle.
If the circle is centred at the origin then a and b are equal to 0 and the equation is reduced to: $x^2 + y^2 = r^2$

Equation of a straight line:
$y = mx + c$ or $y - y_1 = m(x - x_1)$

Equation of a horizontal line:
$y = k$ → where k is a constant

Equation of a vertical line:
$x = k$ → where k is a constant

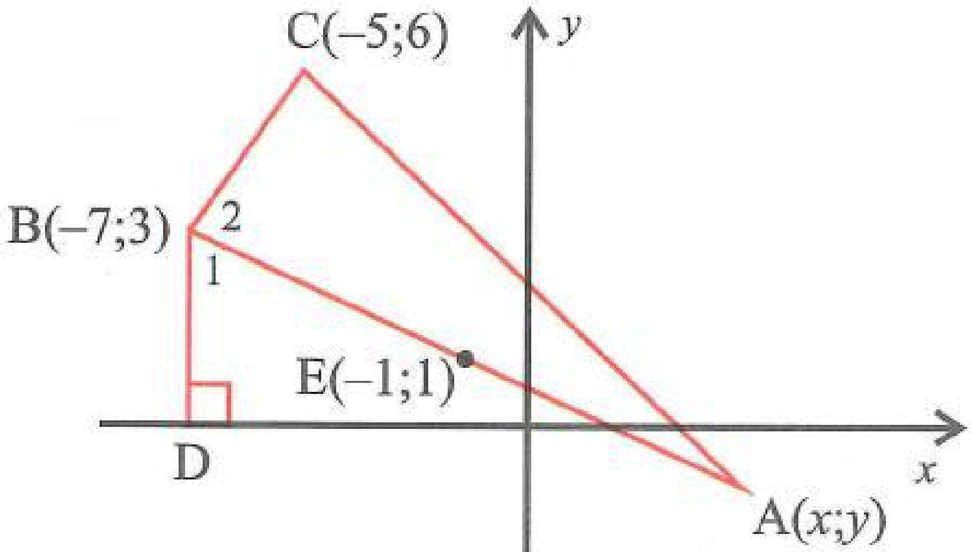
Gradients of perpendicular (⊥) and Parallel (//) lines:

The product of the gradients of perpendicular lines is –1: $m_1 \times m_2 = -1$ or $m_1 = -\dfrac{1}{m_2}$

The gradients of parallel lines are equal: $m_1 = m_2$

B. GRADE 11 REVISION

Example: **In the diagram below, $A(x;\ y)$, $B(-7;3)$ and $C(-5;6)$ are vertices of ΔABC. $E(-1;1)$ is the midpoint of AB. DB is a vertical line.**



a) **Determine the coordinates of M, the midpoint of BC.**

b) **Determine the coordinates of point D.**

c) **Show that the coordinates of A are (5;–1).**

d) **Prove that the point P(0;2) does not lie on line AC.**

e) **Determine the coordinates of F, if F lies in the first quadrant and ABCF is a parallelogram.**

f) **Show that AF is parallel to BC.**

g) **Determine the length of line AB. Leave your answer in simplest surd form .**

h) **Determine the length of line AE.**

i) **Determine the angle of inclination of EB.**

j) **Determine the magnitude of angle B_1.**

k) **Determine the equation of the line joining points C and E.**

l) **Determine the equation of line RS, which is perpendicular to BE and passes through E.**

a) $M_{BC}\left(\frac{x_B+x_C}{2};\frac{y_B+y_C}{2}\right)$

$=M_{BC}\left(\frac{-7-5}{2};\frac{3+6}{2}\right)$

$=M_{BC}\left(-6;\frac{9}{2}\right)$

b) D is the point $(-7;0)$ → DB is a vertical line, ∴ it has the same x-coordinate as B

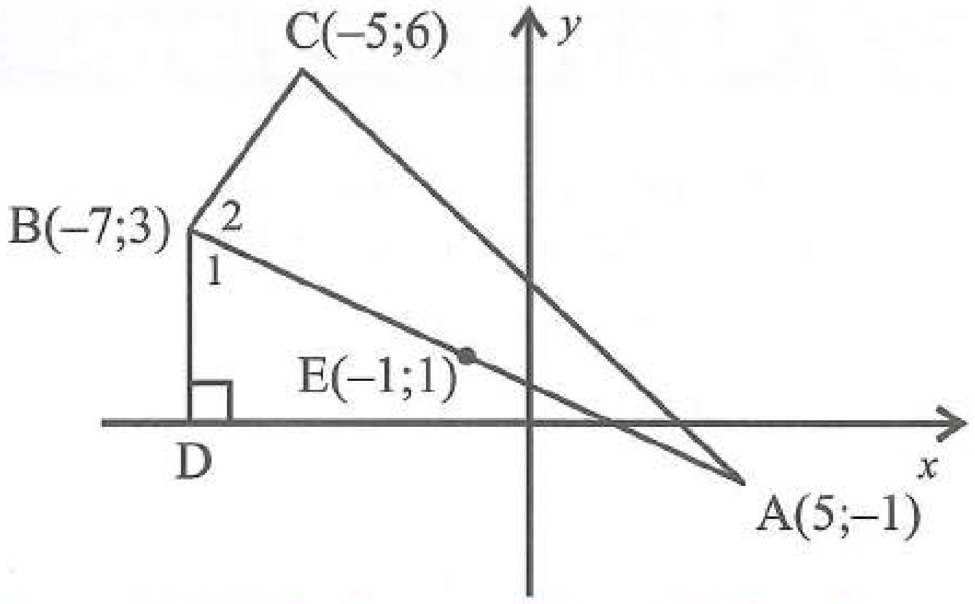
c) $x_E=\frac{x_B+x_A}{2}$ and $y_E=\frac{y_B+y_A}{2}$ → E is the midpoint of AB

$\therefore -1=\frac{-7+x_A}{2}$ and $1=\frac{3+y_A}{2}$ → now solve each equation separately

$\therefore -2=-7+x_A$ and $2=3+y_A$

$\therefore x_A=5$ and $y_A=-1$

∴ A is the point $(5;-1)$



d) In order to prove that point P(0;2) does not lie on AC you have to show A, P and C are not collinear.

$$m_{AP} = \frac{y_P - y_A}{x_P - x_A} = \frac{2-(-1)}{0-5} = -\frac{3}{5} \quad \text{and} \quad m_{CP} = \frac{y_P - y_C}{x_P - x_C} = \frac{2-6}{0-(-5)} = -\frac{4}{5}$$

$\therefore m_{AP} \neq m_{CP}$

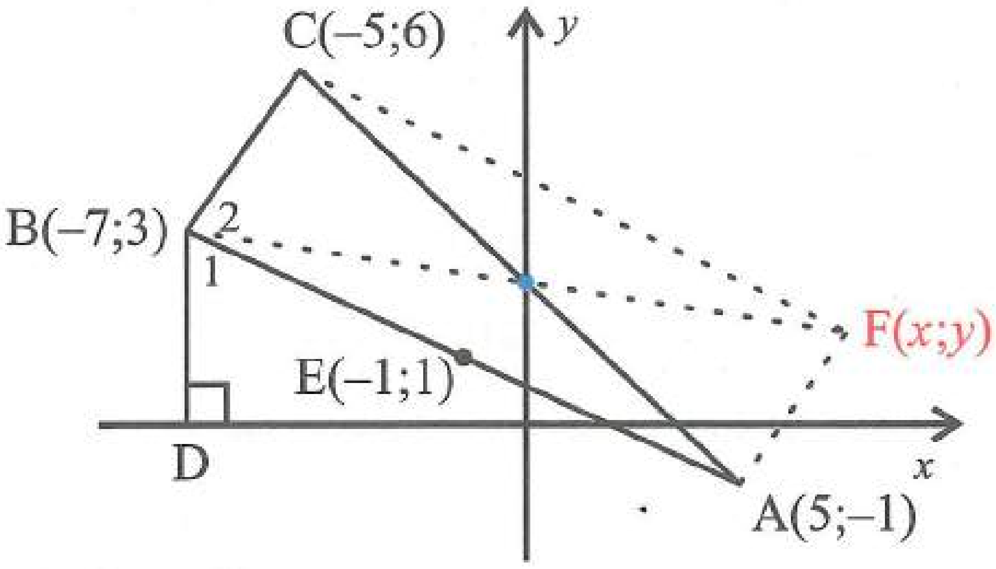
$\therefore$ The points are not collinear. → collinear points all lie on the same straight line

e) If ABCF is a parallelogram, then the midpoint of diagonal AC must be the same as the midpoint of diagonal BF. (Diagonals of a parallelogram bisect each other).

$$M_{AC}\left(\frac{x_A + x_C}{2};\frac{y_A + y_C}{2}\right)$$

$$= M_{AC}\left(\frac{5-5}{2};\frac{-1+6}{2}\right)$$

$$= M_{AC}\left(0;\frac{5}{2}\right)$$



$$\therefore 0 = \frac{-7 + x_F}{2} \text{ and } \frac{5}{2} = \frac{3 + y_F}{2} \rightarrow \text{midpoint formula}$$

$\therefore 0 = -7 + x_F$ and $5 = 3 + y_F$

$\therefore x_F = 7$ and $y_F = 2$

$\therefore$ F is the point $(7;2)$

f) $m_{AF} = \frac{y_F - y_A}{x_F - x_A} = \frac{2+1}{7-5} = \frac{3}{2}$ and $m_{BC} = \frac{y_C - y_B}{x_C - x_B} = \frac{6-3}{-5-(-7)} = \frac{3}{2}$

$m_{AF} = m_{BC}$

$\therefore AF // BC$

g) $AB = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}$ → distance formula

$= \sqrt{(5+7)^2 + (-1-3)^2}$

$= \sqrt{160}$

$= 4\sqrt{10}$ units → always put in units

h) $\text{AE} = \frac{1}{2}\text{AB} \rightarrow$ E is the midpoint of AB

$\therefore \text{AE} = \frac{1}{2}\left(4\sqrt{10}\right) = 2\sqrt{10}$ units

i) $m_{\text{EB}} = \frac{y_{\text{B}} - y_{\text{E}}}{x_{\text{B}} - x_{\text{E}}} = \frac{3-1}{-7-(-1)} = -\frac{1}{3}$

$\tan \theta_{\text{EB}} = m_{\text{EB}}$

$\therefore \tan \theta_{\text{EB}} = -\frac{1}{3}$

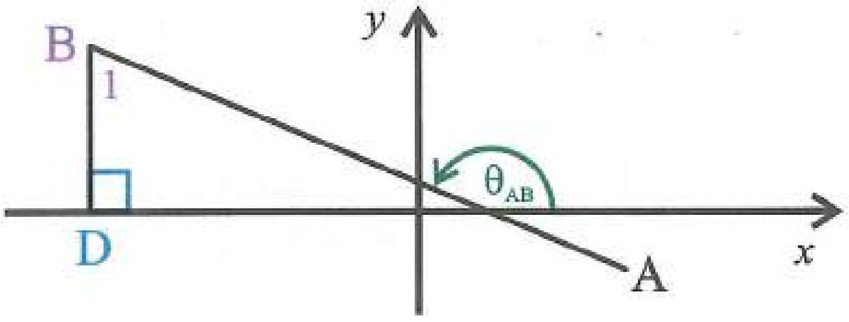
$\therefore \theta_{\text{EB}} = -18,43^\circ + 180^\circ \rightarrow m < 0 \ \therefore$ add 180° as $\theta \in \left[0^\circ;180^\circ\right]$

$\therefore \theta_{\text{EB}} = 161,57^\circ$

j) $\theta_{\text{AB}} = \hat{\text{B}}_1 + \hat{\text{D}} \rightarrow$ exterior angle of a triangle

$\therefore 161,57^\circ = \hat{\text{B}}_1 + 90^\circ \rightarrow \theta_{\text{AB}} = \theta_{\text{EB}}$

$\therefore \hat{\text{B}}_1 = 161,57^\circ - 90^\circ = 71,57^\circ$



k) $y = mx + c$

$m_{\text{CE}} = \frac{y_{\text{C}} - y_{\text{E}}}{x_{\text{C}} - x_{\text{E}}} = \frac{6-1}{-5+1} = -\frac{5}{4}$

$\therefore y = -\frac{5}{4}x + c$

Sub point $\text{E}(-1;1) \rightarrow$ point $\text{C}(-5;6)$ could also be used

$\therefore 1 = -\frac{5}{4}(-1) + c$

$\therefore c = -\frac{1}{4}$

$\therefore y = -\frac{5}{4}x - \frac{1}{4}$

l) $y = mx + c$

$m_{\text{RS}} = -\frac{1}{m_{\text{BE}}} \rightarrow \text{RS} \perp \text{BE}$

$\therefore m_{\text{RS}} = 3 \rightarrow m_{\text{BE}} = -\frac{1}{3}$

$\therefore y = 3x + c \rightarrow$ sub point $\text{E}(-1;1)$

$\therefore 1 = 3(-1) + c$

$\therefore c = 4$

$\therefore y = 3x + 4$

Complete Exercise 1 on page 221.

C. THE CIRCLE

EQUATION OF A CIRCLE

Standard Form: $(x-a)^2+(y-b)^2=r^2$

Where: $a \rightarrow$ **is the** x **- coordinate of the circle centre**

$b \rightarrow$ **is the** y **- coordinate of the circle centre**

$r \rightarrow$ **is the length of the circle's radius**

If a **and** b **are equal to zero, the circle is centred at the origin and the equation reduces to:** $x^2+y^2=r^2$

If the equation of the circle is not given in standard form you have to complete the square. The values of x **and** y **are points on the circumference of the circle.**

Example 1: Determine the coordinates of the centre and the length of the radius of each of the following circles:

a) $x^2+y^2=36$ **b)** $(x+3)^2+(y-2)^2=32$

a) Circle centre at $(0;0)$ and $r=\sqrt{36}=6$ units.

b) Circle centre at $(-3;2)$ and $r=\sqrt{32}=4\sqrt{2}$ units.

Example 2: Determine the coordinates of the centre and the length of the radius of each of the following circles.

a) $x^2+y^2-6y-55=0$ **b)** $x^2+y^2+2x+4y=11$

a) As the equation is not in standard form you have to complete the square:

$x^2+\left(y^2-6y+9\right)=55+9 \rightarrow$ complete the square

$\therefore x^2+(y-3)^2=55+9$

$\therefore x^2+(y-3)^2=64$

$\therefore$ Circle centre at $(0;3)$ and $r=\sqrt{64}=8$ units

b) As the equation is not in standard form you have to complete the square:

$x^2+y^2+2x+4y=11$

$\therefore x^2+2x+1+y^2+4y+4=11+1+4 \rightarrow$ complete the square on x and y separately

$\therefore (x+1)^2+(y+2)^2=16$

$\therefore$ Circle centre at $(-1;-2)$ and $r=\sqrt{16}=4$ units

Example 3: **Write down the equation of the circle with:**

a) Centre at the origin and radius of 20 units.

b) Centre $C(3;-2)$ and radius of 3 units.

a) $(x-a)^2+(y-b)^2=r^2$

$\therefore (x-0)^2+(y-0)^2=20^2$

$\therefore x^2+y^2=400$

b) $(x-a)^2+(y-b)^2=r^2$

$\therefore (x-3)^2+(y+2)^2=3^2$

$\therefore (x-3)^2+(y+2)^2=9$

Example 4: **Determine the value/s of b if $(-3;b)$ is a point on the circle $x^2+y^2=13$.**

$x^2+y^2=13$

Sub $(-3;b)$

$\therefore (-3)^2+(b)^2=13$

$\therefore b^2=4$

$\therefore b=-2$ or $b=2$

Example 5: **Determine the equation of the circle with centre $(1;-1)$ and passing through the point $(2;-5)$.**

$(x-a)^2+(y-b)^2=r^2$

$\therefore (x-1)^2+(y+1)^2=r^2$

Sub $(2;-5)$ and solve for r^2

$\therefore (2-1)^2+(-5+1)^2=r^2$

$\therefore r^2=17$

$\therefore$ The equation of the circle is: $(x-1)+(y+1)^2=17$

Example 6: **Given the equation of a circle with centre O is $x^2-4x+y^2+2y-20=0$:**

a) Determine the coordinates of O and the radius of the circle.

b) If KP is a diameter of the circle and P is the point $(5;3)$, determine the coordinates of K.

a) In order to put the equation in standard form, complete the square on x and y separately:

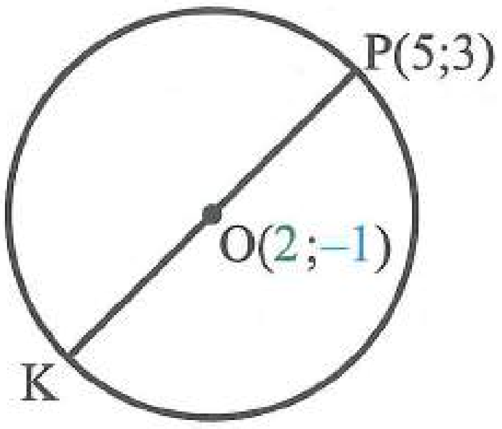
$x^2-4x+y^2+2y=20 \rightarrow$ move the constant to the RHS

$\therefore \left(x^2-4x+4\right)+\left(y^2+2y+1\right)=20+4+1$

$\therefore (x-2)^2+(y+1)^2=25$

$\therefore$ The circle has a centre at $O(2;-1)$ and a radius of 5 units.

b) O is the midpoint of KP:



$\therefore \frac{5+x_K}{2}=2$ *and* $\frac{3+y_K}{2}=-1 \rightarrow$ midpoint formula

$\therefore x_K=-1$ *and* $y_K=-5$

$\therefore$ K is the point $(-1;-5)$

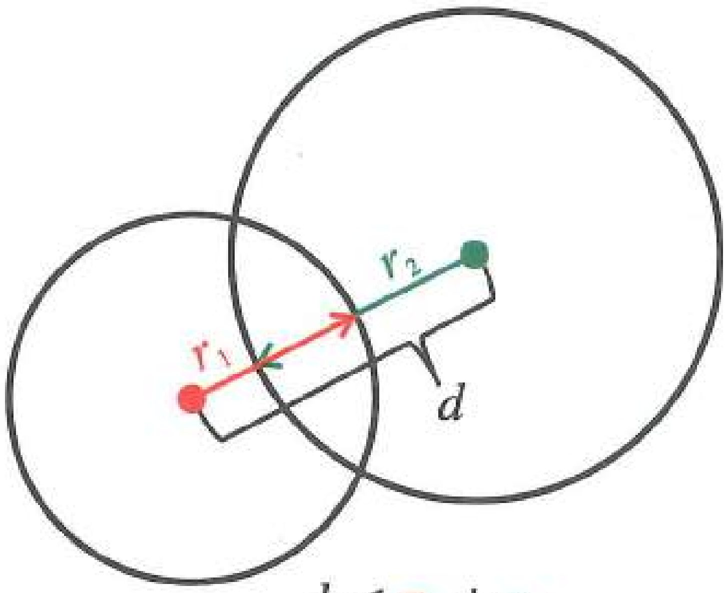
Complete Exercise 2 on page 222.

INTERSECTION OF CIRCLES

Two circles can intersect one another at two points, one point or not at all.

Case 1: Circles intersecting at two points

If the distance between the centres of the two circles is less than the sum of the radii, then the circles intersect one another twice.

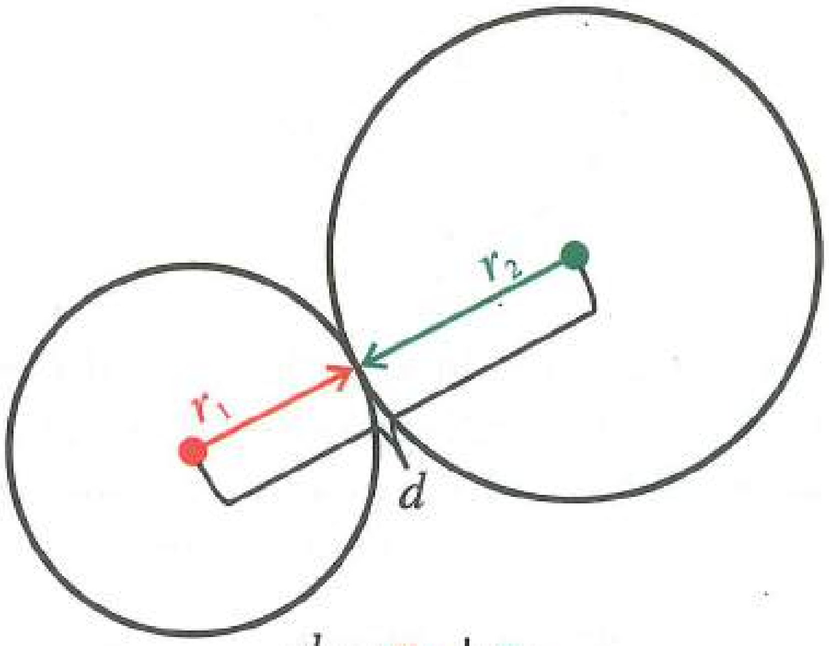


$d < r_1 + r_2$

If the centre of one circle lies inside the other, the above condition will not be true. This case will not be dealt with in Grade 12.

Case 2: Circles intersecting at one point (touching)

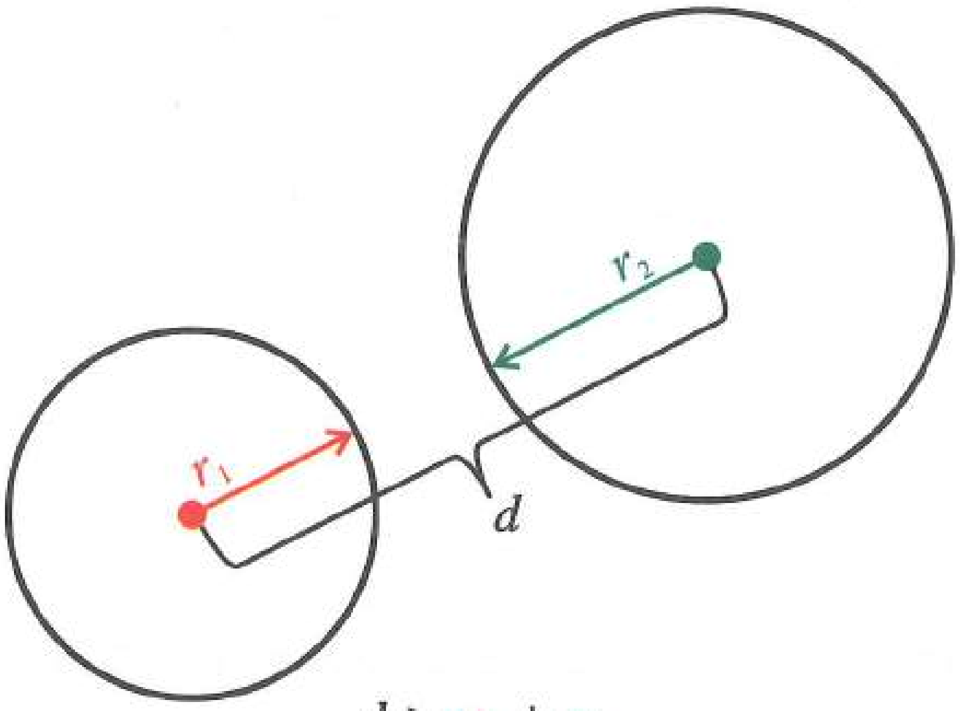
If the distance between the centres of the two circles is equal to the sum of the radii, then the circles intersect one another once.



$$d = r_1 + r_2$$

Case 3: Circles that do not intersect

If the distance between the centres of the two circles is greater than the sum of the radii, then the circles do not intersect one another.



$$d > r_1 + r_2$$

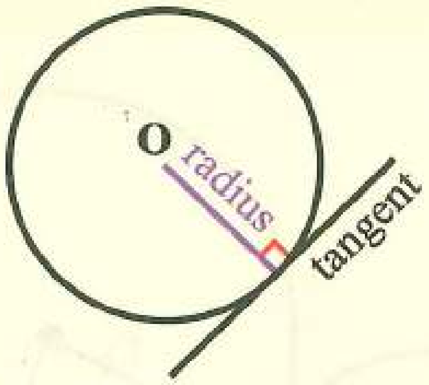
The shortest distance between two circles is referred to as the clearance between the circles.

Shortest distance between two circles $= d - r_1 - r_2$

Complete Exercise 3 on page 222.

STRAIGHT LINES AND TANGENTS TO CIRCLES

A tangent to a circle is always perpendicular (⊥) to the radius at the point of contact.



Steps to finding the equation of a tangent to a circle when the point of contact is given:

1. **Find the gradient of the radius at the given point of contact.**
2. **Using the gradient of the radius, find the gradient of the tangent using the property** $m_{\text{radius}} \times m_{\text{tangent}} = -1$ → tangent is perpendicular to the radius at point of contact
3. **Use the given point of contact and the gradient of the tangent to work out the equation of the tangent.**

Example 1: **Determine the equation of the tangent to the circle $x^2 + y^2 = 8$ at the point $A(2;-2)$.**

$m_{\text{radius}} = \dfrac{-2-0}{2-0} = -1$ → the second point is the circle centre $(0;0)$

$m_{\text{tangent}} = -\dfrac{1}{m_{\text{radius}}} = 1$

$y = mx + c$

$\therefore y = x + c$

Substitute A(2;–2)

$\therefore -2 = 2 + c$

$\therefore c = -4$

$\therefore$ Equation of the tangent is: $y = x - 4$

Example 2: **Find the points of intersection between the circle $x^2 + y^2 = 9$ and the line $y = 2x - 1$. Round your answer to two decimal places.**

To solve for the points of intersection, set the y-values equal to each other and solve simultaneously.

$\therefore x^2 + y^2 = 9 \rightarrow \boxed{1}$ and $y = 2x - 1 \rightarrow \boxed{2}$

Substitute $\boxed{2}$ into $\boxed{1}$:

$\therefore x^2 + (2x-1)^2 = 9$

$\therefore x^2 + 4x^2 - 4x + 1 - 9 = 0$

$\therefore 5x^2 - 4x - 8 = 0 \rightarrow$ use the quadratic formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\therefore x = 1{,}73$ or $x = -0{,}93 \rightarrow \boxed{3}$

Substitute $\boxed{3}$ into $\boxed{2}$:

$\therefore y = 2{,}46$ or $y = -2{,}86$

$\therefore$ Points of intersection are: $(1{,}73\ ;\ 2{,}46)$ and $(-0{,}93\ ;-2{,}86)$

Example 3: The equation of a circle is $x^2 + y^2 - 8x + 6y = 15$.

a) Prove that point $A(2;-9)$ lies on the circumference of the circle.

b) Determine the equation of the tangent at point A.

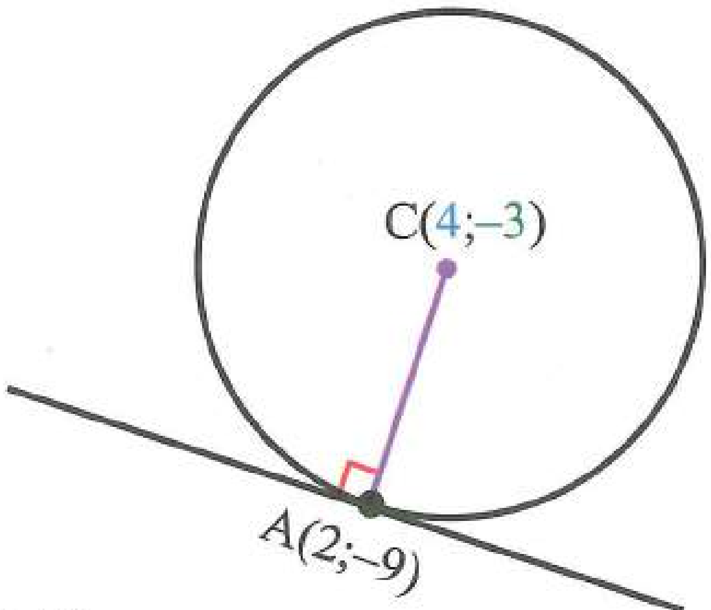
a) In order to prove that the point lies on the circumference of the circle, substitute the point $A(2;-9)$ into the equation of the circle and prove that the LHS = RHS:

$$\text{LHS} = x^2 + y^2 - 8x + 6y$$
$$= (2)^2 + (-9)^2 - 8(2) + 6(-9)$$
$$= 4 + 81 - 16 - 54$$
$$= 15 = \text{RHS}$$

b) In order to determine the equation of the tangent, you need to determine the coordinates of the centre of the circle to obtain the gradient of the radius.

$$x^2 - 8x + y^2 + 6y = 15$$
$$\therefore x^2 - 8x + 16 + y^2 + 6y + 9 = 15 + 16 + 9 \rightarrow \text{complete the square}$$
$$\therefore (x-4)^2 + (y+3)^2 = 40$$

$\therefore$ The centre of the circle is at the point $(4;-3)$



$$\therefore m_{\text{radius}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3+9}{4-2} = 3$$

$$m_{\text{tangent}} = -\frac{1}{m_{\text{radius}}} = -\frac{1}{3} \rightarrow \text{tangent is } \perp \text{ radius}$$

$$y = mx + c$$
$$\therefore y = -\frac{1}{3}x + c$$

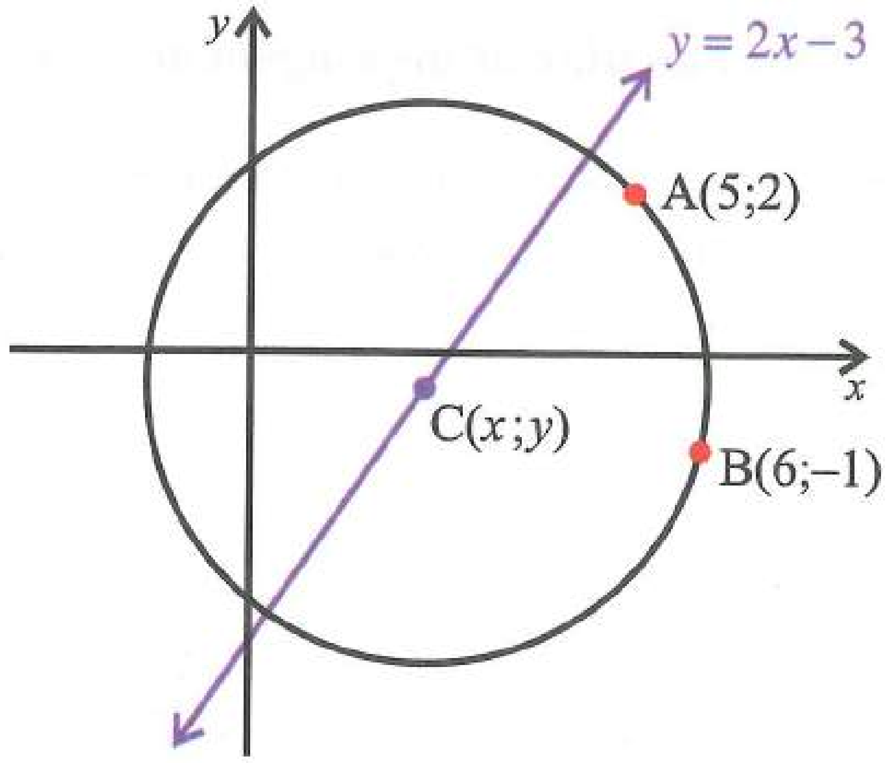
Substitute A(2;–9)

$$\therefore -9 = -\frac{1}{3}(2) + c$$
$$\therefore c = -\frac{25}{3}$$

$\therefore$ Equation of the tangent is: $y = -\frac{1}{3}x - \frac{25}{3}$

A line which cuts the circle in two places is known as a secant. A line which touches the circumference in two places and is not the diameter is known as a chord.

Example 4: **The points A(5;2) and B(6;–1) lie on the circumference of a circle. The centre of the circle lies on the line $y = 2x - 3$.**



a) **Determine the coordinates of C, the circle centre.**
b) **Hence, determine the equation of the circle.**
c) **Determine the equation of the tangent which passes through point A.**

a) In order to find the circle centre you have to use simultaneous equations and write the y-coordinate in terms of x.

As the centre of the circle lies on the line $y = 2x - 3$, the y-coordinate at C is: $2x - 3$

$\therefore$ C is the point $(x; 2x - 3)$

$AC^2 = BC^2 \rightarrow$ AC and BC are radii of the circle

$\therefore (x_A - x_C)^2 + (y_A - y_C)^2 = (x_B - x_C)^2 + (y_B - y_C)^2 \rightarrow$ distance formula

$\therefore (5 - x)^2 + [2 - (2x - 3)]^2 = (6 - x)^2 + [-1 - (2x - 3)]^2$

$\therefore 25 - 10x + x^2 + (5 - 2x)^2 = 36 - 12x + x^2 + (2 - 2x)^2$

$\therefore 25 - 10x + x^2 + 25 - 20x + 4x^2 = 36 - 12x + x^2 + 4 - 8x + 4x^2$

$\therefore 5x^2 - 30x + 50 = 5x^2 - 20x + 40$

$\therefore -10x = -10$

$\therefore x = 1$

$y = 2x - 3 \rightarrow$ substitute $x = 1$

$\therefore y = 2(1) - 3 = -1$

$\therefore$ C is the point $(1; -1)$

b) $(x - a)^2 + (y - b)^2 = r^2$

$(x - 1)^2 + (y + 1)^2 = r^2$

$r^2 = (x_B - x_C)^2 + (y_B - y_C)^2 \rightarrow$ distance formula, $BC^2 = r^2$

$= (6 - 1)^2 + (-1 + 1)^2$

$= 25$

$\therefore$ The equation of the circle is: $(x - 1)^2 + (y + 1)^2 = 25$

c) $m_{\text{radius}} = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{2-(-1)}{5-1} = \dfrac{3}{4}$

$m_{\text{tangent}} = -\dfrac{1}{m_{\text{radius}}} = -\dfrac{4}{3}$

$y = mx + c$

$\therefore y = -\dfrac{4}{3}x + c$

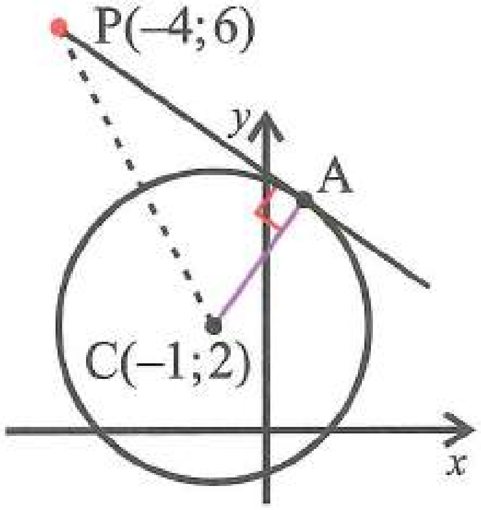
Substitute A(5;2)

$\therefore 2 = -\dfrac{4}{3}(5) + c$

$\therefore c = \dfrac{26}{3}$

$\therefore$ Equation of the tangent is: $y = -\dfrac{4}{3}x + \dfrac{26}{3}$

Example 5: The circle $(x+1)^2 + (y-2)^2 = 9$ has a tangent drawn from the point P(–4;6) to point A on the circle. Determine the length of PA.



In this case, it is not necessary to determine the point of contact of the tangent. PA can be solved using Pythagoras, as the tangent is perpendicular to the radius.

$\text{PC} = \sqrt{(x_P - x_C)^2 + (y_P - y_C)^2} \rightarrow$ determine the length of PC

$\therefore \text{PC}^2 = [-4-(-1)]^2 + (6-2)^2 \rightarrow$ both sides were squared

$\therefore \text{PC}^2 = 25 \rightarrow$ you do not need to solve for PC as PC^2 is used in Pythagoras

$\text{PC}^2 = \text{PA}^2 + \text{AC}^2 \rightarrow$ Pythag

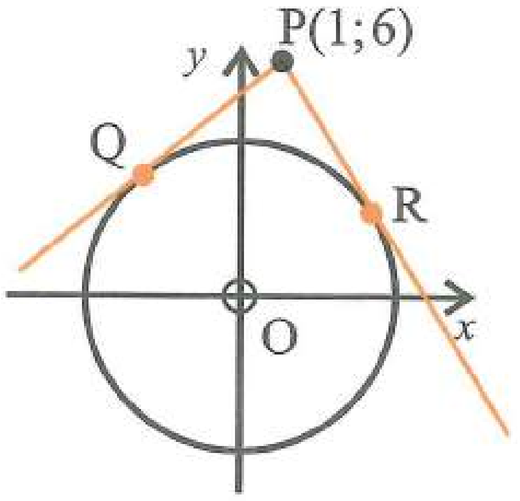
$\therefore 25 = \text{PA}^2 + 3^2 \rightarrow$ AC is the radius of the circle

$\therefore \text{PA}^2 = 25 - 9$

$\therefore \text{PA}^2 = 16$

$\therefore \text{PA} = 4\,\text{units}$

Example 6: **Tangents to the circle $x^2+y^2=15$ are drawn from point P(1;6). Determine the x-coordinates of the points of intersection of the circle and tangents at Q and R correct to one decimal place.**



From the equation of the circle, the y-coordinate at any point on the circumference can be expressed as:

$$y^2=15-x^2$$

$$\therefore y=\pm\sqrt{15-x^2}$$

$\therefore$ The coordinate of R is $\left(x;\sqrt{15-x^2}\right)\rightarrow$ positive as R is in the 1st quadrant

$$m_{\text{OR}}=\frac{y_{\text{R}}-y_{\text{O}}}{x_{\text{R}}-x_{\text{O}}}=\frac{\sqrt{15-x^2}-0}{x-0}=\frac{\sqrt{15-x^2}}{x}$$

$\therefore m_{\text{PR}}=-\dfrac{x}{\sqrt{15-x^2}}\rightarrow$ tangent is perpendicular $(\perp)$ to the radius

but the gradient of PR is also:

$$m_{\text{PR}}=\frac{y_{\text{R}}-y_{\text{P}}}{x_{\text{R}}-x_{\text{P}}}=\frac{\sqrt{15-x^2}-6}{x-1}$$

$$\therefore\frac{\sqrt{15-x^2}-6}{x-1}=-\frac{x}{\sqrt{15-x^2}}$$

$$\therefore\sqrt{15-x^2}\left(\sqrt{15-x^2}-6\right)=-x(x-1)$$

$$\therefore 15-x^2-6\sqrt{15-x^2}=-x^2+1x$$

$\therefore 15-x=6\sqrt{15-x^2}\rightarrow$ square both sides

$$\therefore 225-30x+x^2=36\left(15-x^2\right)$$

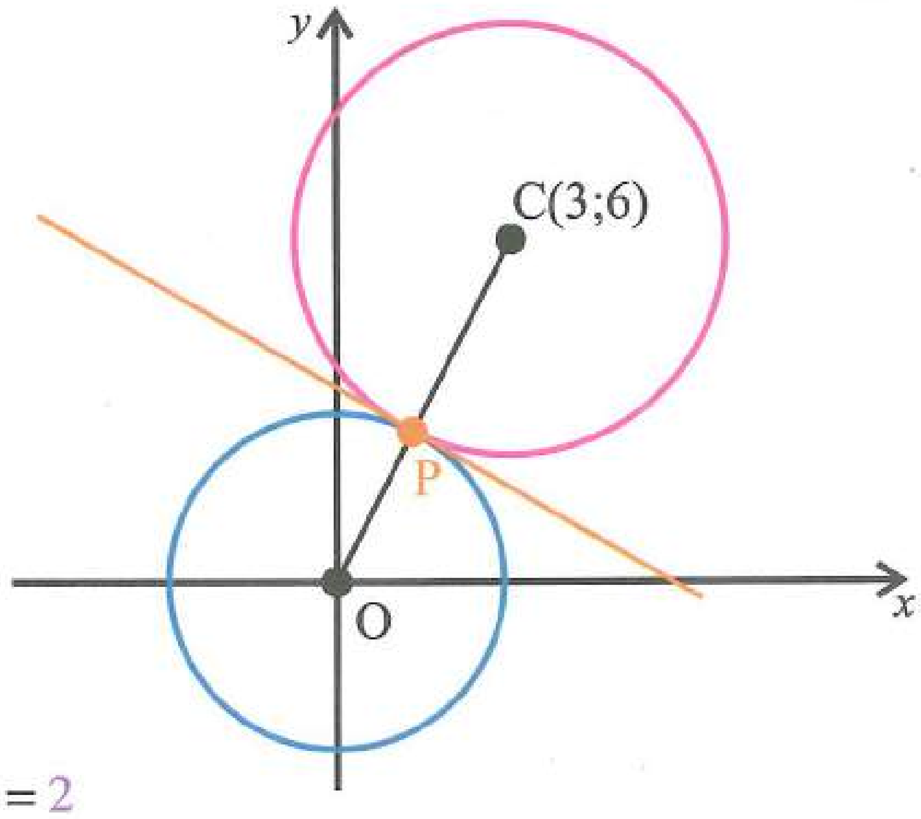
$$\therefore x^2-30x+225=540-36x^2$$

$\therefore 37x^2-30x-315=0\rightarrow$ use the quadratic formula

$$\therefore x=\frac{-(-30)\pm\sqrt{(-30)^2-4(37)(-315)}}{2(37)}\rightarrow\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\therefore x_{\text{Q}}=-2,5\quad\text{and}\quad x_{\text{R}}=3,4$$

Example 7: **The circles $x^2+y^2=5$ and $(x-3)^2+(y-6)^2=20$ touch each other at point P. Determine the equation of their common tangent.**



$$m_{\text{OC}}=\frac{y_{\text{C}}-y_{\text{O}}}{x_{\text{C}}-x_{\text{O}}}=\frac{6-0}{3-0}=2$$

$\therefore m_{\text{tangent}}=-\frac{1}{2}\rightarrow$ tangent is perpendicular $(\perp)$ to the radius

Determine the equation of OC as it passes through point P

$y=2x+c\rightarrow$ the y-intercept of OC is $(0;0)$, $\therefore c=0$

$\therefore y=2x$

substitute $y=2x$ into $x^2+y^2=5$ in order to determine the x-value at P:

$\therefore x^2+(2x)^2=5\rightarrow y=2x$

$\therefore x^2+4x^2=5$

$\therefore 5x^2=5$

$\therefore x^2=1$

$\therefore x=1$ or $x=-1\rightarrow$ N/A as P is in the 1st quadrant

Determine the y-value at P by substituting $x=1$ into $y=2x$

$\therefore y=2(1)$

$\therefore y=2$

$\therefore$ The coordinates of P are $(1;2)$

Equation of the tangent: $y=mx+c$

$\therefore y=-\frac{1}{2}x+c\rightarrow$ substitute point $\text{P}(1;2)$

$\therefore 2=-\frac{1}{2}(1)+c$

$\therefore c=\frac{5}{2}$

$\therefore y=-\frac{1}{2}x+\frac{5}{2}$

Complete Exercise 4 on pages 222 and 223 and the
Mixed Exercise on pages 223 and 224.

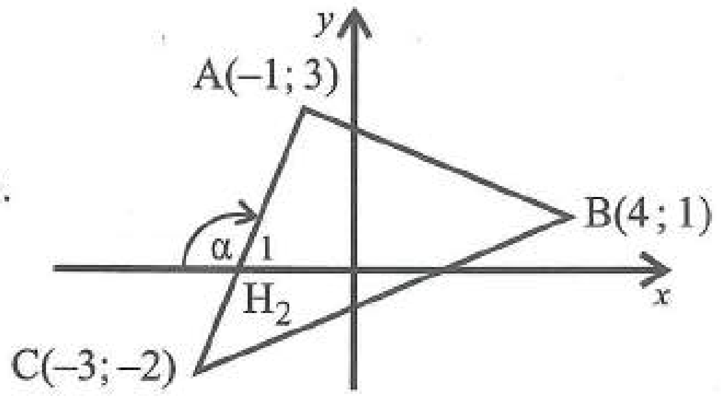
Chapter 10 Analytical Geometry (Exercises)

Grade 11 Revision

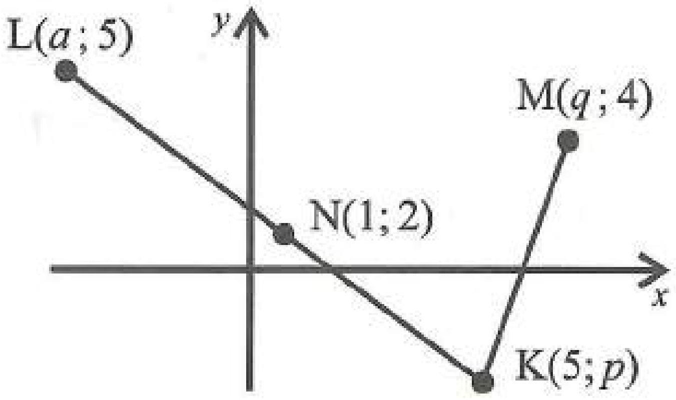
Exercise 1: (answers on page 360)

1) **Given the points $P(-2;2)$, $Q(0;1)$ and $R(-4;-2)$:**
   - a) Determine the equation of the straight line through R parallel to PQ.
   - b) Determine the equation of the straight line through Q perpendicular to PQ.
   - c) Show that $(-2;-3)$ is the point of intersection of these two lines.

2) **Consider the diagram alongside:**
   - a) Determine the length of AB.
   - b) Determine the value of α.
   - c) If ABDC is a parallelogram, determine the coordinates of D.
   - d) Prove that $\hat{A} = 90^\circ$.
   - e) What type of parallelogram can ABDC be?
   - f) Prove that ABDC is a square.

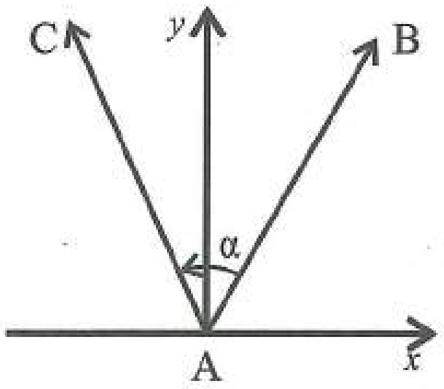


3) **In the diagram below, N(1; 2) is the midpoint of line segment LK, with K(5; p) and L(a ; 5). NM = $2\sqrt{10}$ units.**

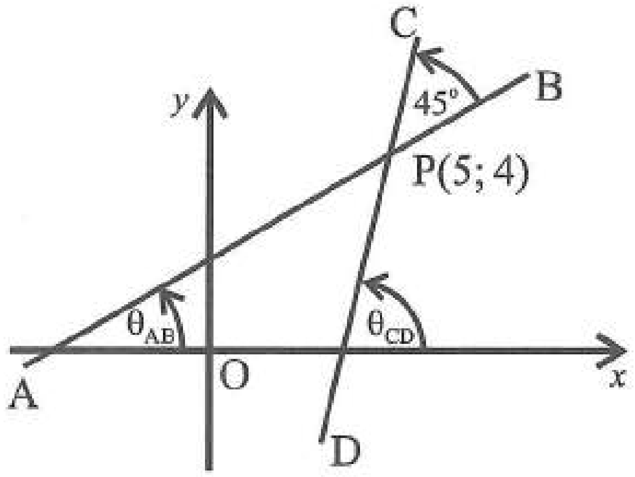


   - a) Determine the values of a and p.
   - b) Determine the value of q.

4) **The line AB with equation $y = \frac{3x}{2}$ is rotated so that its new equation (line AC) is $y = -2x$. Determine θ, the angle through which the line was rotated.**



5) **The straight line AB has the equation $5y - 3x - 5 = 0$. Another straight line CD is drawn to intersect AB at point $P(5;4)$ such that the acute angle between AB and CD is 45°.**



   - a) Determine the gradient of line CD.
   - b) Determine the equation of line CD.

The Circle

Exercise 2: (answers on page 360)

1) **Determine the equations of the following circles with centres at:**

   a) The origin and radius of 11 units.

   b) $(1;0)$ and radius of $2\sqrt{2}$ units.

   c) $\left(-\frac{1}{2};3\right)$ and radius of $\sqrt{7}$ units.

   d) $(a;b)$ and radius of $2r$ units.

2) **Determine the coordinates of the circle centre and length of the radius of each of the following circles:**

   a) $x^2+y^2=20,25$

   b) $(x-1)^2+y^2=100$

   c) $(x+5)^2+(y+1)^2=45$

   d) $(x-a)^2+(y-b)^2=t^2+16$

   e) $3x^2+3y^2=75$

   f) $x^2+2x+y^2-12=0$

   g) $x^2+10x+y^2+24=0$

   h) $x^2+y^2-6y+8x-27=0$

   i) $x^2+3x+y^2+\frac{5}{4}=0$

   j) $6x^2+6y^2+12x+18y-12=0$

   k) $2x^2+2y^2-16x-4y-40=0$

   l) $x^2+y^2-2ax=0$

   m) $x^2+10x+y^2-y-m=0$

3) **Given the circle $x^2+y^2=20$, determine the value/s of *p*, if:**

   a) $(p;2)$ is a point on the circle.

   b) $(3;p)$ is a point on the circle.

4) **Determine the equation of the circles with:**

   a) Centre $(5;-5)$ and passing through the point $(-3;1)$.

   b) A radius of 5 units and circle centre $C(a;b)$, cutting the *y*-axis at $A(0;1)$ and $B(0;7)$ if $a<0$.

5) **Determine the equation of the circles whose diameter is the line joining the points:**

   a) $A(2;5)$ and $B(-2;3)$.

   b) $A(-3;2)$ and $B(1;6)$.

6) **The equation of a circle with radius $5\sqrt{3}$ units is given by $x^2+y^2-10x+8y-p=0$.**

   a) Rewrite the equation in the form $(x-a)^2+(y-b)^2=r^2$.

   b) Hence, determine the value of *p*.

Intersection of Circles

Exercise 3: (answers on page 361)

1) **Given two circles $x^2+y^2=4$ and $(x-2)^2+(y+3)^2=9$, determine:**

   a) The circle centres and lengths of the radii of each circle.

   b) Whether or not the circles intersect each other.

2) **Given two circles $x^2-12x+y^2-10y+16=0$ and $x^2+y^2+4x-2y=0$, prove that the circles touch each other.**

Straight Lines and Tangents to Circles

Exercise 4: (answers on page 361)

1) **Determine the equation of the tangent which touches the circle:**

   a) $x^2+y^2=34$ at the point $(-3;5)$.

   b) $(x+2)^2+(y-1)^2=17$ at the point $(-3;-3)$.

   c) With centre at the origin, at the point $(4;2)$.

   d) With centre at $(1;-1)$, at the point $(3;-2)$.

   e) $x^2+2x+y^2-2y=0$ at the origin.

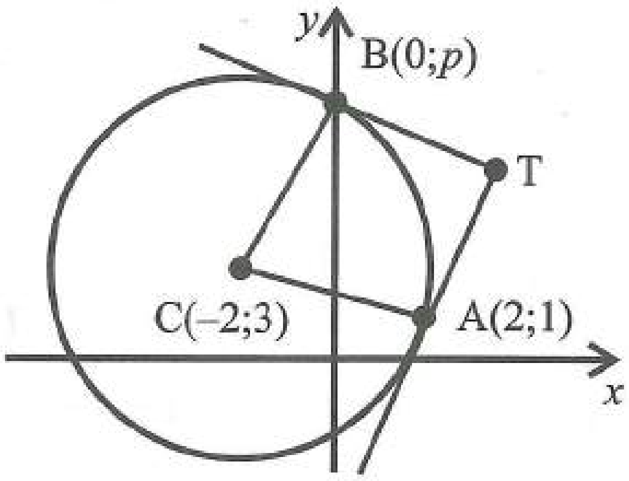
   f) $x^2-4x+y^2+2y=95$ at the point $(2;9)$.

2) **The equation of a circle is $(x+4)^2+(y+1)^2=10$.**
   - a) What is the length of the radius?
   - b) Prove that the point $P(-6;4)$ lies outside the circle.
   - c) Hence, calculate the length of the tangent PQ drawn from $P(-6;4)$ to the circle.

3) **Determine the length of a tangent from the point $R(-2;2)$ to the point of contact P on the circle $x^2-8x+12+y^2=0$.**

4) **The circle with equation $x^2+y^2=18$ and the line $x+y=6$ are given:**
   - a) Find the point(s) of intersection between the line and the circle.
   - b) Is the line a tangent, a secant or neither?

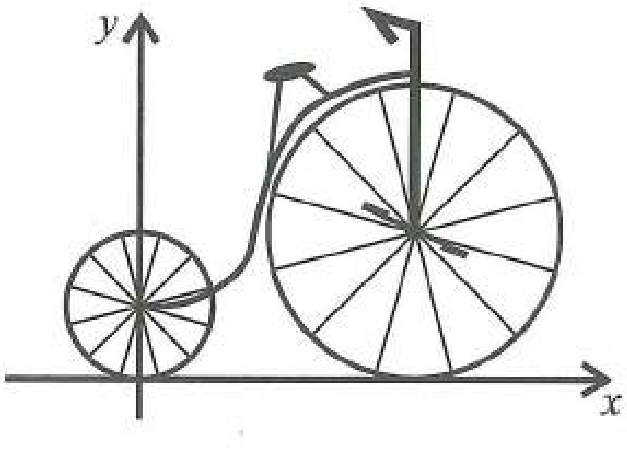
5) **A circle with centre C(–2;3) has the points A(2;1) and B(0;p) on its circumference. If the tangents at A and B meet at T, determine:**



   - a) The equation of the circle.
   - b) The value of p.
   - c) The equation of AT.
   - d) The coordinates of T.

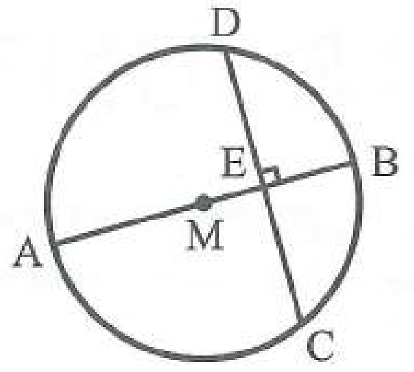
Mixed Exercise: (answers on page 361 and 362)

1) **An old Penny-Farthing bicycle is on display in a museum. A representation of the bicycle is shown below on the set of axes. The equation of the rear wheel is $x^2+y^2-6y=0$ and the equation of the front wheel is $x^2+y^2-28x-20y+196=0$. B(6;16) is a point on the front wheel and 1 unit is equal to 5cm.**



   - a) Calculate the distance between the centres of the two wheels. Give your answer in centimetres correct to one decimal place,
   - b) Hence, determine the smallest distance between the front and rear wheels correct to the nearest millimetre.
   - c) Determine the equation of the tangent to the wheel at B.
   - d) Hence, determine the point/s of intersection of the tangent and rear wheel.
   - e) What conclusion can you draw from your result in d)?

2) **AB is a diameter of a circle with equation $x^2+4x+y^2-6y-37=0$.**



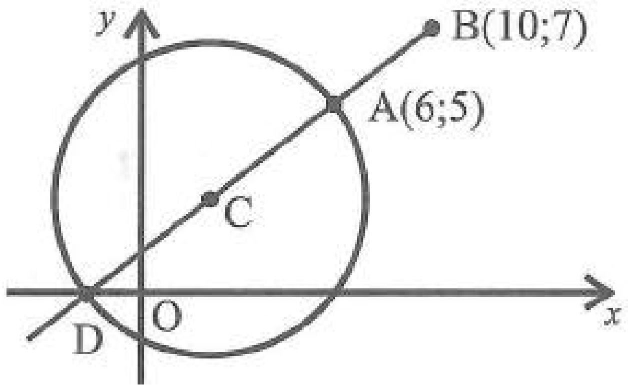
a) Determine the length of the radius in simplest surd form and the coordinates of M, the circle centre.

b) If AB is perpendicular to DC, determine the equation of CD, a chord of the circle which intersects AB at E(1;6).

c) Determine the coordinates of D, if $x_C > x_D$.

d) A tangent is drawn to point D from a point P outside the circle. Determine the equation of tangent PD.

3) **A(6;5) and B(10;7) are two points in a Cartesian plane. BA produced intersects the x-axis at D. AD is a diameter of the circle centred at C.**



a) Show that the equation of the line through A and B can be given as $2y-x-4=0$.

b) Determine the coordinates of D.

c) Determine the coordinates of C.

d) Determine the equation of the circle.

e) Determine the equation of the tangent passing through A(6;5).

4) **A circle with equation $x^2+ax+y^2-10y+22=0$ has a radius of $2\sqrt{3}$.**

a) Calculate the possible values of a.

b) Write down the coordinates of the centre of one of the possible circles if $a<0$.

5) **The straight line $y=x+3$ cuts the circle $x^2+(y-1)^2=20$ at A and B.**

a) Determine the coordinates of A and B if the x-value at A is less than at B.

b) Determine the length of chord AB.

c) Determine the coordinates of N, the midpoint of AB.

d) Show that ON ⊥ AB if O is the circle centre.

e) Determine the equation of the tangent to the circle at A.

f) If the equation of the tangent at B is $y=-0,5x+6$, determine the coordinates of C, the point of intersection of the tangents to the circle at A and B.

6) **The circles $(x+8)^2+(y+4)^2=5$ and $x^2+y^2=45$ touch each other at P.**

a) Determine the coordinates of P.

b) Hence, determine the equation of the common tangent which the circles share.

7) **A circle with centre O lying on the line $y=\frac{x}{3}+1$ passes through the points P(3;6) and Q(7;2).**

a) Determine the equation of the circle.

b) Determine the equation of the tangent passing through point P.

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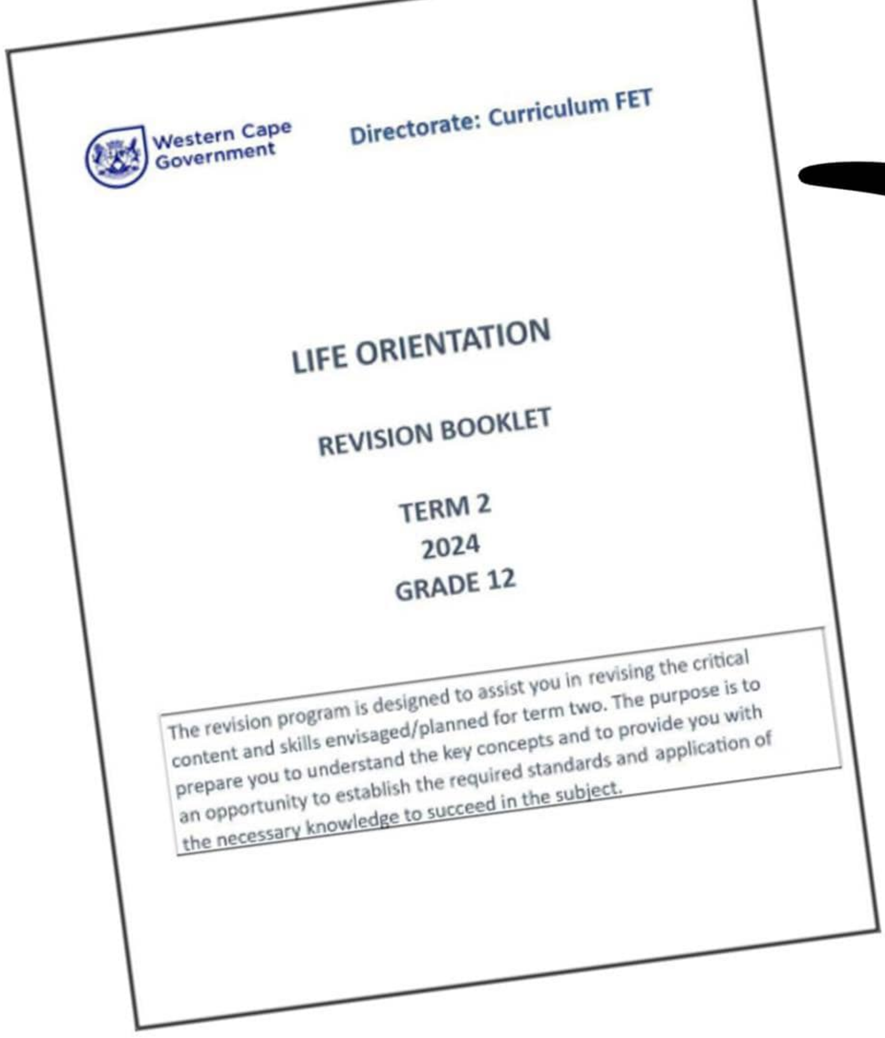


Western Cape Government

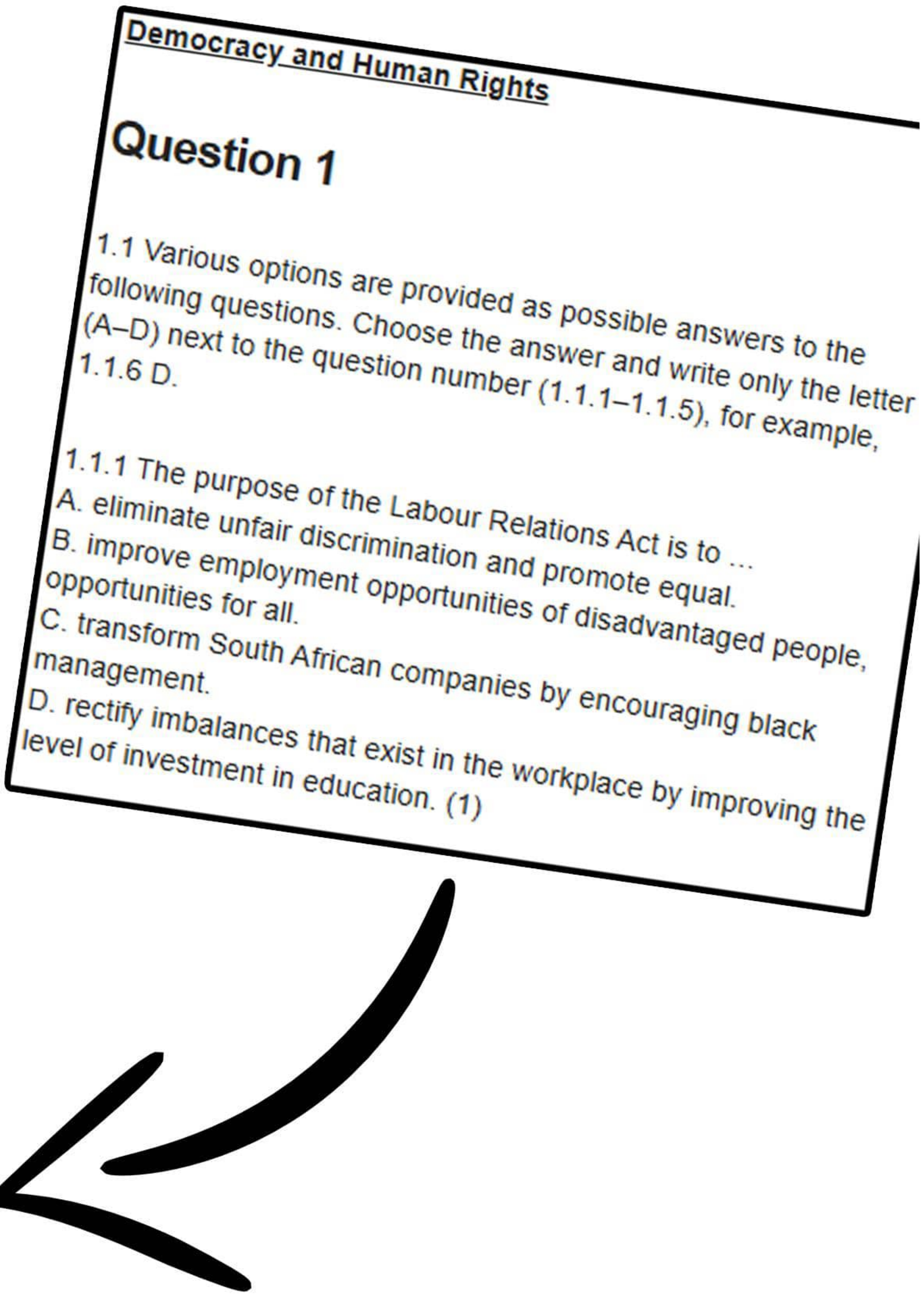
Directorate: Curriculum FET

LIFE ORIENTATION

REVISION BOOKLET

TERM 2
2024
GRADE 12

The revision program is designed to assist you in revising the critical content and skills envisaged/planned for term two. The purpose is to prepare you to understand the key concepts and to provide you with an opportunity to establish the required standards and application of the necessary knowledge to succeed in the subject.

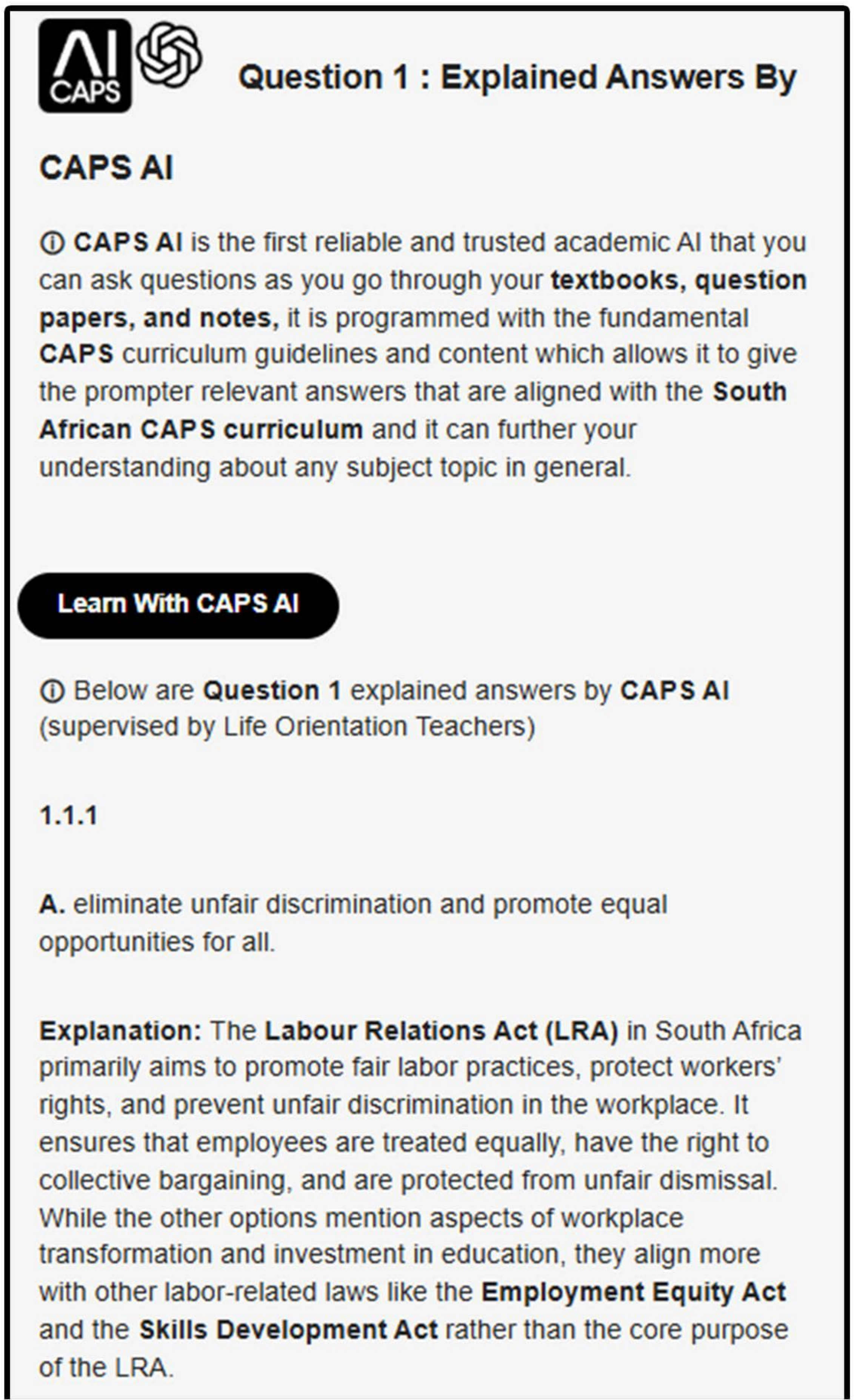


Democracy and Human Rights

Question 1

1.1 Various options are provided as possible answers to the following questions. Choose the answer and write only the letter (A–D) next to the question number (1.1.1–1.1.5), for example, 1.1.6 D.

1.1.1 The purpose of the Labour Relations Act is to …
A. eliminate unfair discrimination and promote equal opportunities for all.
B. improve employment opportunities of disadvantaged people,
C. transform South African companies by encouraging black management.
D. rectify imbalances that exist in the workplace by improving the level of investment in education. (1)



Question 1 : Explained Answers By CAPS AI

ⓘ **CAPS AI** is the first reliable and trusted academic AI that you can ask questions as you go through your **textbooks, question papers, and notes,** it is programmed with the fundamental **CAPS** curriculum guidelines and content which allows it to give the prompter relevant answers that are aligned with the **South African CAPS curriculum** and it can further your understanding about any subject topic in general.

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ⓘ Below are **Question 1** explained answers by **CAPS AI** (supervised by Life Orientation Teachers)

1.1.1

A. eliminate unfair discrimination and promote equal opportunities for all.

Explanation: The **Labour Relations Act (LRA)** in South Africa primarily aims to promote fair labor practices, protect workers' rights, and prevent unfair discrimination in the workplace. It ensures that employees are treated equally, have the right to collective bargaining, and are protected from unfair dismissal. While the other options mention aspects of workplace transformation and investment in education, they align more with other labor-related laws like the **Employment Equity Act** and the **Skills Development Act** rather than the core purpose of the LRA.

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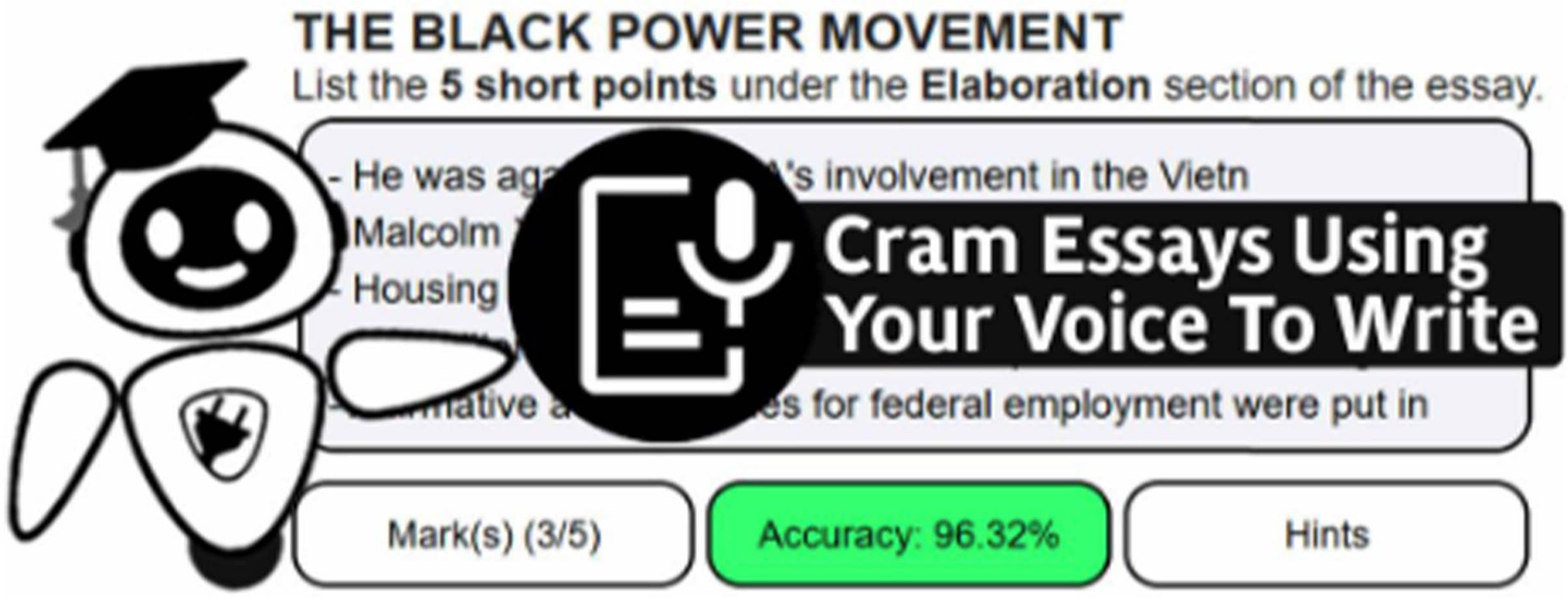

THE BLACK POWER MOVEMENT

List the **5 short points** under the **Elaboration** section of the essay.

- He was ag ... A's involvement in the Vietn
- Malcolm
- Housing
- ...mative a ... es for federal employment were put in

Mark(s) (3/5)

Accuracy: 96.32%

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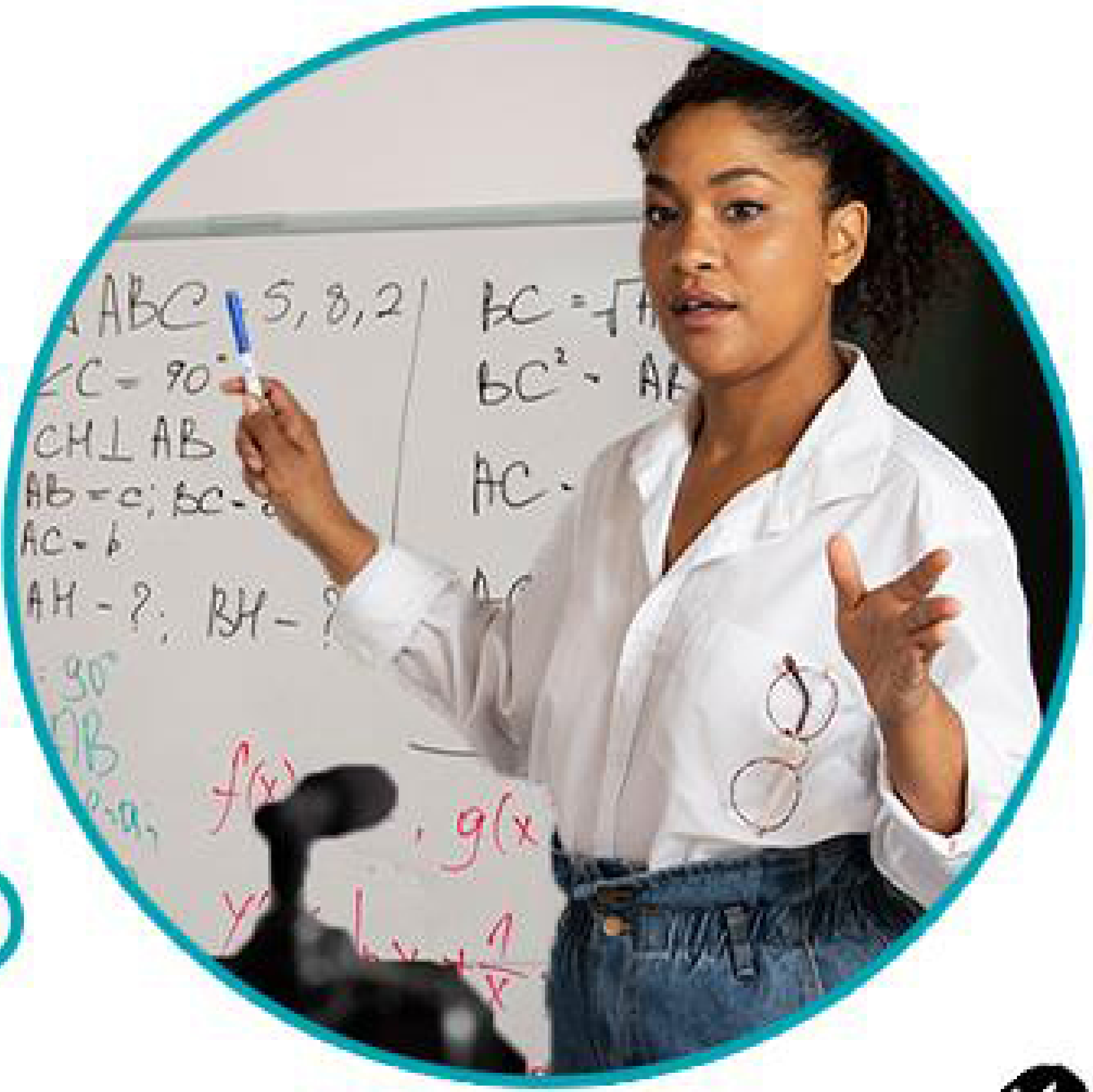

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CHAPTER 11
EXEMPLAR JUNE EXAMS

A. PAPER 1

Marks: 150
Time: 3 hours

INSTRUCTIONS AND INFORMATION

Read the following instructions carefully before answering the questions:

1. This question paper consists of 13 questions.
2. Answer ALL the questions.
3. Clearly show ALL calculations, diagrams, graphs, et cetera that you have used in determining the answers.
4. Answers only will not necessarily be awarded full marks.
5. An approved scientific calculator (non-programmable and non-graphical) may be used, unless stated otherwise.
6. Round your answers off to TWO decimal places if necessary, unless stated otherwise.
7. Diagrams are NOT necessarily drawn to scale.
8. Number the answers correctly according to the numbering system used in this question paper.
9. Write legibly and present your work neatly.

Question 1

1.1 Solve for x, rounded off to TWO decimal places where necessary:

1.1.1 $x^2 = x + 6$ (3)

1.1.2 $x(x+5) = 12$ (4)

1.1.3 $6 - 2x < 4x^2$ (5)

1.2 Determine the values of x and y if they satisfy both the following equations simultaneously:

$2x + y = 10$

$4x + y = y^2$ (8)

1.3 Solve for p and q if $(x+2)(x+p) = x^2 + qx - 10$. (4)

[24]

Question 2

A group of workers is erecting a fence around a nature reserve. They store their tools in a shed at the entrance to the reserve. Each day they collect their tools and erect 0,8km of new fence. They then lock up their tools in the shed and return the next day.

2.1 If the fence takes 40 days to erect, how far would the workers have travelled in total? (6)

2.2 How far do the workers travel on the 30th day? (2)

[8]

Question 3

Given: $\sum_{p=2}^{99}(2p+2)$

3.1 Write down the first FOUR terms of the series. (1)

3.2 Calculate the sum of the series. (5)

[6]

Question 4

The sum of the first n terms of a sequence is given by $S_n = 3^{n+1} - 6$.

4.1 Determine the sum of the first 12 terms. (1)

4.2 Determine the 12th term. (2)

4.3 Prove that the nth term of the sequence is divisible by 2. (3)

[6]

Question 5

5.1 A motor car costing R230 000 depreciates at a rate of 8% per annum on the reducing balance method. Calculate how long it will take for the car to depreciate to a half of its initial value. (2)

5.2 Rx is invested into a savings account which offers an interest rate of 7,5% per annum compounded quarterly. After 4 years, a further R$2x$ is deposited into the account. 10 years after the initial deposit, there is R16 879,76 in the account. Determine the value of the initial deposit. (6)

5.3 A loan of R27 000 is repaid over 5 years by means of equal monthly payments, starting one month after the loan was granted. Determine the monthly repayments if the interest on the loan is 9% p.a. compounded monthly. (4)

5.4 In order to plan for retirement, Mr. Pillay invests R7 000 at the end of each month into a savings account which offers a fixed interest rate of 11,6% per annum, compounded monthly, for a period of 25 years. After 25 years, he plans to stop investing R7 000 each month and intends to let the investment grow at 12% compounded semi-annually until he has R25 000 000 in the account. How long in total, to the nearest month, will it take him to realise his goal? (4)

[16]

Question 6

It is found that for the first 12 hours of running, airflow in a specific air conditioner operates at 100% efficiency. Every 12 hours thereafter, airflow in the air conditioner is reduced by 2% due to dust accumulation on the filters. In order to prevent damage to the motor, the air conditioner automatically stops operation if airflow is reduced to less than 30%.

6.1 A clean filter light turns on after the air conditioner has been running for 72 hours. What is the efficiency of the airflow when the light turns on? Give your answer correct to the nearest percent. (4)

6.2 How many full days can the air conditioner run continuously, without being cleaned, before it automatically shuts down? (4)

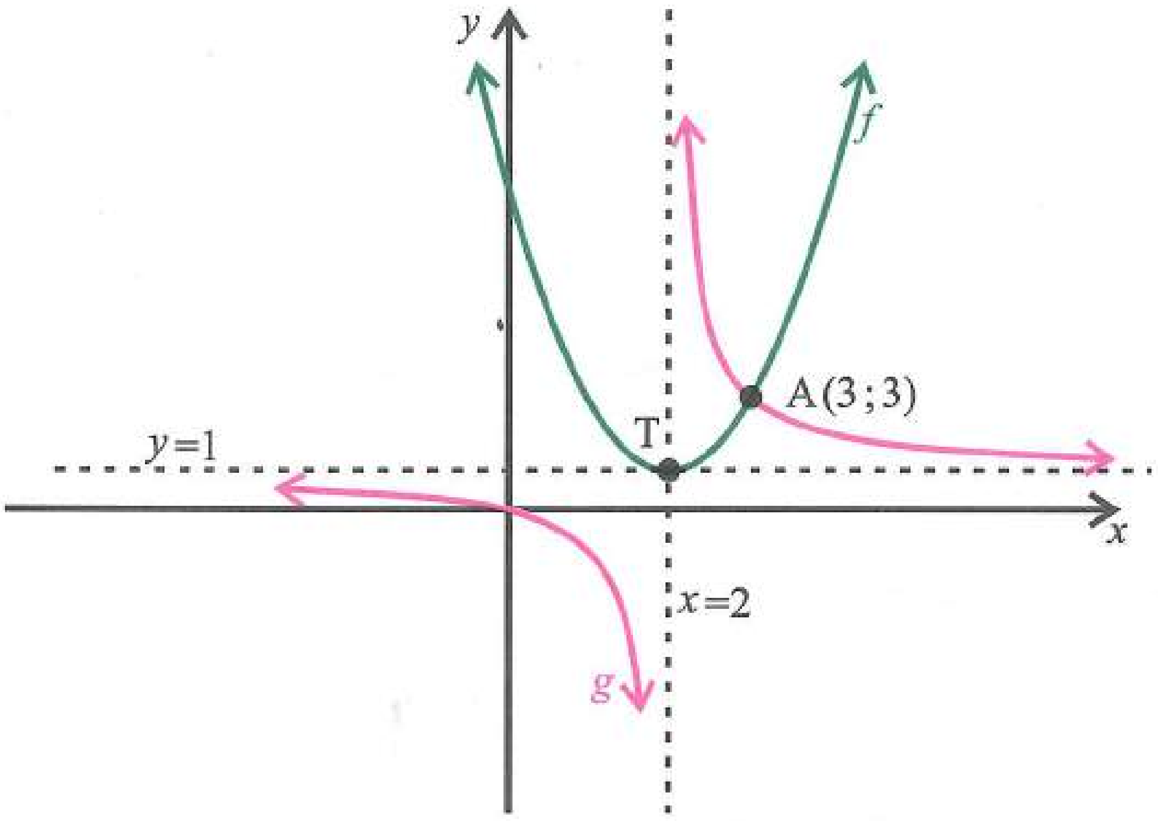
[8]

Question 7

In the diagram below, the graphs of the following functions have been sketched:

$$f(x) = a(x-p)^2 + q \quad \text{and} \quad g(x) = \frac{a}{x-p} + q$$

The graphs intersect at $A(3;3)$ and the turning point of the parabola lies at the point of intersection of the asymptotes of the hyperbola. The line $x = 2$ is the axis of symmetry of the parabola.



7.1 Determine the equation of *f* in the form $y = a(x-p)^2 + q$ (3)

7.2 Determine the equation of *g* in the form $y = \dfrac{a}{x-p} + q$ (3)

7.3 State the range of *f*. (2)

7.4 Write down the equation of $p(x)$ if *p* is the graph of *f* shifted 1 unit right and 3 units up. (2)

7.5 Determine the values of *x* for which $g(x) - f(x) \geq 0$. (2)

[12]

Question 8

Consider the function $f(x) = -2^{-x} + 1$.

8.1 Calculate the coordinates of the intercepts of *f* with the axes. (4)

8.2 Write down the equation of the asymptote of *f*. (1)

8.3 Sketch the graph of *f*. (3)

8.4 Write down the equation of *g*, if *g* is the graph of *f* shifted 1 unit left. (1)

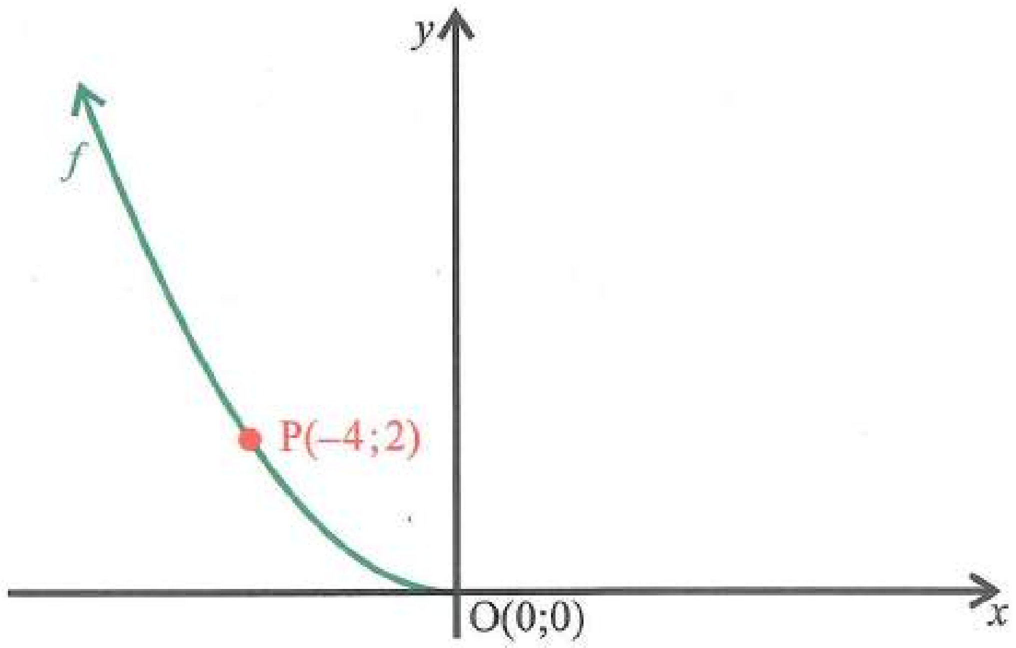
8.5 Solve for *x* if $f(x) = -5$. Give your answer correct to 1 decimal place. (3)

[12]

Question 9

The graph of $f(x) = ax^2$, $x \leq 0$ is sketched below.

The point P$(-4;2)$ lies on the graph of f.



9.1 Determine the value of a. (2)

9.2 Determine the equation of f^{-1} in the form $y = ...$ (3)

9.3 Write down the range of f^{-1}. (1)

9.4 Sketch the graph of f^{-1}. Indicate the coordinates of a point on the graph other than $(0;0)$. (2)

9.5 The graph of f is reflected across the line $y = x$ and is then reflected across the y-axis. Determine the equation of the new function in the form $y = ...$ Remember to include all restrictions on the domain. (3)

[11]

Question 10

10.1 Determine $f'(x)$ from first principles if $f(x) = -x^2$. (5)

10.2 Determine $\frac{dy}{dx}$ if: $y = \frac{5x + \sqrt{x^5}}{2x^2}$ (4)

[9]

Question 11

The graph of f has the equation $f(x) = x^3 - 3x - 2$.

11.1 Determine the coordinate of the y-intercept of f. (1)

11.2 Determine the coordinates of the x-intercepts of f. (4)

11.3 Determine the stationary points of f. (5)

11.4 Determine the point of inflection of f. (3)

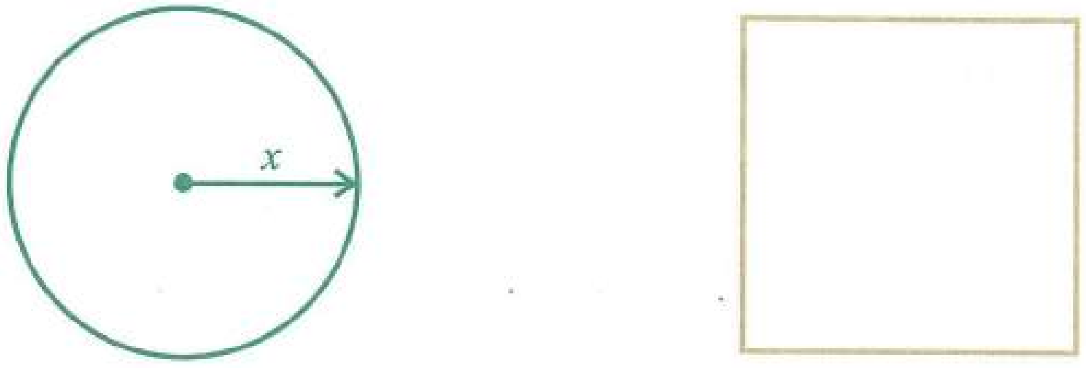
11.5 Draw a clear sketch graph of f. Show and label all intercepts with the axes, the stationary points and the point of inflection. (5)

11.6 For which values of x is $f'(x).f''(x) < 0$? (2)

[20]

Question 12

A length of wire 200 metres long is cut into two pieces. One piece is used to form a circle with radius x metres in length. The other piece is used to form a square.



12.1 Show that the sides of the square are $50-\frac{\pi x}{2}$ metres in length. (2)

12.2 Show that the sum of the areas of the circle and square can be given as:

$$\pi x^2+2500-50\pi x+\frac{\pi^2x^2}{4}\ \text{m}^2 \qquad (3)$$

12.3 Determine the value of x so that the area in 12.2 above is a minimum. (4)

12.4 If copper wire is used to make the square at a cost of R200 per metre, what will it cost to make the square in question 12.3 above? (3)

[12]

Question 13

13.1 Without the use of a calculator, determine the exact value of:

$123456788\times123456790-(123456789)^2$ (3)

13.2 For which value of n does $\frac{2+4+6+8+...+2n}{3+5+7+...+(2n+1)}=\frac{1998}{1999}$? (3)

[6]

Total: [150]

Answers to this paper can be found on pages 363 and 364.

A. PAPER 2

Marks: 150
Time: 3 hours

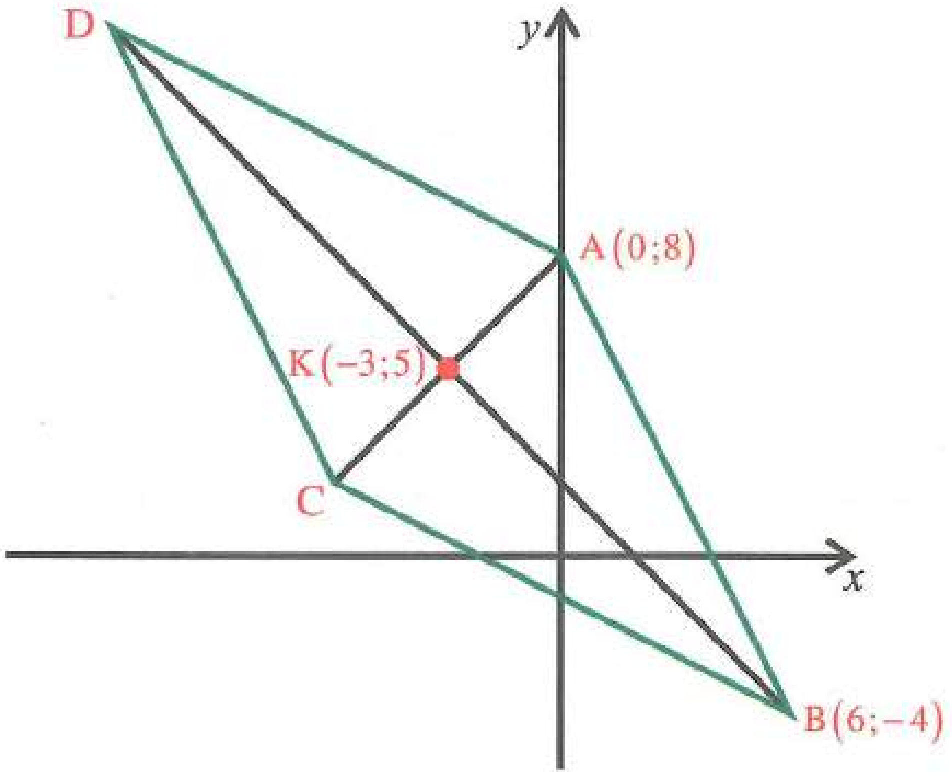
INSTRUCTIONS AND INFORMATION

Read the following instructions carefully before answering the questions:

1. This question paper consists of 11 questions.
2. Answer ALL the questions.
3. Clearly show ALL calculations, diagrams, graphs, et cetera that you have used in determining the answers.
4. Answers only will not necessarily be awarded full marks.
5. An approved scientific calculator (non-programmable and non-graphical) may be used, unless stated otherwise.
6. Round your answers off to TWO decimal places if necessary, unless stated otherwise.
7. Diagrams are NOT necessarily drawn to scale.
8. Number the answers correctly according to the numbering system used in this question paper.
9. Write legibly and present your work neatly.

Question 1

In the figure below, ABCD is a rhombus with diagonals meeting at $K(-3;5)$.
Points $A(0;8)$ and $B(6;-4)$ are vertices of the rhombus.



1.1 Determine the coordinates of C and D. (4)

1.2 Determine $A\hat{B}K$. (5)

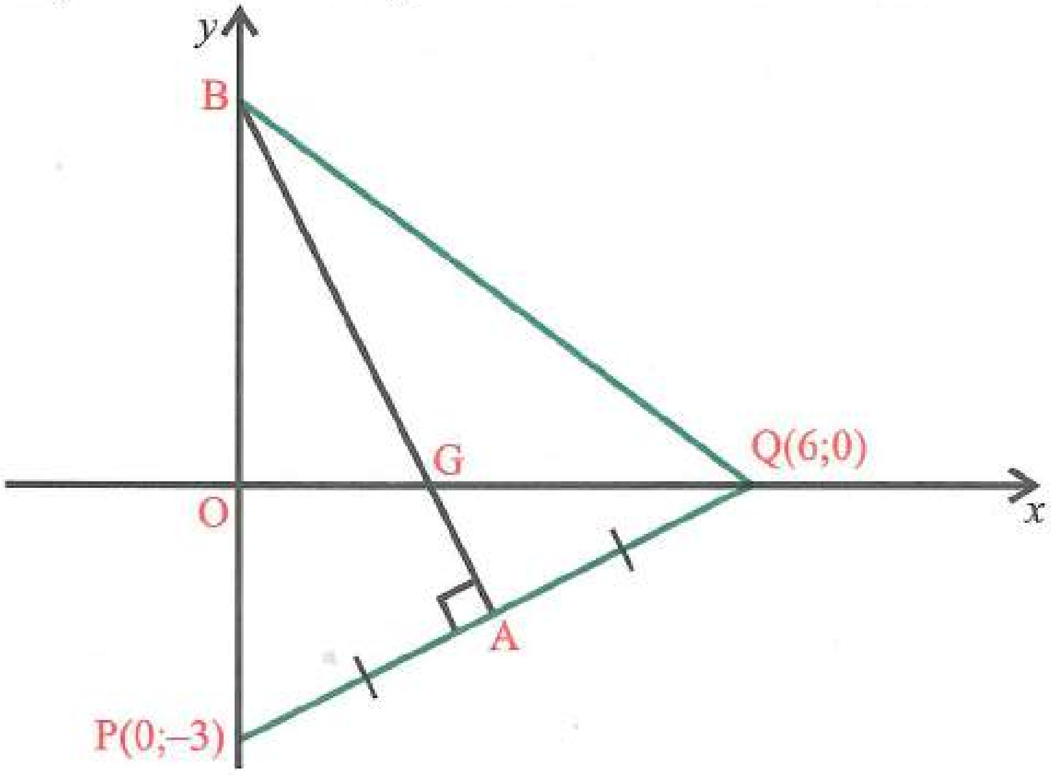
1.3 Determine the equation of circle centre B which passes through C. (4)

1.4 Determine the equation of the tangent at point A, if A lies on the circle. (3)

[16]

Question 2

The diagram below shows the points $P(0;-3)$ and $Q(6;0)$. Point A is the midpoint of PQ. The line AB is perpendicular to PQ and intersects the *x*-axis at G and the *y*-axis at B.



2.1 Show that the gradient of PQ is 0,5. (1)

2.2 Determine the coordinates of A. (2)

2.3 Determine the equation of line AB. (3)

2.4 Determine the length of BQ. (2)

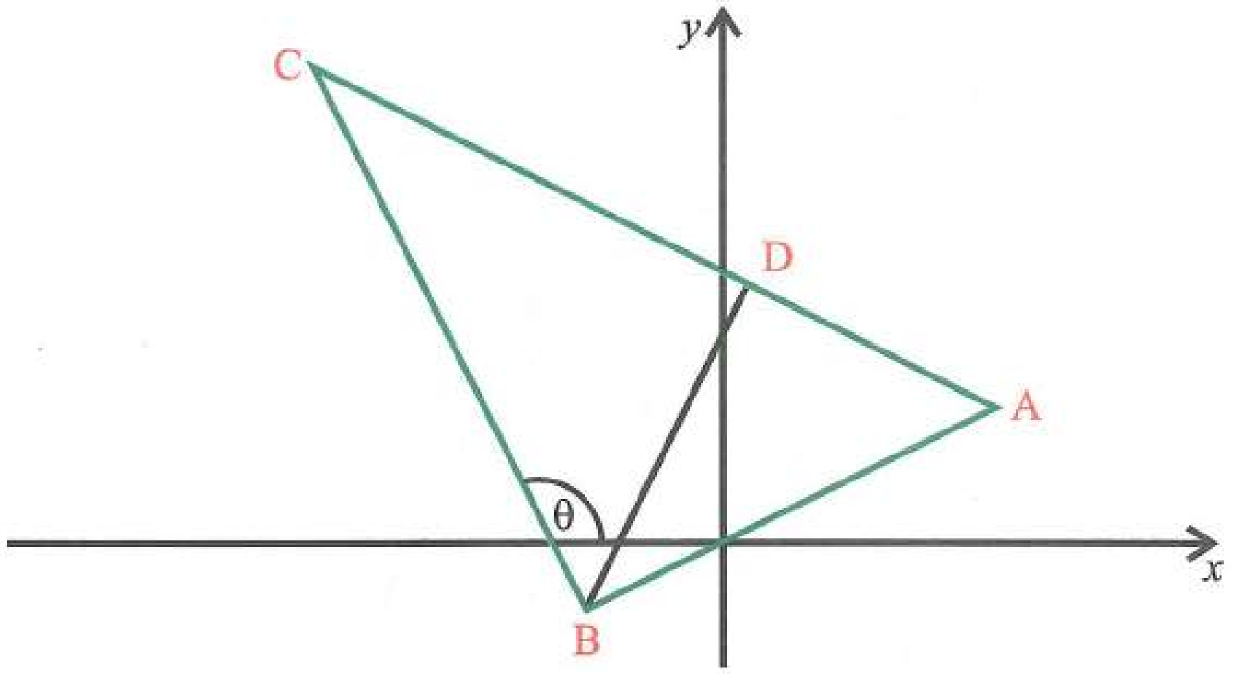
2.5 Show that $\triangle BPQ$ is an isosceles triangle. (2)

2.6 If BQRP is a rhombus, determine the coordinates of R. (2)

[12]

Question 3

The diagram below shows the points $A(4;2)$, $B(-2;-1)$ and $C(-6;7)$ in the Cartesian plane.



3.1 Prove that $A\hat{B}C$ is a right angle. (3)

3.2 Determine the equation of the line parallel to BC passing through A. (5)

3.3 D is a point on AC such that $AC \perp BD$. Determine the equation of BD. (4)

3.4 Determine the coordinates of D. (6)

3.5 Calculate the size of θ, the angle of inclination of BC. (2)

3.6 Hence or otherwise, calculate the size of $B\hat{C}A$. (4)

[24]

Question 4

Fifteen members of a netball team took part in a tournament. The points scored by each player at the end of the tournament are shown below. You may assume that each member played for the same amount of time in the tournament.

17	18	20	22	24	28	31	32	33	33	34	36	43	52	42

4.1 Determine the median of the data set. (1)

4.2 Determine the interquartile range for the data. (3)

4.3 Draw a box and whisker diagram to represent the data. (3)

4.4 Use the box and whisker diagram to comment on the points scored by the players in this team. (2)

[9]

Question 5

It is given that $\tan 27^\circ = p$. With aid of a diagram and without the use of a calculator, express the following in terms of p:

5.1 $\tan 153^\circ$ (2)

5.2 $\sin 63^\circ$ (3)

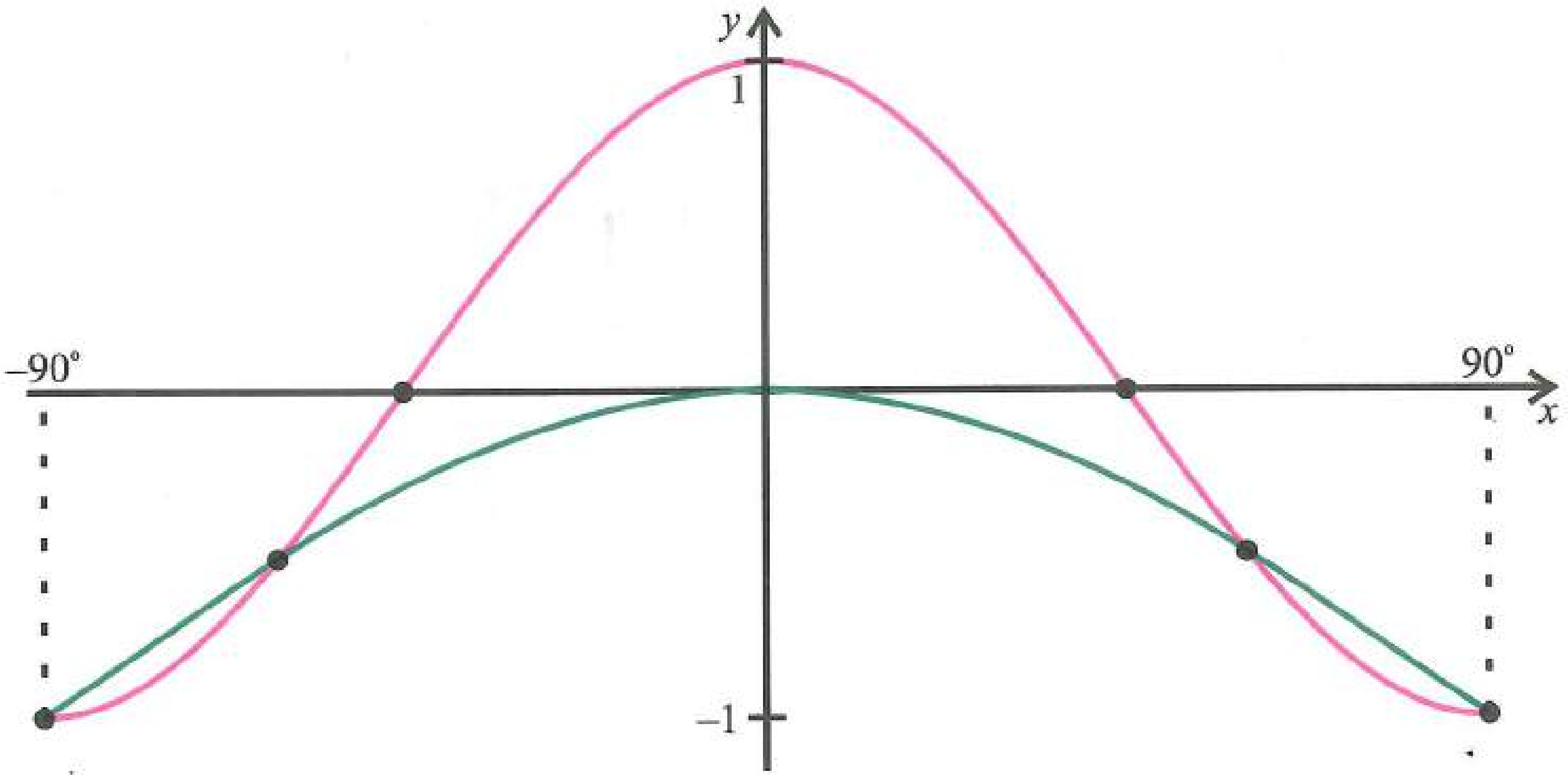
5.3 $\cos\left(-126^\circ\right)$ (4)

5.4 $\cos 18^\circ$ (4)

[13]

Question 6

The graphs of $f(x) = \cos x + b$ and $g(x) = \cos kx$ are sketched below for $x \in \left[-90^\circ; 90^\circ\right]$:



6.1 Determine:

6.1.1 the value of b if f touches the x-axis at the origin. (1)

6.1.2 the amplitude of f. (1)

6.1.3 value of k if the period of g is half the period of f. (1)

6.1.4 the coordinates of the x-intercepts of g. (2)

6.1.5 the maximum value of f. (1)

6.2 Use the graphs to determine the values of x in the interval $x \in \left[-90^\circ; 90^\circ\right]$ for which:

6.2.1 $g(x).f(x) \leq 0$. (2)

6.2.2 $\dfrac{g(x)}{f(x)}$ is undefined. (1)

6.2.3 $\dfrac{g(x)}{f(x)} < 1$ given that f and g intersect at $x = \pm 60^\circ$ and $x = \pm 90^\circ$. (4)

[13]

Question 7

7.1 Simplify $-\cos\left(\theta-90^\circ\right)\cos^2(-\theta)-\sin^2\left(180^\circ+\theta\right)\sin\left(360^\circ+\theta\right)$. (7)

7.2 Without using a calculator, prove that: $\dfrac{-\cos 225^\circ \sin 60^\circ}{\cos 765^\circ}+\dfrac{\sin\left(-210^\circ\right)}{\tan 210^\circ}=\sqrt{3}$. (6)

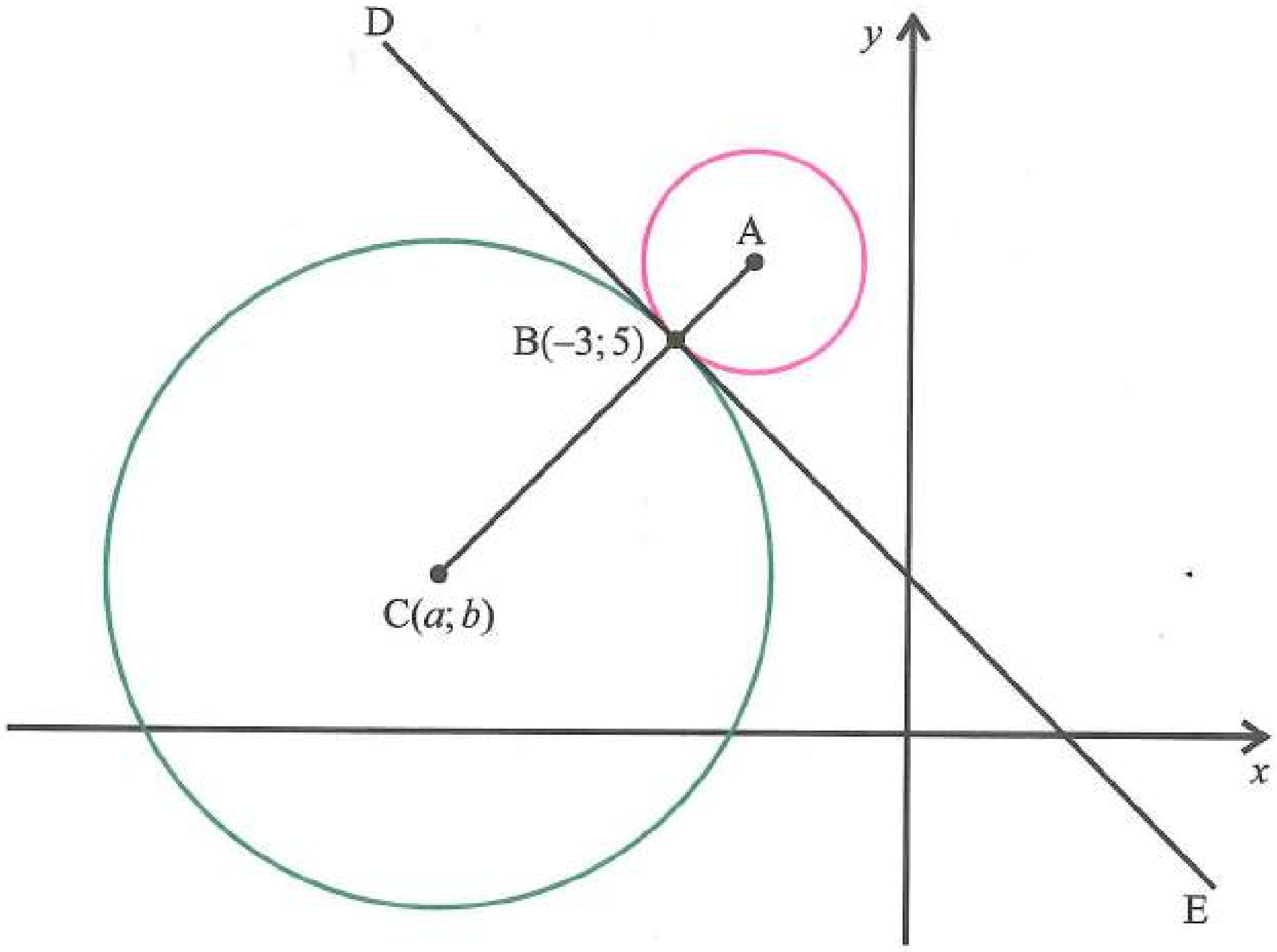
7.3 Determine the general solution, correct to ONE decimal place:

$$\frac{3+2\cos x}{8}=0,325 \qquad (5)$$

[18]

Question 8

In the figure below circles with centres A and C are drawn. The equation of the circle with centre A is $(x+2)^2+(y-6)^2=r^2$. DE is a common tangent to circles A and C at point $B(-3;5)$. It is given that C lies in the second quadrant.



8.1 Write down the coordinates of A and the radius of the circle centre A. (4)

8.2 Determine the equation of the tangent DE in the form $y=mx+c$. (4)

8.3 If $\dfrac{AB}{BC}=\dfrac{1}{3}$, determine the coordinates of C. (8)

[16]

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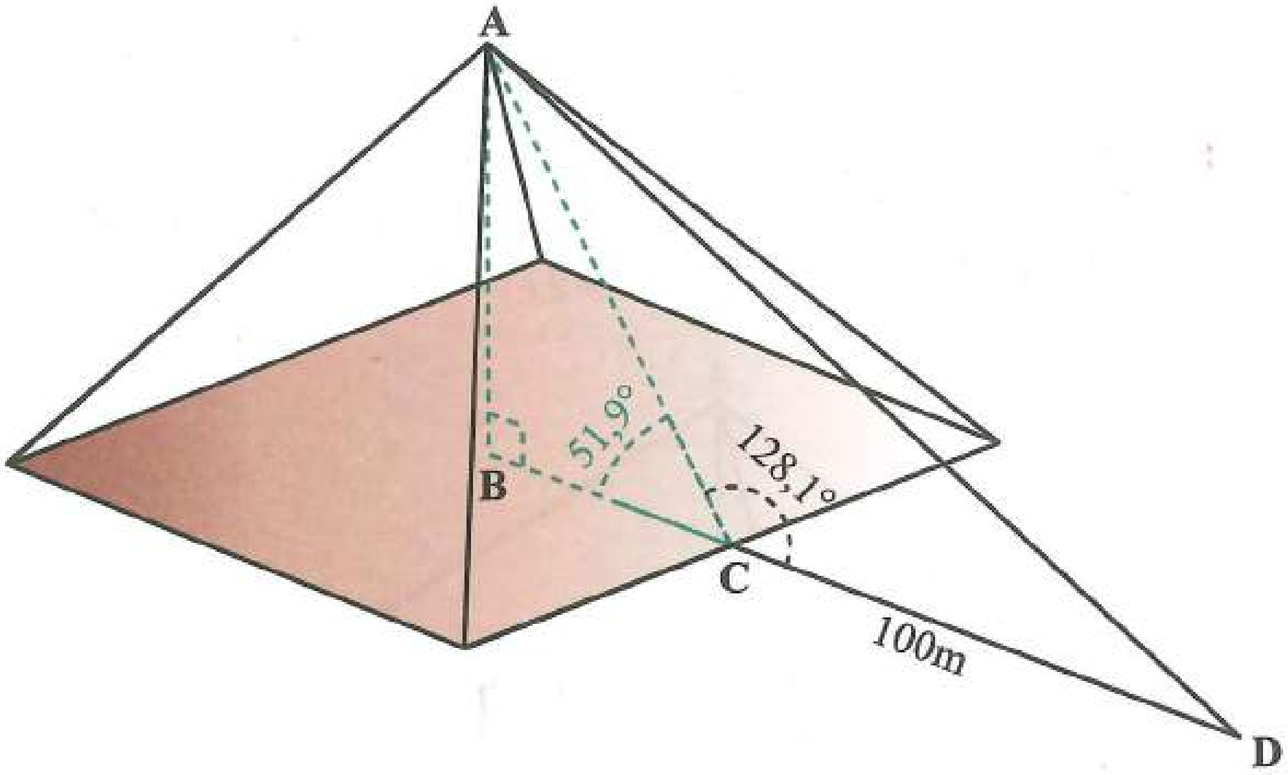
Question 9

The Great Pyramid of Giza was one of the first wonders of the world. For over 4000 years, it was the tallest monument ever built and to this day it is still the largest stone monument in the world. It has been estimated that approximately 2 300 000 blocks were used to build the great pyramid, each averaging 2,5 tons, making up a total mass of 5,75 million tons.

The pyramid has a height of 146 metres and a slope angle $\hat{ACB} = 51{,}9^\circ$ and $\hat{ACD} = 128{,}1^\circ$.



9.1 Determine the slant height, AC, of the pyramid. (3)

9.2 An archaeologist stands at point D, 100 meters from the foot of the pyramid and measures AD to be 120 meters. Is he correct? (4)

9.3 The volume of the great pyramid of Giza is 2 574 466,67 m^3. Determine the length of the side of the base of the pyramid if it is given that the base is a square and that the volume of a pyramid $= \frac{1}{3} \times$ area of base $\times \perp$ height . (4)

9.4 The Empire State building in America was the tallest building in the world between 1931 and 1972. The area of its base is approximately 8093 m^2. Determine how many Empire State buildings would fit on the base of the Great Pyramid of Giza. (2)

[13]

Question 10

Given the data set: $\{2k-4; 2k+1; 2k+6; 2k+10; 2k-3\}$.

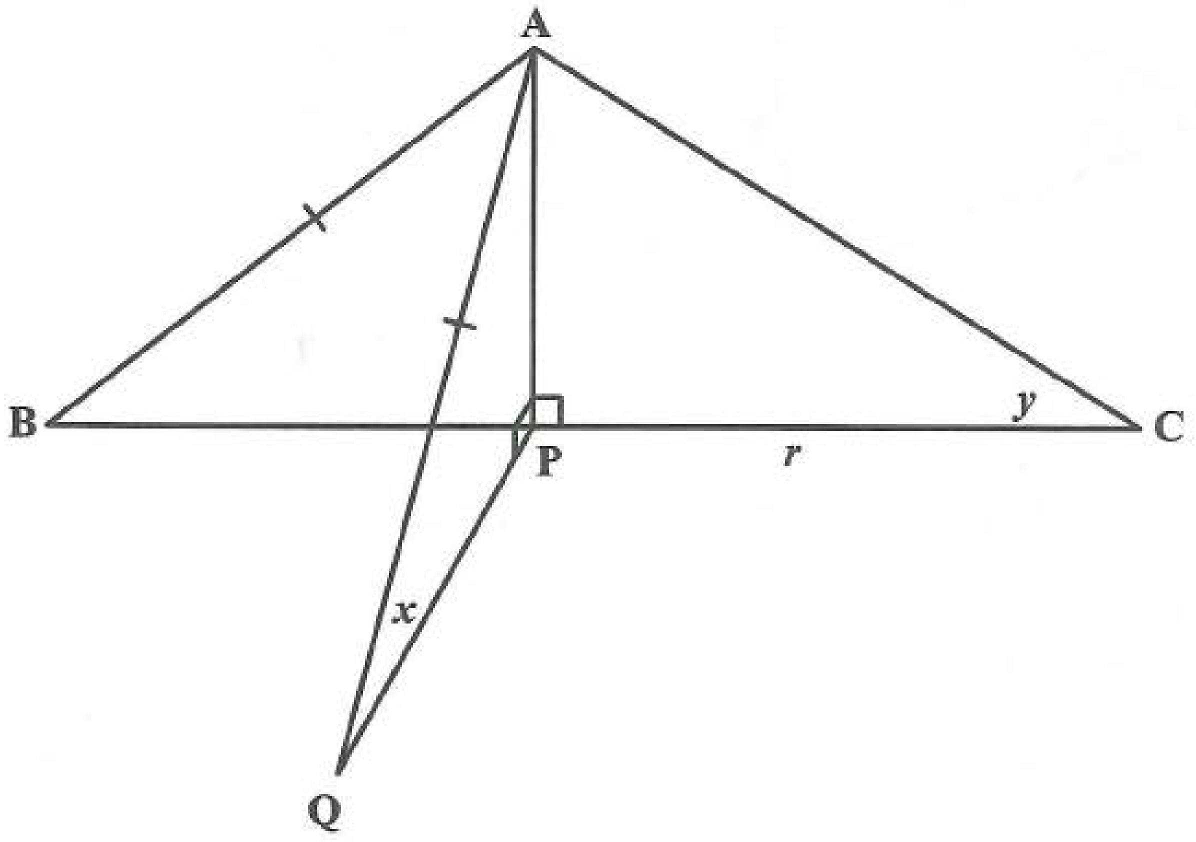
10.1 Showing all calculations, determine the standard deviation for the given set of 5 numbers. (6)

10.2 How would adding 5 to each number affect the mean and standard deviation? (2)

[8]

Question 11

B, C, P and Q are four points on the ground such that B, P and C lie in a straight line. AP is a vertical flagpole. The angle of elevation of A from Q is x, while the angle of elevation of A from C is y. The length of PC is r and $AB = AQ$.



11.1 Show that $AQ = \dfrac{r \tan y}{\sin x}$. (4)

11.2 Calculate the straight line distance between B and Q if:
$Q\hat{P}C = 120^\circ$ and $PQ = 32\,\text{m}$. (4)

[8]

Total: [150]

Answers to this paper can be found on pages 365 and 366.

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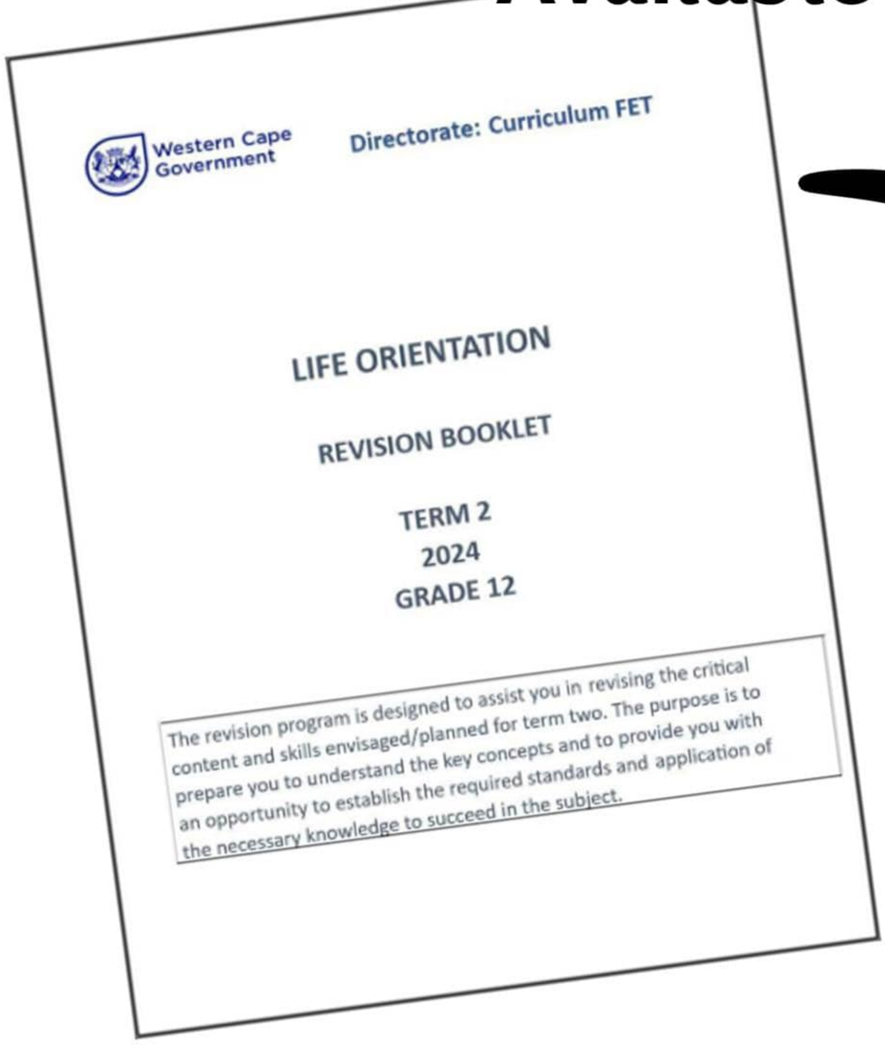

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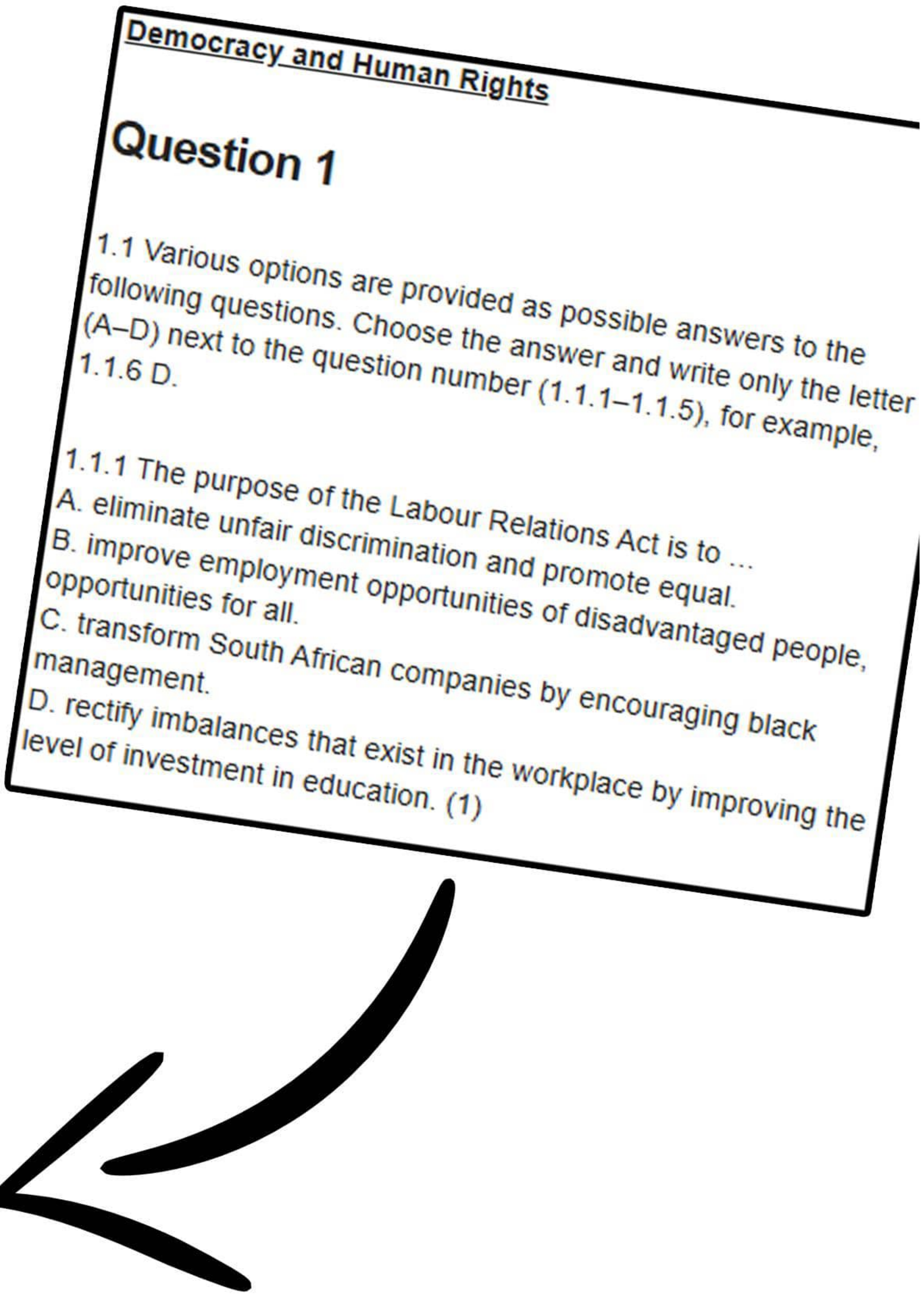
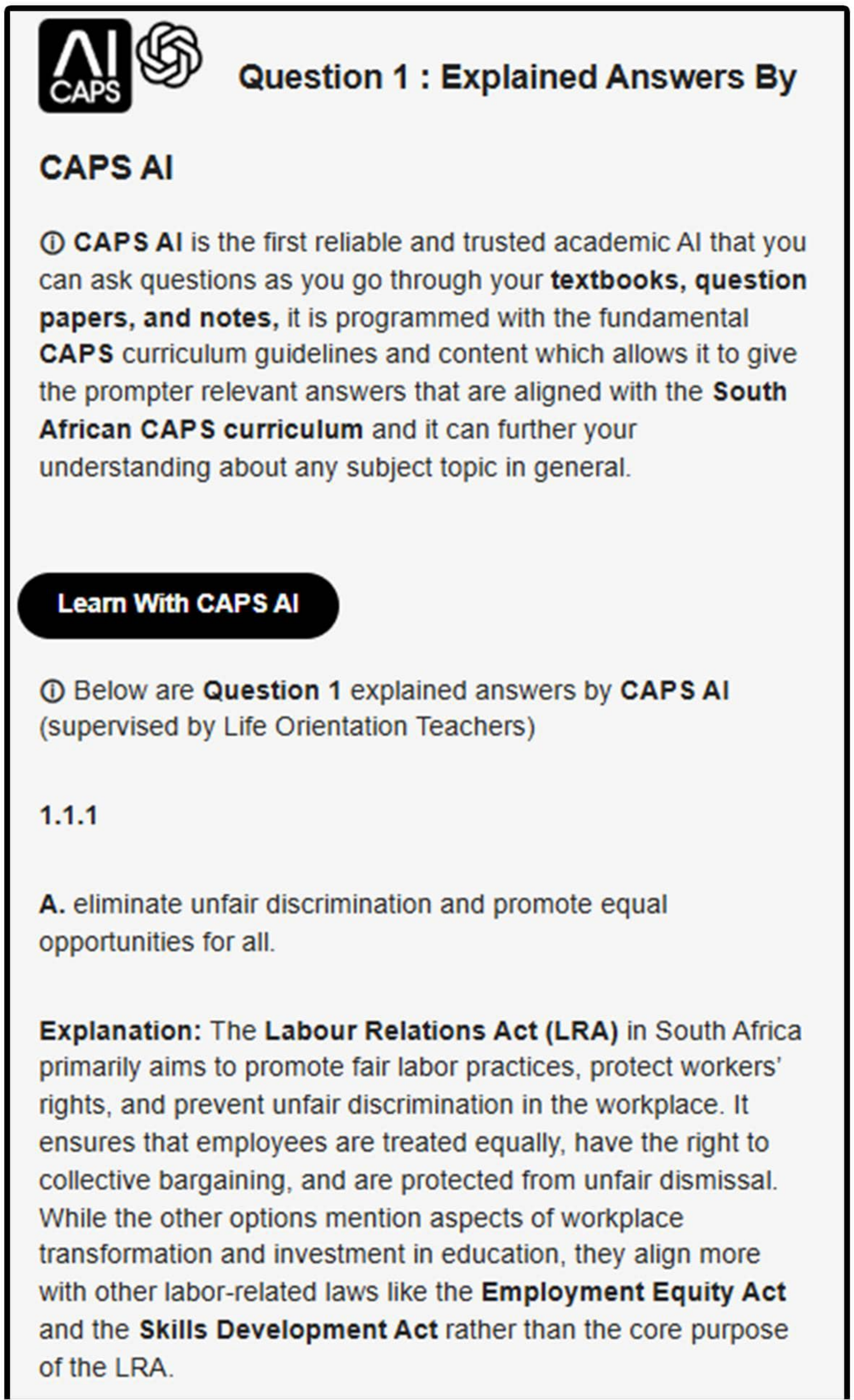


Democracy and Human Rights

Question 1

1.1 Various options are provided as possible answers to the following questions. Choose the answer and write only the letter (A–D) next to the question number (1.1.1–1.1.5), for example, 1.1.6 D.

1.1.1 The purpose of the Labour Relations Act is to …
A. eliminate unfair discrimination and promote equal opportunities for all.
B. improve employment opportunities of disadvantaged people,
C. transform South African companies by encouraging black management.
D. rectify imbalances that exist in the workplace by improving the level of investment in education. (1)



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1.1.1

A. eliminate unfair discrimination and promote equal opportunities for all.

Explanation: The **Labour Relations Act (LRA)** in South Africa primarily aims to promote fair labor practices, protect workers' rights, and prevent unfair discrimination in the workplace. It ensures that employees are treated equally, have the right to collective bargaining, and are protected from unfair dismissal. While the other options mention aspects of workplace transformation and investment in education, they align more with other labor-related laws like the **Employment Equity Act** and the **Skills Development Act** rather than the core purpose of the LRA.

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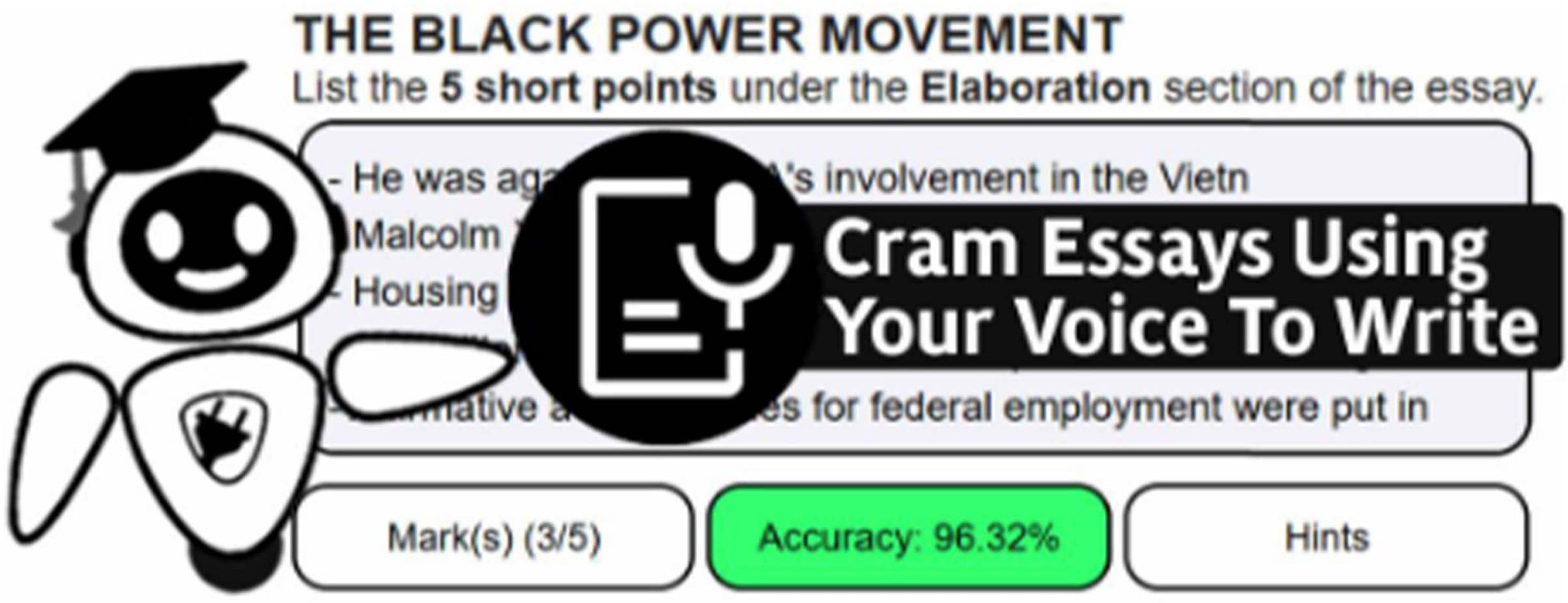

THE BLACK POWER MOVEMENT

List the **5 short points** under the **Elaboration** section of the essay.

- He was ag 's involvement in the Vietn
- Malcolm
- Housing
- mative a es for federal employment were put in

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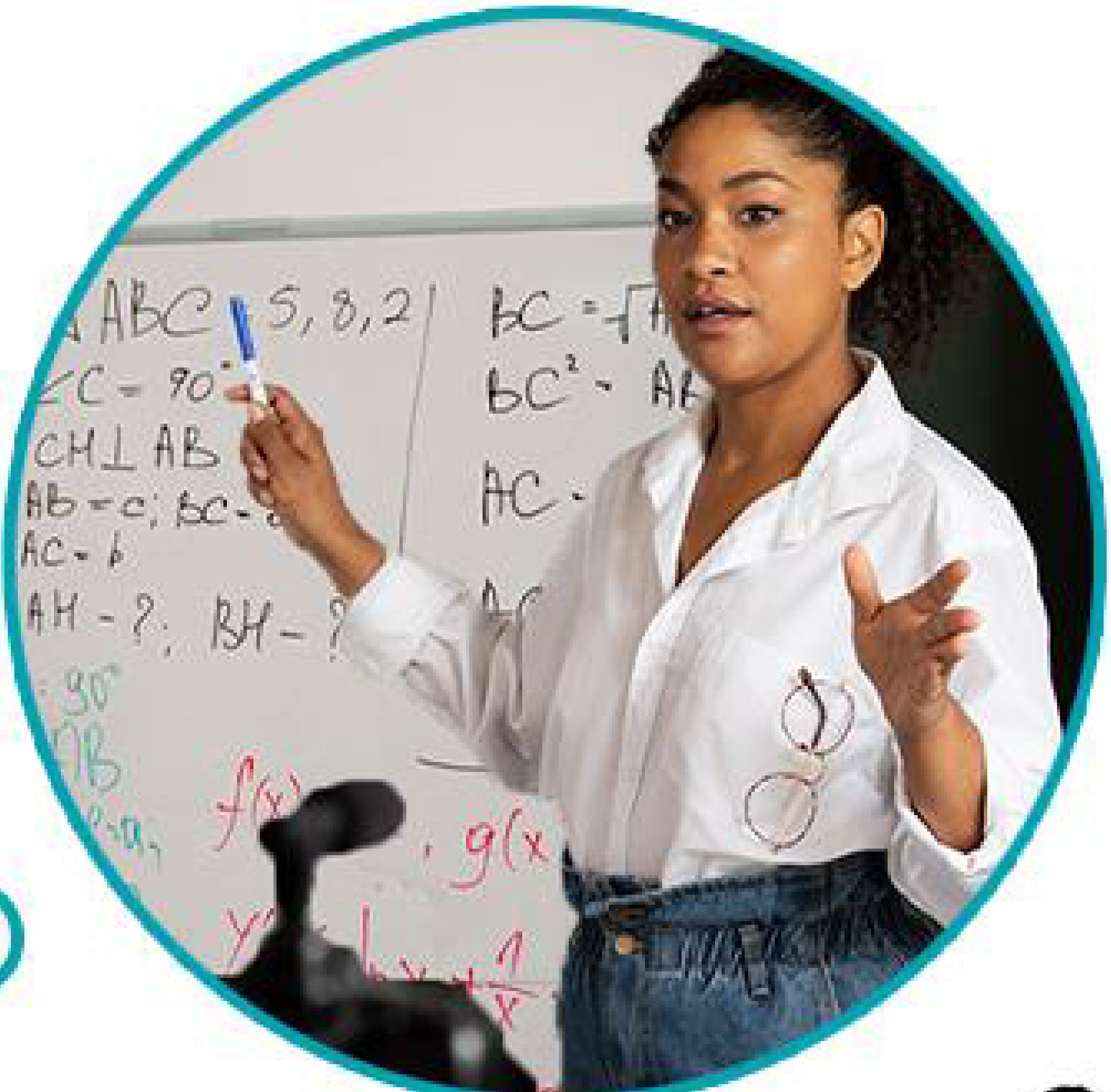

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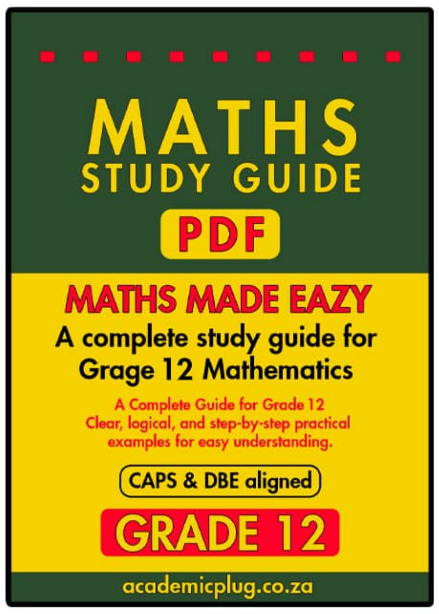
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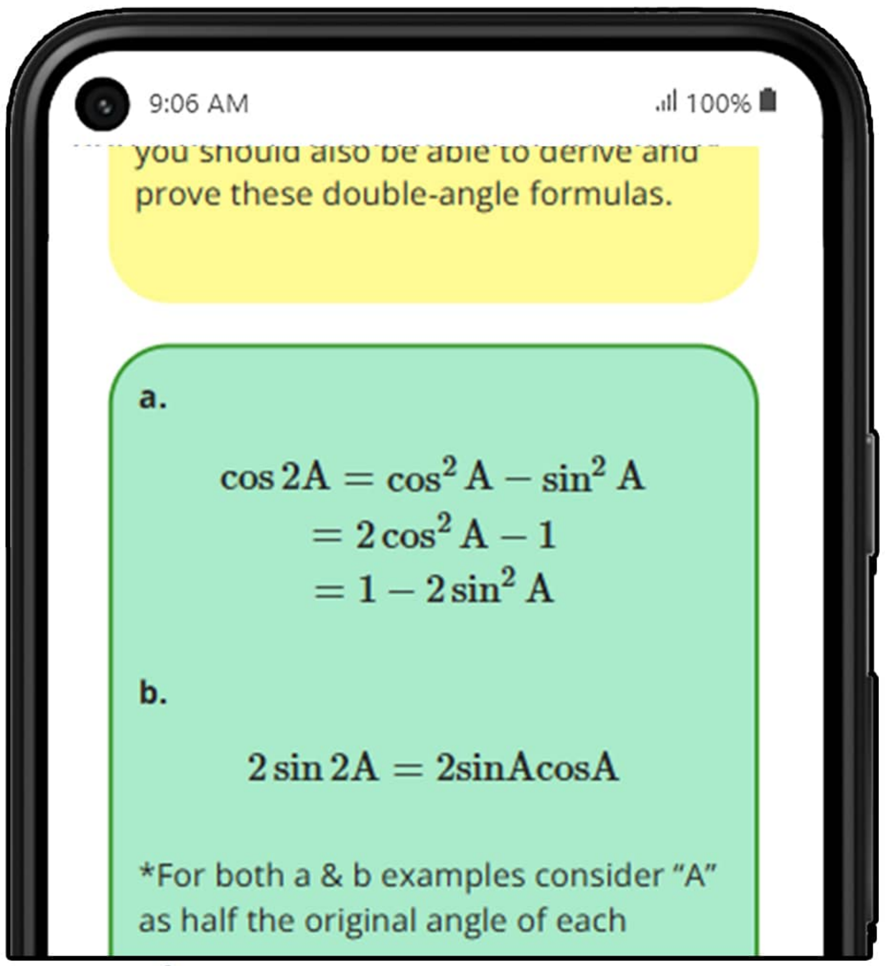

you should also be able to derive and prove these double-angle formulas.

a.

$$\cos 2A = \cos^2 A - \sin^2 A$$
$$= 2\cos^2 A - 1$$
$$= 1 - 2\sin^2 A$$

b.

$$2\sin 2A = 2\sin A\cos A$$

*For both a & b examples consider "A" as half the original angle of each

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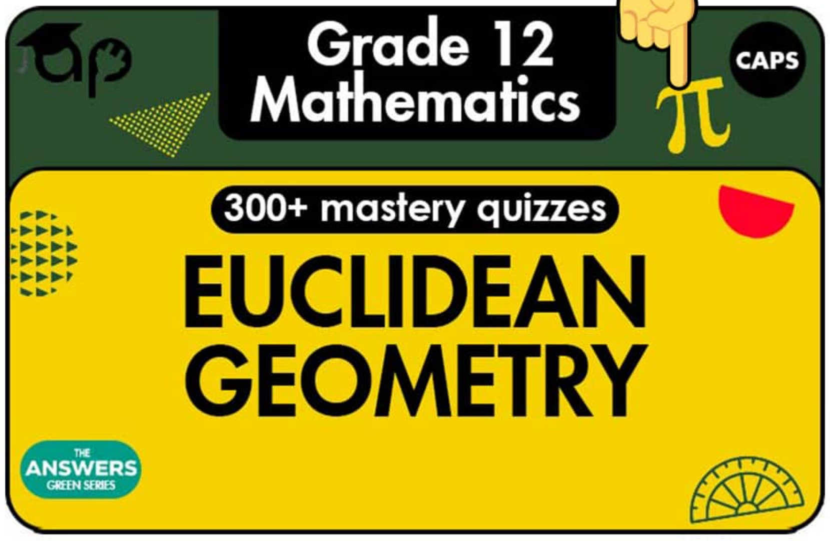
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CHAPTER 12
EUCLIDEAN GEOMETRY

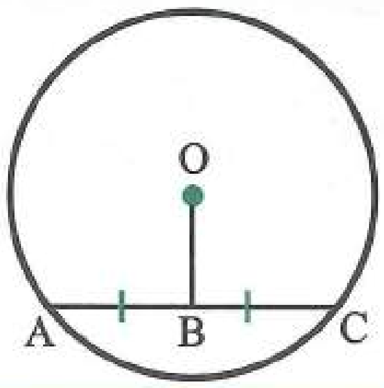
WHAT YOU WILL LEARN IN THIS CHAPTER

PAGE	CONTENT	CAPS	ASSESSMENTS YOU WILL BE ABLE TO:
239	**A. Revision**	7	• Understand the conditions necessary for a polygon to be similar. • Prove that a line drawn parallel to one side of triangle divides the other 2 sides in proportion. • Prove that equiangular triangles are similar. • Prove that triangles with sides in proportion are similar. • Prove the Pythagorean Theorem using similar triangles • Solve geometry riders.
239	• Grade 11 Circle Geometry		
241	• Straight Lines		
241	• Parallel Lines		
242	• Triangles		
242	• Congruency		
243	• Properties of Quadrilaterals		
244	**B. Ratios and Proportion**		
245	• Area of Triangles		
246	• Theorem 1: Line drawn parallel to one side of a triangle		
246	• Theorem 1: Corollaries		
251	• Theorem 1: Converse		
253	• The Midpoint Theorem		
253	• Converse of the Midpoint Theorem		
255	**C. Similarity**		
255	• Similarity of Polygons		
255	• Theorem 2: Equiangular Triangles are Similar		
257	• Theorem 2: Converse		
259	• The Theorem of Pythagoras		

A. REVISION

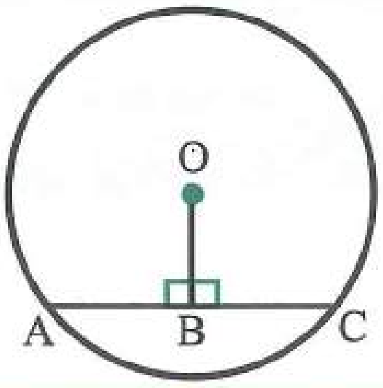
GRADE 11 CIRCLE GEOMETRY

The following section revises the circle theorems which you learnt in Grade 11:



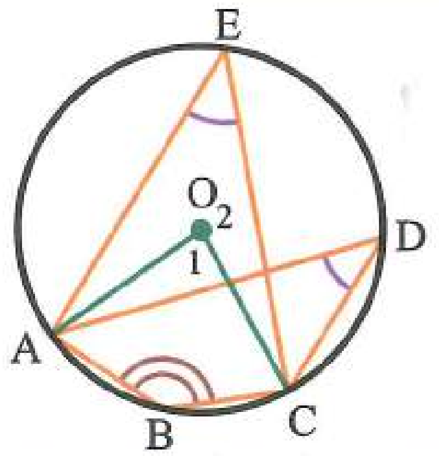
Given:	**Conclusion:**
AC ⊥ OB	AB = BC

Reason:
Line from centre ⊥ to chord



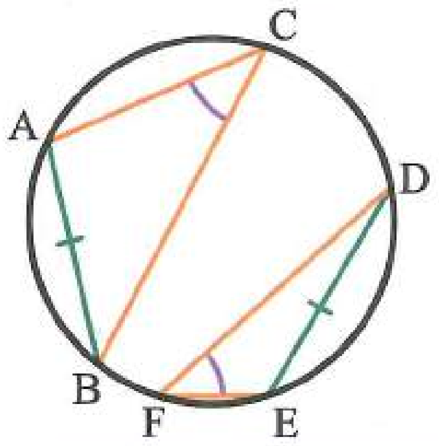
Given:	**Conclusion:**
AB = BC	AC ⊥ OB

Reason:
Line from centre to midpoint of chord



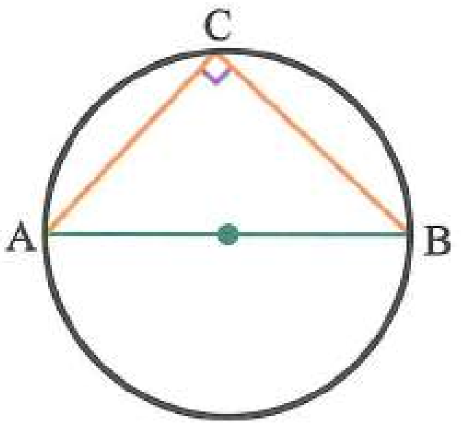
Given:	**Conclusion:**
A, B, C, D and E are points on the circle with centre O.	$\hat{O}_1 = 2\hat{E} = 2\hat{D}$ $\hat{O}_2 = 2\hat{B}$

Reason:
∠ at centre = 2∠ at circumference



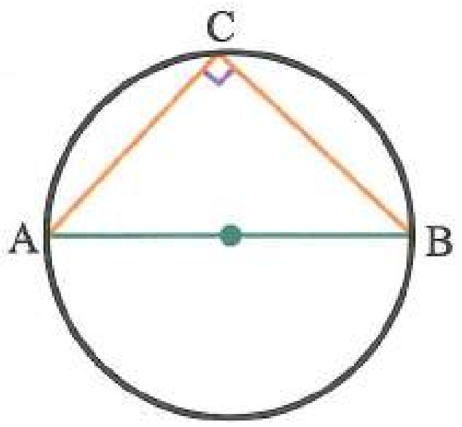
Given:	**Conclusion:**
AB = DE	$\hat{C} = \hat{F}$

Reason:
= chords subtend = ∠'s



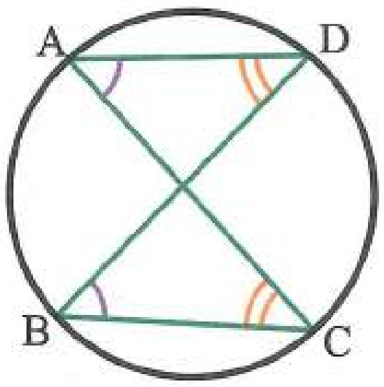
Given:	**Conclusion:**
AB is a diameter	$\hat{C} = 90^\circ$

Reason:
∠ in a semi - circle



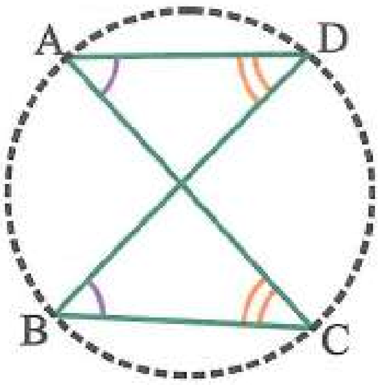
Given:	**Conclusion:**
$\hat{C} = 90^\circ$	AB is a diameter

Reason:
Chord subtends a 90° ∠



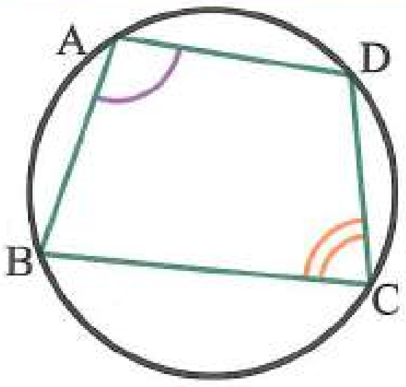
Given:	Conclusion:
ABCD is a cyclic quadrilateral	$\hat{A} = \hat{B}$ and $\hat{D} = \hat{C}$

Reason:
∠'s in same segment



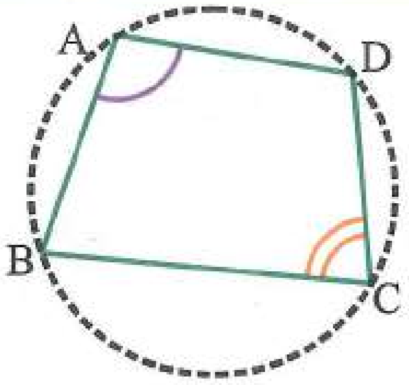
Given:	Conclusion:
$\hat{A} = \hat{B}$ and $\hat{D} = \hat{C}$	ABCD is a cyclic quadrilateral

Reason:
Line seg. subt. = ∠'s on the same side



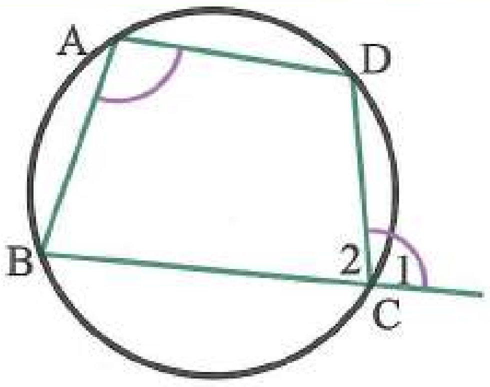
Given:	Conclusion:
ABCD is a cyclic quadrilateral	$\hat{A} + \hat{C} = 180^\circ$

Reason:
Opp. ∠'s of a cyclic quad.



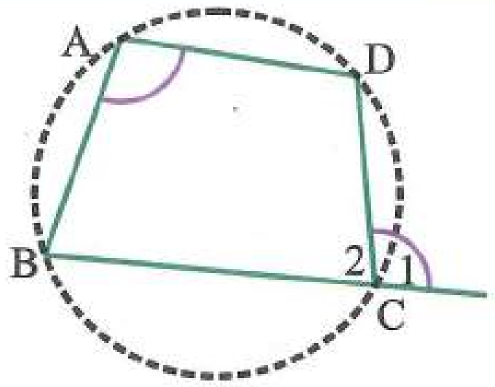
Given:	Conclusion:
$\hat{A} + \hat{C} = 180^\circ$	ABCD is a cyclic quadrilateral

Reason:
Opp. ∠'s supplementary



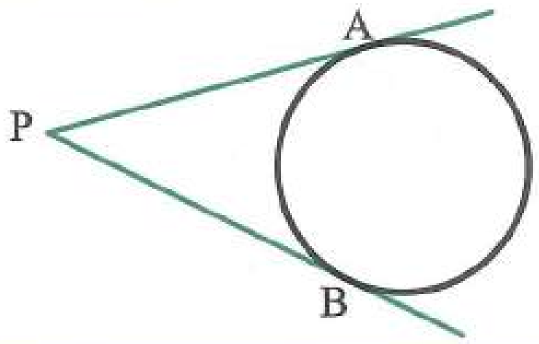
Given:	Conclusion:
ABCD is a cyclic quadrilateral	$\hat{A} = \hat{C}_1$

Reason:
Ext. ∠ of a cyclic quad.



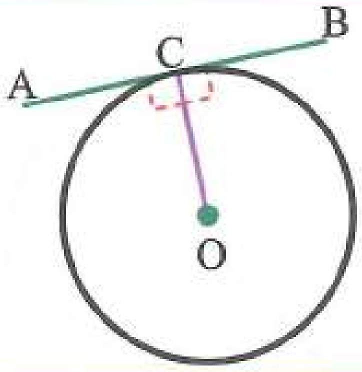
Given:	Conclusion:
$\hat{A} = \hat{C}_1$	ABCD is a cyclic quadrilateral

Reason:
Ext. ∠ = opp. interior ∠



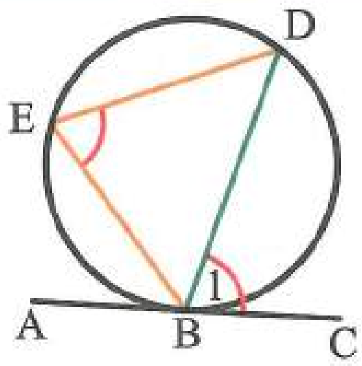
Given:	Conclusion:
AP and BP are tangents	AP = BP

Reason:
Tangents from same pt.



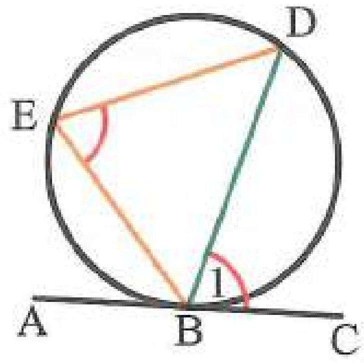
Given:	Conclusion:
AB is a tangent at C	$OC \perp AB$
$OC \perp AB$	AB is a tangent at C

Reason:
Tangent ⊥ to radius
Line to ⊥ radius



Given:	Conclusion:
ABC is a tangent	$\hat{B}_1 = \hat{E}$

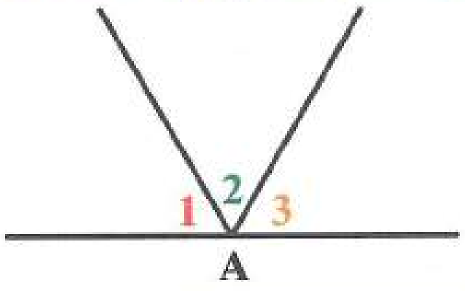
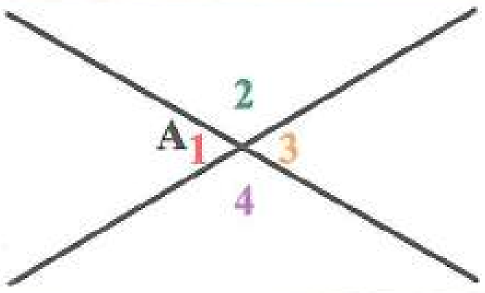
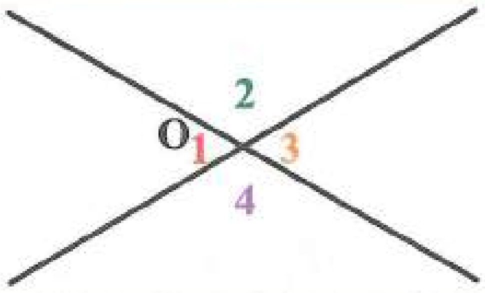
Reason:
∠ between tan. and chord



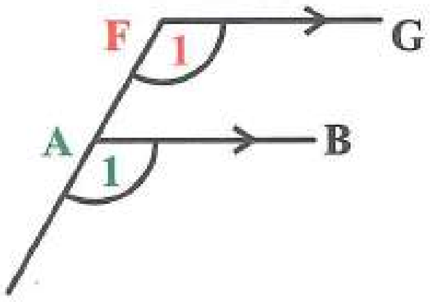
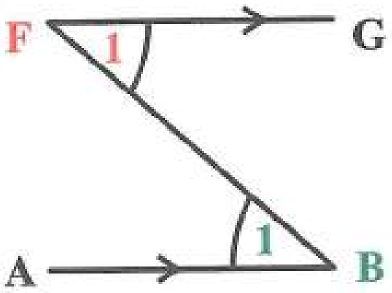
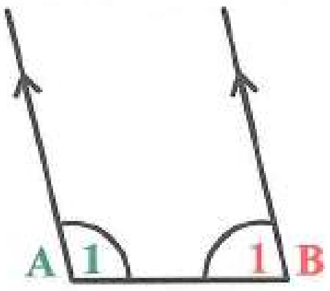
Given:	Conclusion:
$\hat{B}_1 = \hat{E}$	ABC is a tangent

Reason:
∠ between line and chord

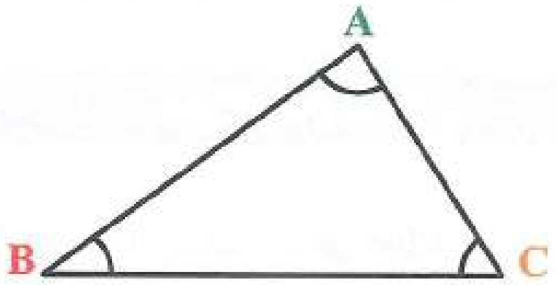
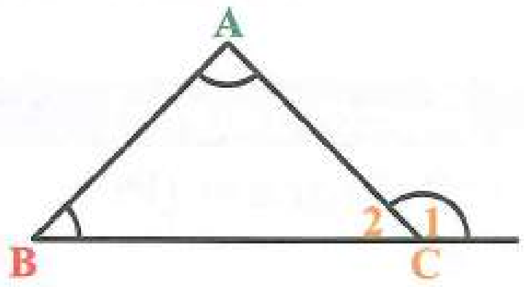
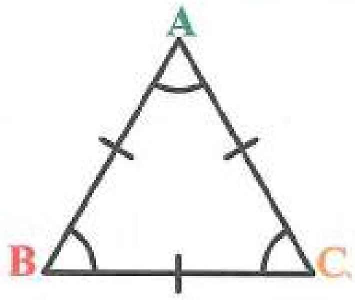
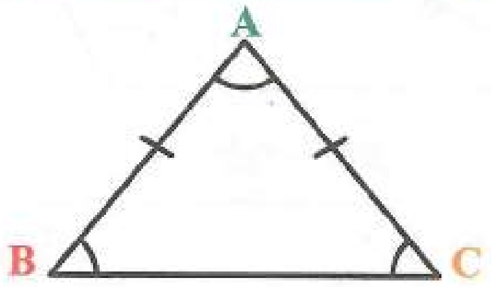
STRAIGHT LINES

Adjacent angles on a straight line are supplementary.	Angles around a point sum to 360°.	Vertically opposite angles are equal.
		
$\hat{A}_1+\hat{A}_2+\hat{A}_3=180^\circ$	$\hat{A}_1+\hat{A}_2+\hat{A}_3+\hat{A}_4=360^\circ$	$\hat{O}_1=\hat{O}_3$ and $\hat{O}_2=\hat{O}_4$

PARALLEL LINES

Corresponding angles are equal (F shape).	Alternate angles are equal (Z or N shape).	Co-interior angles are supplementary (U shape).
		
$\hat{F}_1=\hat{A}_1$	$\hat{F}_1=\hat{B}_1$	$\hat{A}_1+\hat{B}_1=180^\circ$

TRIANGLES

The interior angles of a triangle are supplementary.	The exterior angle of a triangle is equal to the sum of the interior opposite angles.
	
$\hat{A}+\hat{B}+\hat{C}=180^\circ$	$\hat{C}_1=\hat{A}+\hat{B}$
In an equilateral triangle all sides are equal and all angles are equal to 60°.	**Angles opposite equal sides are equal. Sides opposite equal angles are equal.**
	
$\hat{A}=\hat{B}=\hat{C}=60^\circ$ and $AB=AC=BC$	$\hat{B}=\hat{C}$ and $AB=AC$

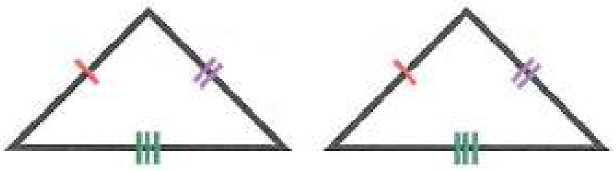
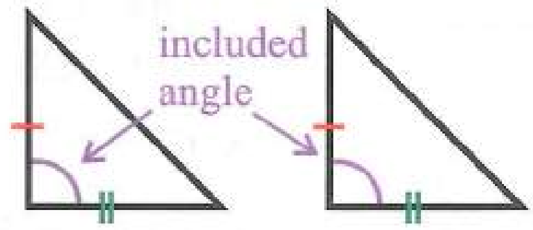
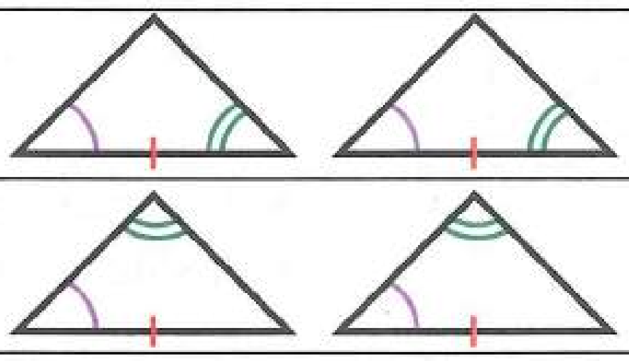
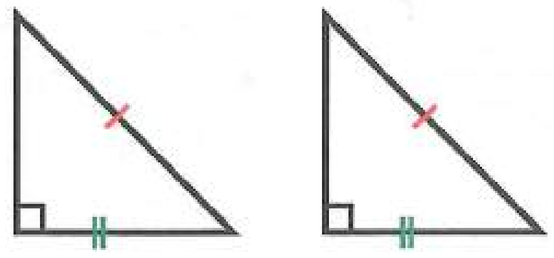
CONGRUENCY

If triangles are congruent it means that they are exactly the same. In other words, all corresponding angles and sides in each triangle are equal.

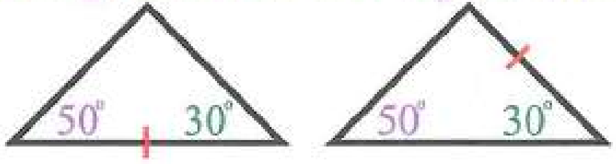
There are 4 conditions for triangles to be congruent.

If any of the conditions listed below are true then the triangles are congruent.

The symbol ≡ is used for congruency.

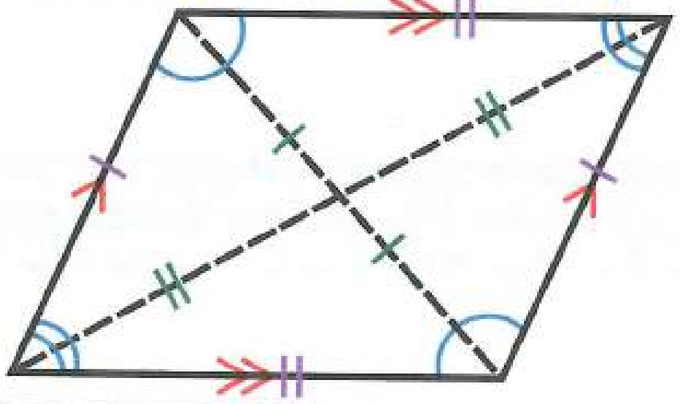
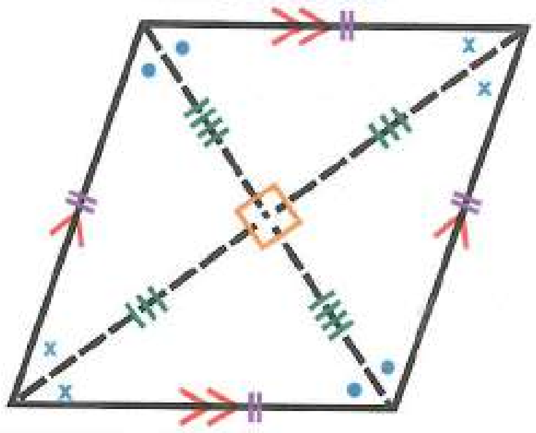
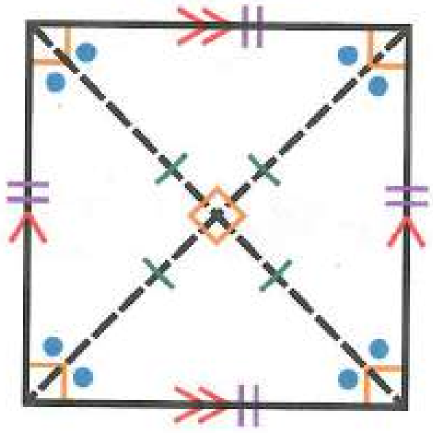
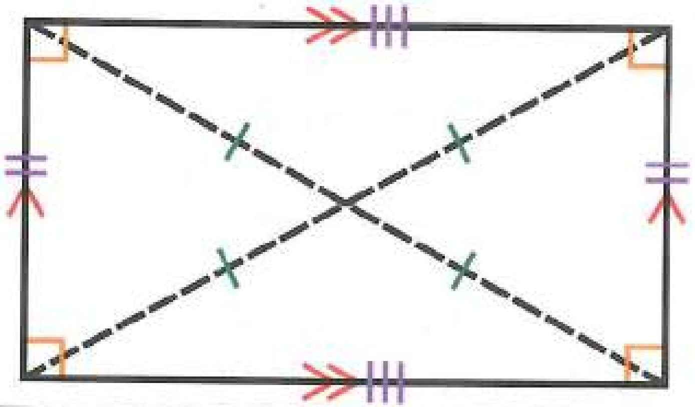
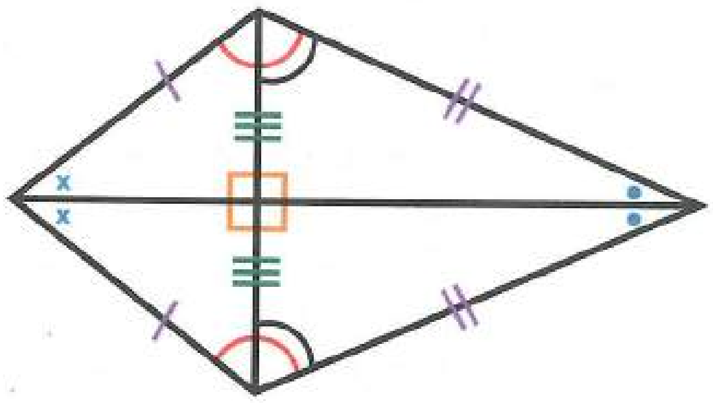
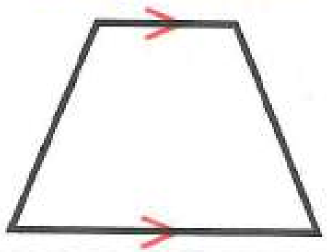
Condition / Reason	Example	Explanation
Side Side Side (SSS)		If triangles have three corresponding sides that are equal, the triangles are congruent.
Side Angle Side (SAS)		If triangles have two corresponding sides and included angles that are equal, the triangles are congruent.
Angle Angle Side Side Angle Angle Angle Side Angle (AAS) / (SAA) / (ASA)		If triangles have two corresponding angles and single corresponding sides that are equal, the triangles are congruent.
Right angle Hypotenuse Side (RHS)		If the hypotenuse and one other corresponding side are equal in right-angled triangles, the triangles are congruent.

Using AAS, the sides that are equal have to be the corresponding sides. The triangles below are not congruent as the equal sides are not the same relative to the equal angles.



PROPERTIES OF QUADRILATERALS

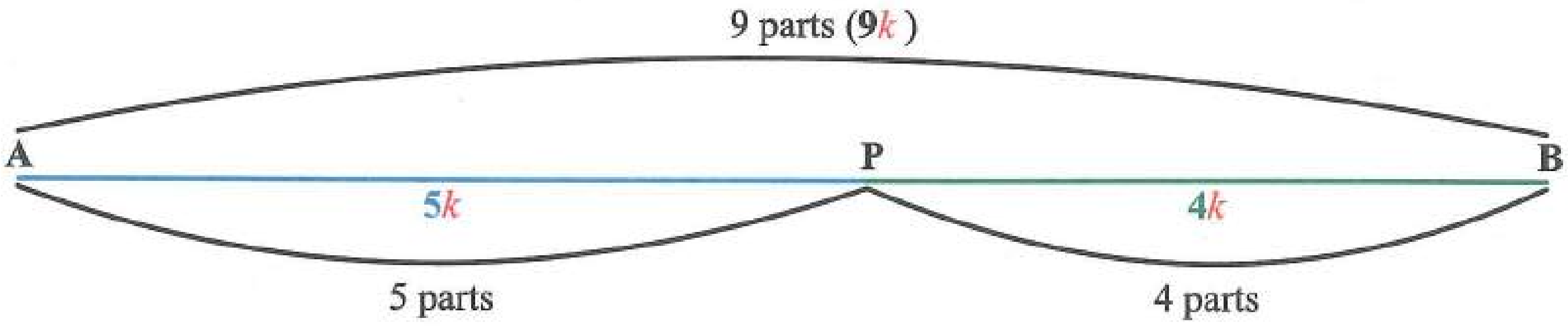
A quadrilateral is defined as a closed four-sided figure whose interior angles add up to 360°. The table below shows some common quadrilaterals and their properties.

Quadrilateral	Properties
Parallelogram (Parm) 	• Opposite sides are **parallel** and **equal** in length • Opposite angles are **equal** • **Diagonals bisect** each other
Rhombus 	• All **sides** are **equal** in length • Opposite sides are **parallel** • Opposite angles are **equal** • Diagonals **bisect** each other at **90°** • Diagonals bisect the corner **angles**
Square 	• All **sides** are **equal** in length • Opposite sides are **parallel** • Corner **angles** equal **90°** • Diagonals are **equal** and **bisect** each other at **90°** • Diagonals **bisect** the corner **angles** •
Rectangle 	• Opposite sides are **parallel** and **equal** in length • Corner **angles equal 90°** • **Diagonals** are **equal** and **bisect** each other
Kite 	• Two pairs of **adjacent sides** are **equal** • A single pair of **opposite angles are equal** • Diagonals intersect each other at **90°** • One diagonal **bisects** the corner **angles** • Shorter diagonal is **bisected** by the longer diagonal
Trapezium 	• One pair of **opposite sides** are **parallel**

B. RATIOS AND PROPORTION

Example 1: **If P divides line segment AB in a ratio of $5:4$, determine the ratios of:**

a) $\dfrac{\mathbf{AP}}{\mathbf{AB}}$ **b)** $\dfrac{\mathbf{AB}}{\mathbf{AP}}$ **c)** $\dfrac{\mathbf{PB}}{\mathbf{AB}}$



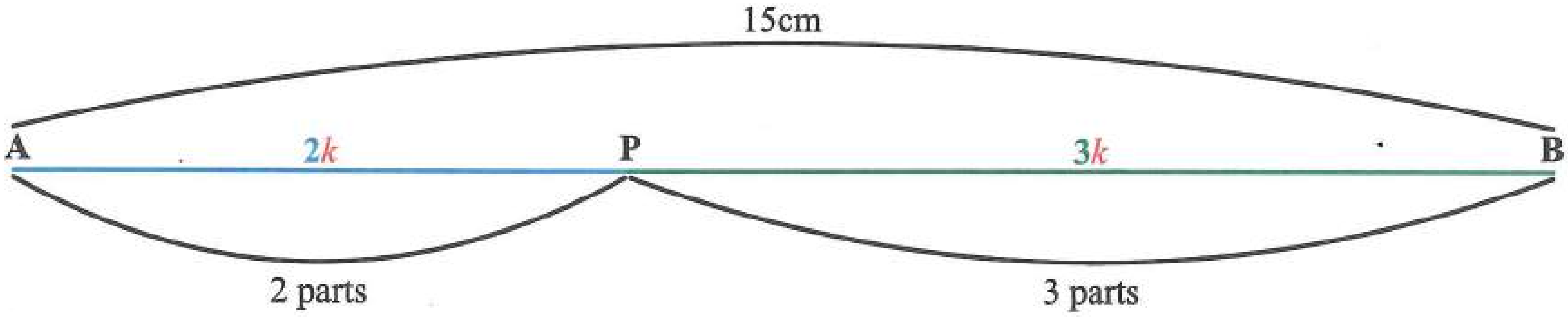
As P divides AB in the ratio of 5:4, let $\text{AP} = 5k$ units and $\text{PB} = 4k$ units.

$\therefore \text{AB} = 9k$

a) $\dfrac{\text{AP}}{\text{AB}} = \dfrac{5k}{9k} = \dfrac{5}{9}$ b) $\dfrac{\text{AB}}{\text{AP}} = \dfrac{9k}{5k} = \dfrac{9}{5}$ c) $\dfrac{\text{PB}}{\text{AB}} = \dfrac{4k}{9k} = \dfrac{4}{9}$

It is important to note that the ratios do not necessarily represent the lengths of sides. This is the reason for the adding *k* into the ratios.

Example 2: **The line segment AB below is 15cm in length. P divides AB in the ratio $\text{AP} : \text{PB} = 2 : 3$. Determine the lengths of AP and PB.**



As P divides AB in the ratio of 2:3, let $\text{AP} = 2k$ units and $\text{PB} = 3k$ units.

$\therefore \text{AB} = 5k$

Method 1:

$\dfrac{\text{AP}}{\text{AB}} = \dfrac{2k}{5k}$ and $\dfrac{\text{PB}}{\text{AB}} = \dfrac{3k}{5k}$

$\therefore \dfrac{\text{AP}}{15} = \dfrac{2}{5}$ and $\dfrac{\text{PB}}{15} = \dfrac{3}{5}$

$\therefore \text{AP} = \dfrac{2}{5} \times 15$ and $\text{PB} = \dfrac{3}{5} \times 15$

$\therefore \text{AP} = 6\text{cm}$ and $\text{PB} = 9\text{cm}$

Method 2:

$2k + 3k = 15\text{cm}$

$\therefore 5k = 15$

$\therefore k = 3\text{cm}$

$\therefore 1 \text{ part} = 3\text{cm}$

$\therefore \text{AP} = 2k$ and $\text{PB} = 3k$

$\therefore \text{AP} = 2(3)$ and $\text{PB} = 3(3)$

$\therefore \text{AP} = 6\text{cm}$ and $\text{PB} = 9\text{cm}$

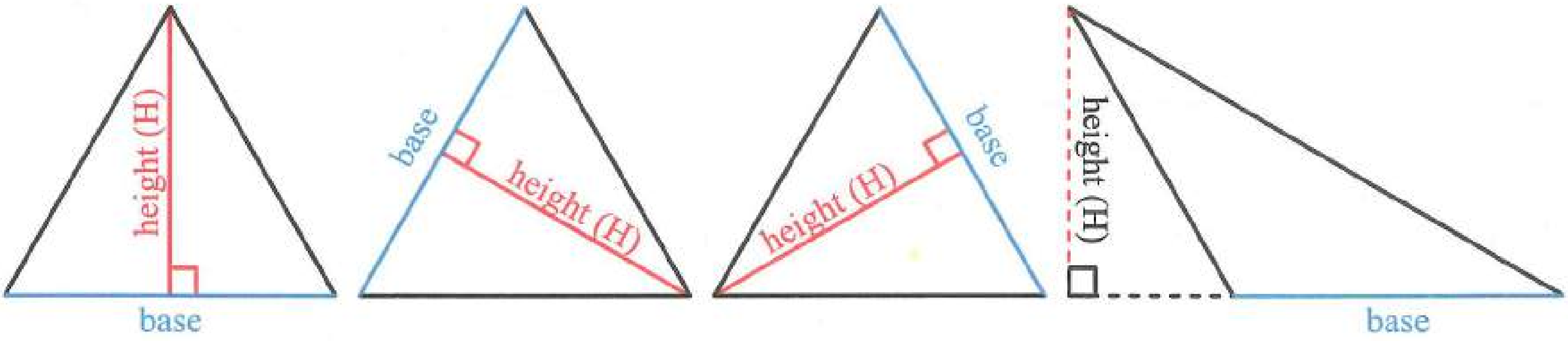
It is important to note that the ratios do not necessarily represent the lengths of sides. This is the reason for the adding *k* into the ratios.

Complete Exercise 1 on page 264.

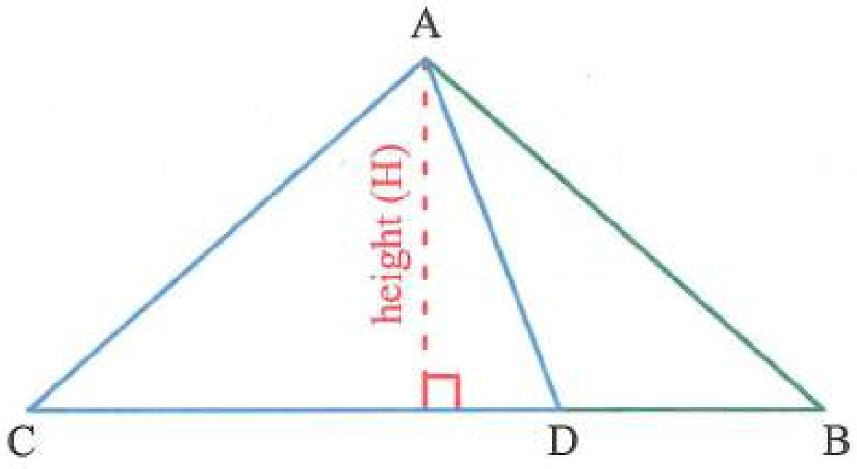
AREA OF TRIANGLES

The three concepts shown below are important to understand before dealing with proportionality theorems and triangles:

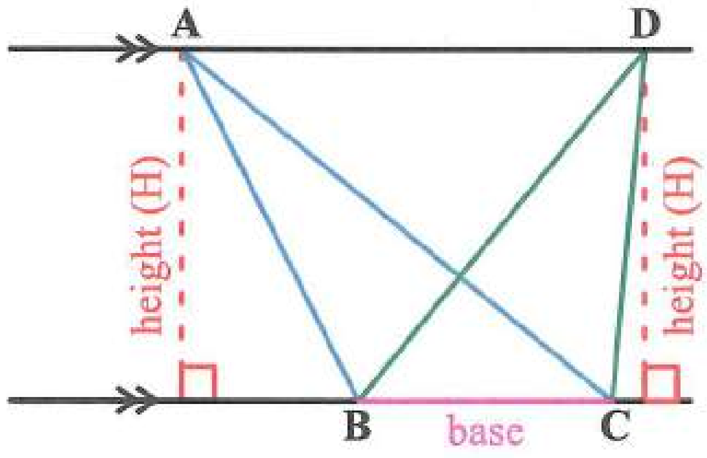
1. **The height (altitude) of a triangle is always relative to the side chosen as the base when using the formula $\text{Area} = \frac{1}{2} \times \text{base} \times \perp \text{height}$. All of the triangles below have the same area.**

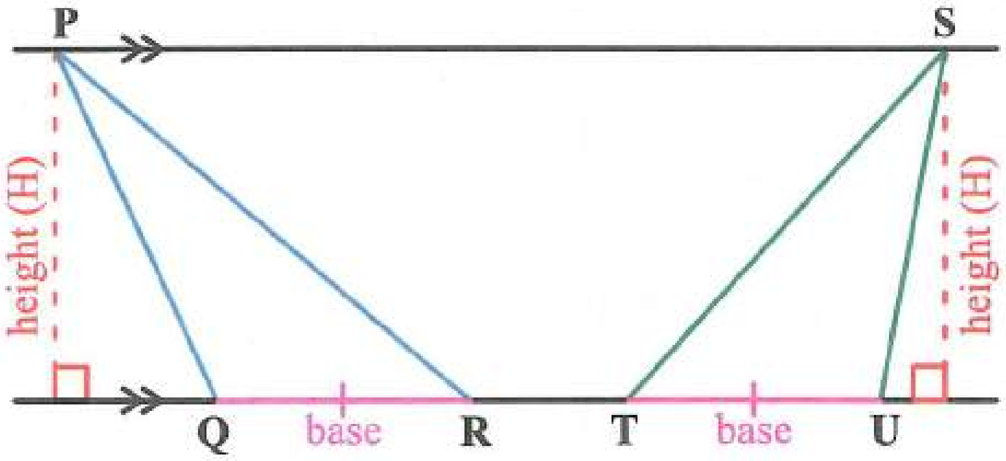

height (H)
base
base
height (H)
height (H)
base
height (H)
base

2. **Triangles which share a common vertex have the same height. In the diagram below triangles ACB, ABD and ACD all have the same perpendicular height.**


A
height (H)
C
D
B

3. **Triangles on the same base (or with equal bases) between parallel lines are equal in area.**


A
D
height (H)
height (H)
B
base
C


P
S
height (H)
height (H)
Q
base
R
T
base
U

THEOREM 1: LINE DRAWN PARALLEL TO ONE SIDE OF A TRIANGLE

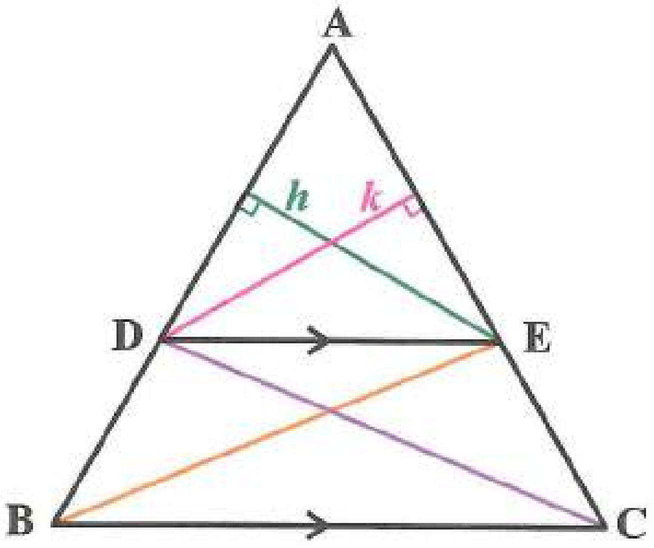
A line drawn parallel to one side of a triangle divides the other two sides proportionally.

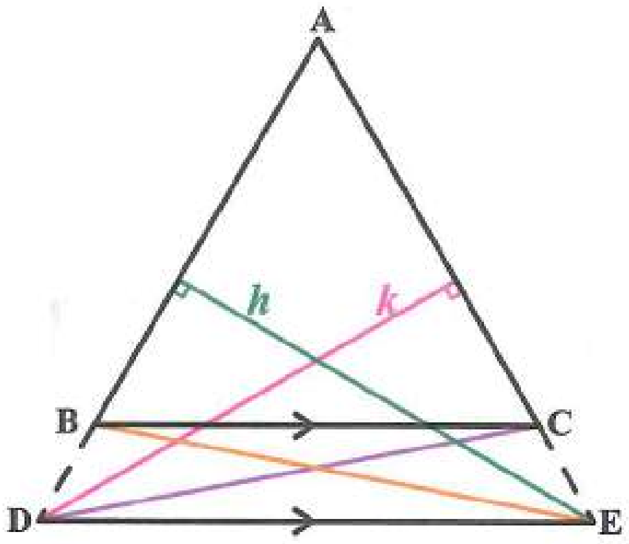
Proof:

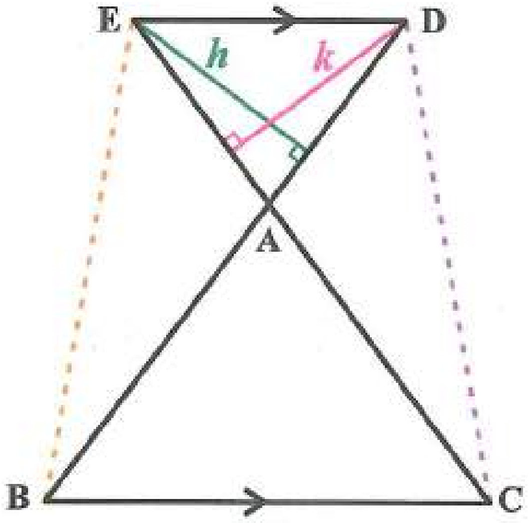
Given: In △ABC, D lies on AB and E lies on AC (or AB and AC are produced in either direction) and DE//BC.

Required to prove (RTP): $\frac{AD}{DB}=\frac{AE}{EC}$.

Construction: Draw altitudes h and k for the altitudes of △ADE relative to bases AD and AE. Join DC and BE







$$\frac{\text{Area }\triangle ADE}{\text{Area }\triangle BDE}=\frac{\cancel{\frac{1}{2}}.AD.\cancel{h}}{\cancel{\frac{1}{2}}.DB.\cancel{h}}=\frac{AD}{DB}$$ →△ADE and △BDE have the same height

$$\frac{\text{Area }\triangle ADE}{\text{Area }\triangle CED}=\frac{\cancel{\frac{1}{2}}.AE.\cancel{k}}{\cancel{\frac{1}{2}}.EC.\cancel{k}}=\frac{AE}{EC}$$ →△ADE and △CED have the same height

But area △BDE = △CED → same base and height lying between // lines DE and BC

$$\therefore\frac{\text{Area }\triangle ADE}{\text{Area }\triangle BDE}=\frac{\text{Area }\triangle ADE}{\text{Area }\triangle CED}$$

$$\therefore\frac{AD}{DB}=\frac{AE}{EC}$$

THEOREM 1: COROLLARIES

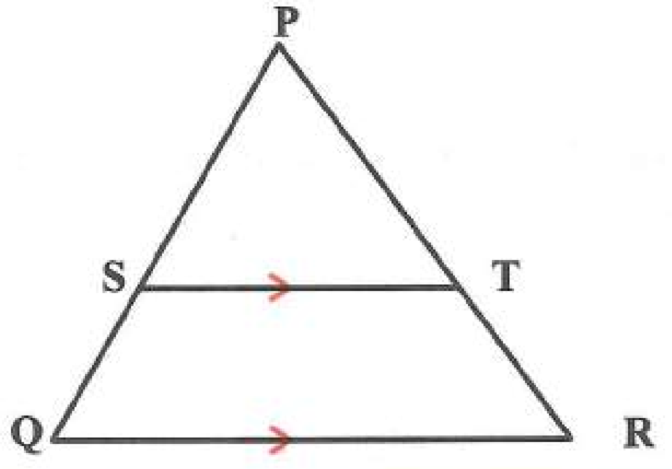
1. $\frac{AB}{AD}=\frac{AC}{AE}$	**2.** $\frac{AB}{DB}=\frac{AC}{EC}$
3. $\frac{BD}{DA}=\frac{CE}{EA}$	**4.** $\frac{BD}{BA}=\frac{CE}{CA}$

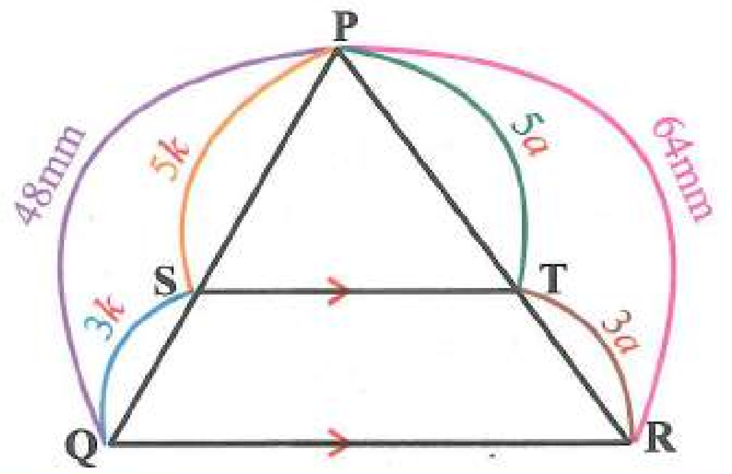
When using this theorem the reason used is:

Line // to one side of △.

Example 1: S and T are points on sides PQ and PR of $\triangle PQR$ so that ST//QR. $PQ = 48\text{mm}$ and $PS : SQ = 5 : 3$. It is further given that $PR = 64\text{mm}$.

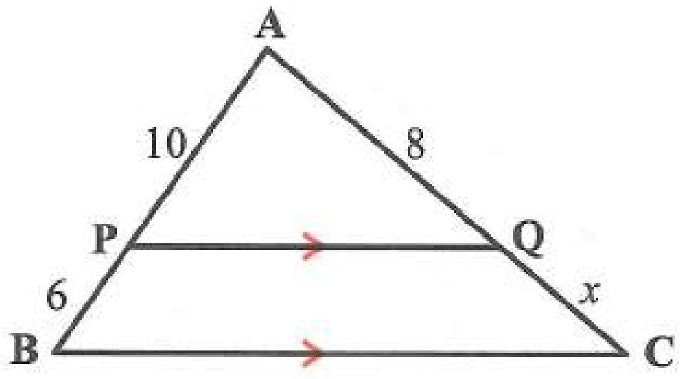
a) Determine the length of PS.
b) Determine the length of SQ.
c) What is the ratio of PT:TR?
d) Determine the length of PT.
e) Determine the length of TR.





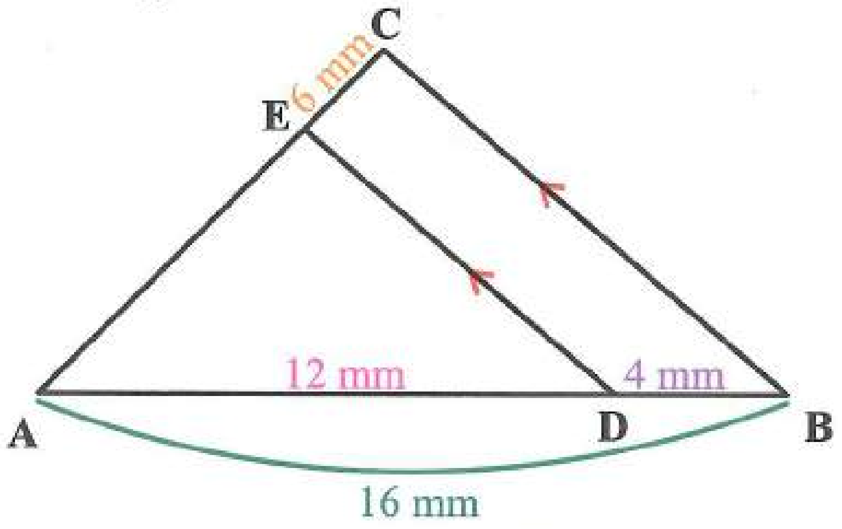
	Statement	Reason
a)	$\frac{PS}{PQ}=\frac{5k}{8k}$ → put the side you're looking for on 'top' $\therefore \frac{PS}{48}=\frac{5}{8}$ $\therefore PS=\frac{5}{8}\times 48=30\text{ mm}$	
b)	$\frac{SQ}{PQ}=\frac{3k}{8k}$ → put the side you're looking for on 'top' $\therefore \frac{SQ}{48}=\frac{3}{8}$ $\therefore SQ=\frac{3}{8}\times 48=18\text{ mm}$ **This question could also be answered using the lengths of the sides:** $SQ = PQ - PS = 48 - 30 = 18\text{ mm}$	
c)	$PT : TR = 5 : 3$	Line // to one side of Δ
d)	$\frac{PT}{PR}=\frac{5a}{8a}$ → put the side you're looking for on 'top' $\therefore \frac{PT}{64}=\frac{5}{8}$ $\therefore PT=\frac{5}{8}\times 64=40\text{ mm}$	
e)	$\frac{TR}{PR}=\frac{3a}{8a}$ → put the side you're looking for on 'top' $\therefore \frac{TR}{64}=\frac{3}{8}$ $\therefore TR=\frac{3}{8}\times 64=24\text{ mm}$ **This question could also be answered using the lengths of the sides:** $TR = PR - PT = 64 - 40 = 24\text{ mm}$	

Example 2: **In △ABC, PQ//BC and each side has the dimensions shown on the diagram. Determine the value of x.**



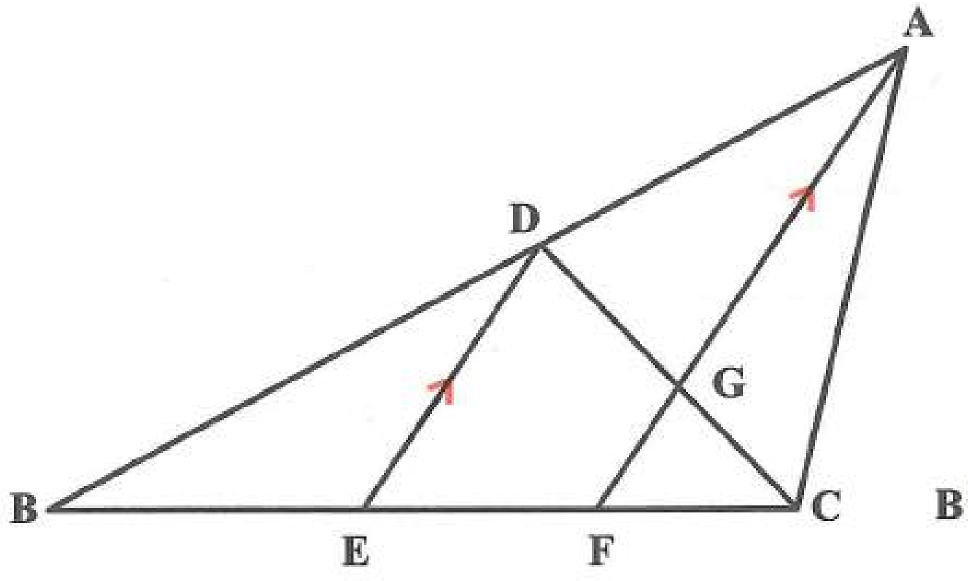
Statement	Reason
$\frac{QC}{AQ}=\frac{PB}{AP}$ → put the side you're looking for on 'top' $\therefore \frac{x}{8}=\frac{6}{10}$ → multiply both sides by 8 $\therefore x=\frac{24}{5}$ $\therefore x=4{,}8$ units	Line // to one side of Δ

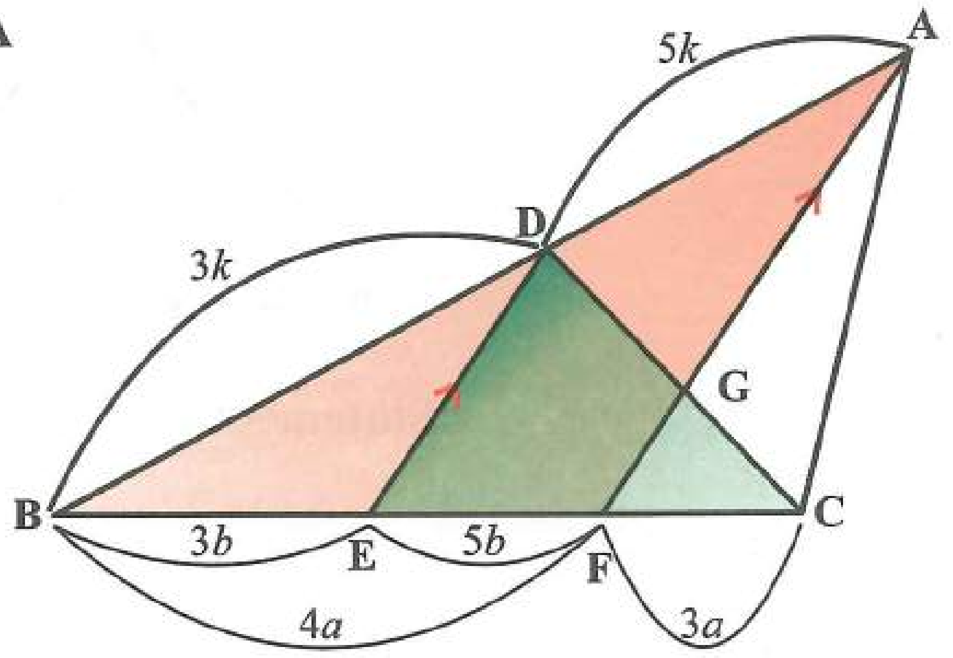
Example 3: **In △ABC, DE//BC. $AB=16\text{mm}$, $AD=12\text{mm}$ and $EC=6\text{mm}$. Determine the length of AE and AC.**



Statement	Reason
$DB = AB - AD = 16 - 12 = 4$	
$\frac{AE}{EC}=\frac{AD}{DB}$ → put the side you're looking for on 'top' $\therefore \frac{AE}{6}=\frac{12}{4}$ → multiply both sides by 6 $\therefore AE = 18\text{mm}$ $AC = AE + EC$ $\therefore AC = 18 + 6 = 24\text{mm}$ **Alternatively the problem could have been solved by determining AC first:**	Line // to one side of Δ
$\frac{AC}{EC}=\frac{AB}{DB}$ → put the side you're looking for on 'top' $\therefore \frac{AC}{6}=\frac{16}{4}$ → multiply both sides by 6 $\therefore AC = 24\text{mm}$ $AE = AC \div EC$ $\therefore AE = 24 - 6 = 18\text{mm}$	Line // to one side of Δ

Example 4: In the diagram below D and F are points on sides AB and BC such that $AD:DB=5:3$ and $BF:FC=4:3$. Determine $\frac{CG}{GD}$:

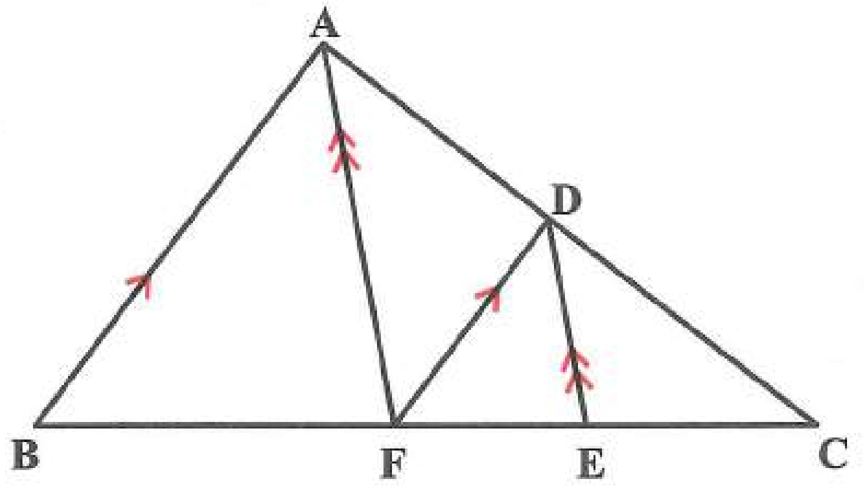


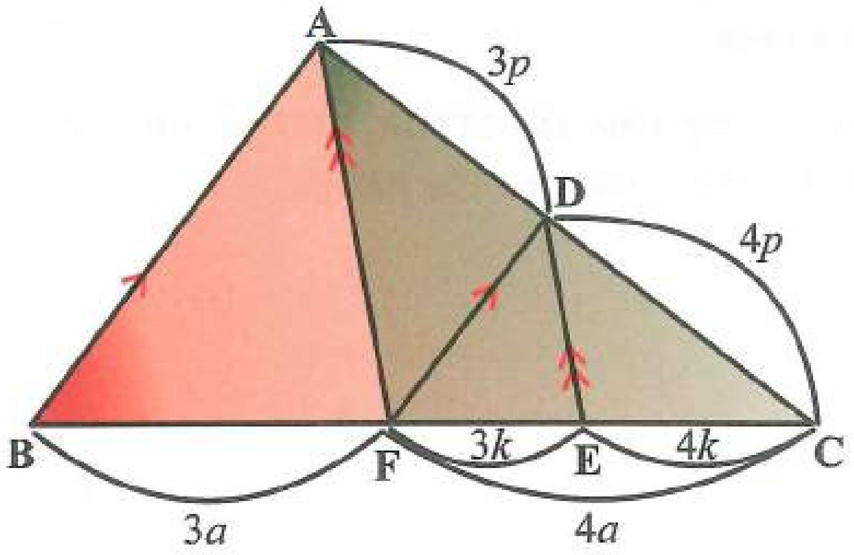


In this type of example, you have to relate the ratios in ΔABF and ΔDEC to each other. Remember that the values given in the question are ratios not lengths so you assign variables to them.
In other words, let $AD=5k$, $DB=3k$, $BF=4a$ and $FC=3a$.

Statement	Reason
$\frac{AD}{DB}=\frac{5k}{3k}$	
$\therefore \frac{FE}{EB}=\frac{5b}{3b} \rightarrow$ you cannot use k as the variable	Line // to one side of Δ
$3b+5b=4a$	
$\therefore 8b=4a$	
$\therefore b=\frac{1}{2}a$	
$\therefore EF=5b=5\left(\frac{1}{2}a\right)=\frac{5}{2}a$	
$\therefore \frac{CF}{FE}=\frac{3a}{\frac{5}{2}a}=\frac{6}{5}$	
$\frac{CF}{FE}=\frac{CG}{GD}$	Line // to one side of Δ
$\therefore \frac{CG}{GD}=\frac{6}{5}$	

Example 5: **In ΔABC, $AF // DE$, $AB // DF$ and $FE : EC = 3 : 4$. Determine the ratio of $EC : BC$.**





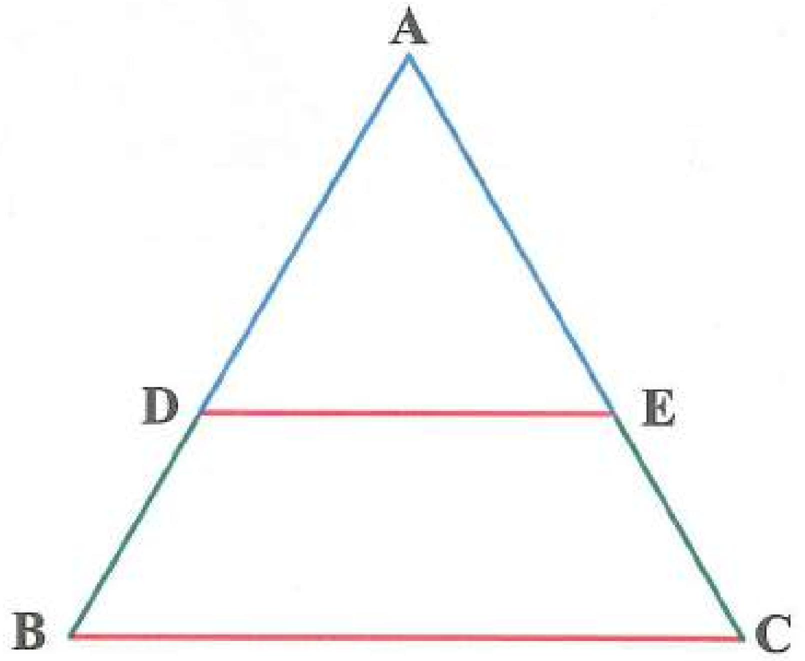
Statement	Reason
$\frac{FE}{EC}=\frac{3k}{4k}$	
$\therefore \frac{AD}{DC}=\frac{3p}{4p}$	Line // to one side of Δ
$\therefore \frac{BF}{FC}=\frac{3a}{4a}$	Line // to one side of Δ
$4a=3k+4k$	
$\therefore 4a=7k$	
$\therefore a=\frac{7}{4}k$	
$BC=7a$	
$\therefore BC=7\left(\frac{7}{4}k\right)=\frac{49}{4}k$	
$\therefore \frac{EC}{BC}=\frac{4k}{\frac{49}{4}k}=\frac{16}{49}$	
$\therefore EC:BC=16:49$	

Complete Exercise 2 on pages 264 and 265.

THEOREM 1: CONVERSE

If a line divides two sides of a triangle proportionally, then the line is parallel to the third side.

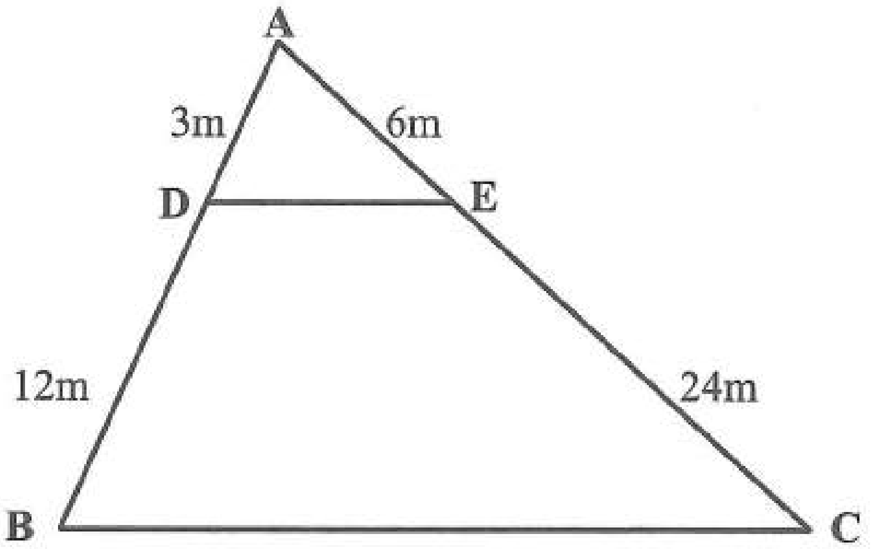
When using this theorem the reason used is:
Line divides sides of △ prop.



Given: In △ABC $\frac{BD}{DA} = \frac{CE}{EA}$.

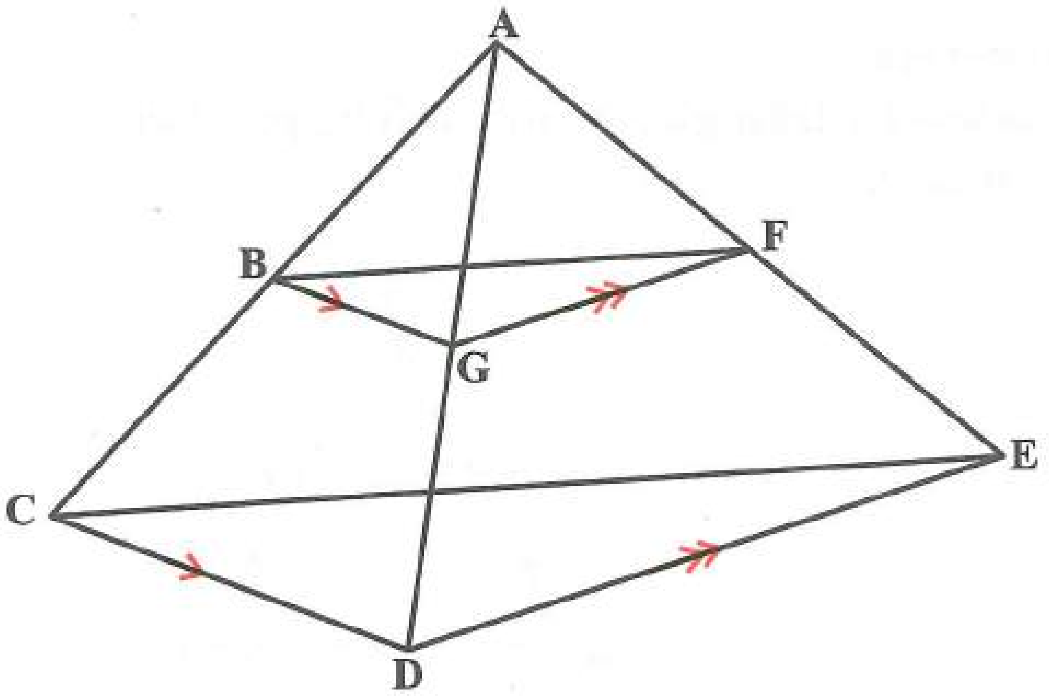
Conclusion: DE//BC

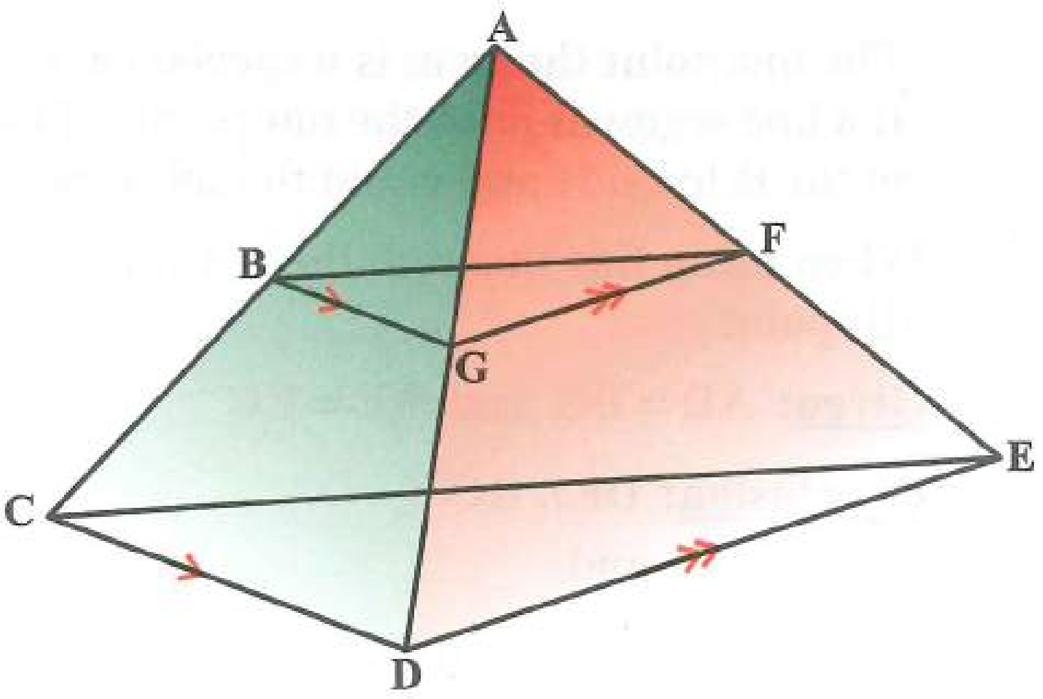
Example 1: In ΔABC, $AD = 3m$, $DB = 12m$, $AE = 6m$ and $EC = 24m$.
Prove that DE//BC.



Statement	Reason
$\frac{DB}{AD} = \frac{12}{3} = 4$	
$\frac{EC}{AE} = \frac{24}{6} = 4$	
$\therefore \frac{DB}{AD} = \frac{EC}{AE} = 4$	
$\therefore$ DE//BC	Line divides sides of △ prop.

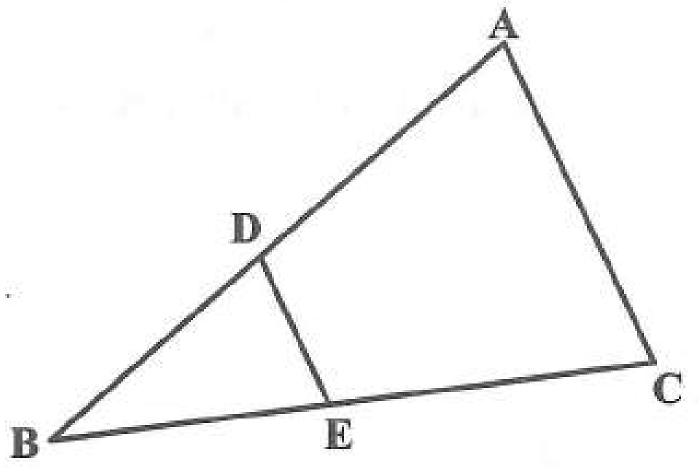
Example 2: **In ΔACD and ΔADE, BG//CD and GF//DE. Prove that BF//CE.**

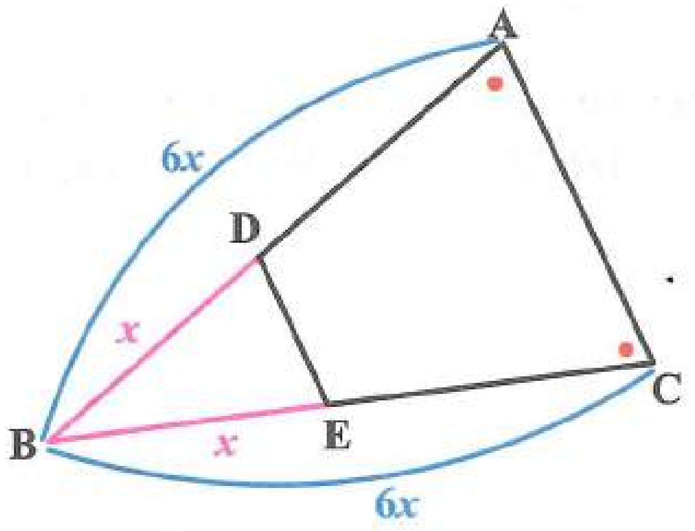




Statement	Reason
$\frac{AB}{BC}=\frac{AG}{GD}$	Line // to one side of Δ
$\frac{AF}{FE}=\frac{AG}{GD}$	Line // to one side of Δ
$\therefore \frac{AB}{BC}=\frac{AF}{FE}$	
∴ BF//CE	Line divides sides of △ prop.

Example 3: **In ΔABC, $\hat{A}=\hat{C}$, $BD=BE=x$ and $AB=6x$. Prove that DE//AC.**





Statement	Reason
$AB=BC=6x$	= sides opp. = ∠'s
$\therefore DA=EC=6x-x=5x$	
$\therefore \frac{BD}{DA}=\frac{x}{5x}=\frac{1}{5}$ and $\frac{BE}{EC}=\frac{x}{5x}=\frac{1}{5}$	
$\therefore \frac{BD}{DA}=\frac{BE}{EC}$	
∴ DE//AC	Line divides sides of △ prop.

Complete Exercise 3 on page 266.

THE MIDPOINT THEOREM

The midpoint theorem is a special case of theorem 1.
If a line segment joins the midpoints of two sides of a triangle, the line will be parallel to the third side and equal to half of the third side.

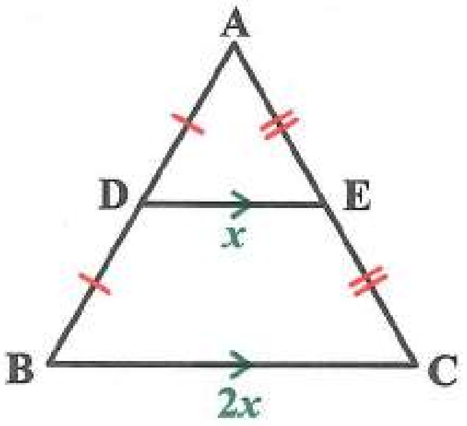
When using this theorem the reason used is:
Midpoint thm.

Given: AD = DB and AE = EC

Conclusion: DE // BC

and

BC = 2DE → also written as DE = $\frac{1}{2}$BC



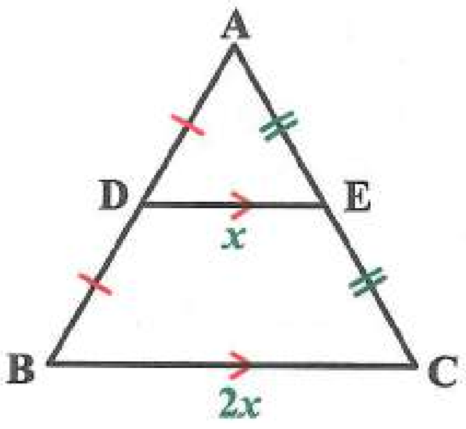
CONVERSE OF THE MIDPOINT THEOREM

If a line joins the midpoint of one side of a triangle and is drawn parallel to another side, it bisects the third side and is equal to half of the length of the side that it is parallel to.

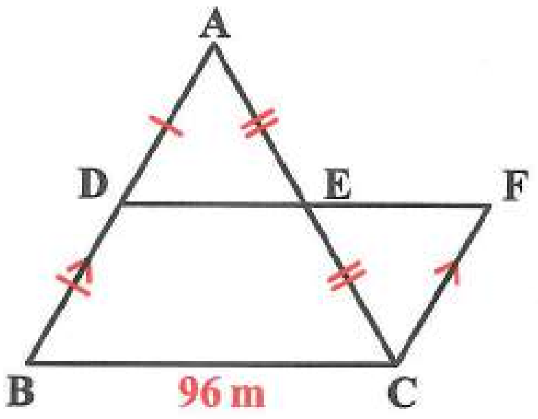
When using this theorem the reason used is:
Converse Midpt. Thm.

Given: AD = DB and DE//BC

Conclusion: AE = EC and BC = 2DE



Example 1: In ΔABC, AD = DB and AE = EC. Point E lies on line DF and AC. DB//FC and BC = 96 metres.

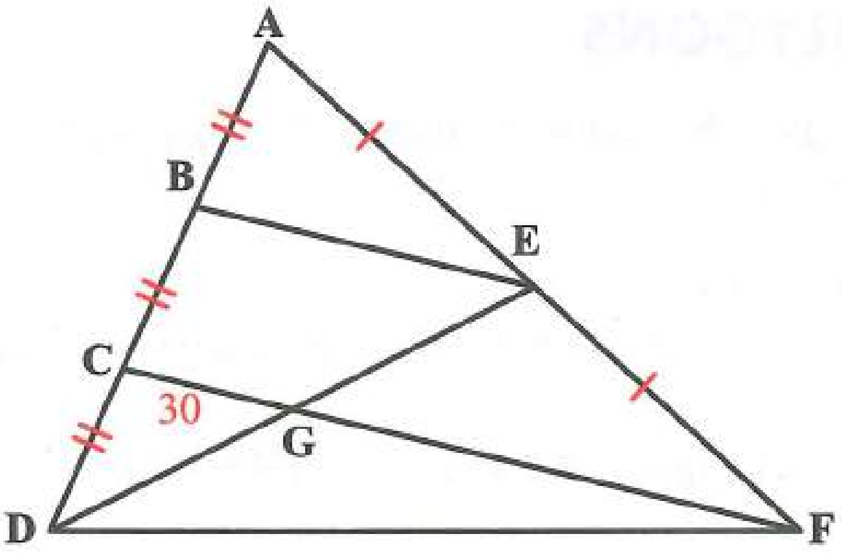


a) Prove that BCFD is a parallelogram.
b) Determine the length of DE.

	Statement	Reason
a)	DB//FC DE//BC ∴ BCFD is a parallelogram	Given Midpoint thm. Both pairs of opp. sides //
b)	$DE = \frac{1}{2}BC$ $DE = \frac{1}{2}BC$ $\therefore DE = \frac{1}{2}(96)$ $\therefore DE = \frac{1}{2}(96)$ ∴ DE = 48 metres ∴ DE = 48 metres	Midpoint thm.

Note that the proof of the above converse theorem is not examinable.

Example 2: In ΔADF, $AE = EF$ and $CG = 30$ units. Calculate the length of FG if $AB = BC = CD$.



Statement	Reason
$AB = BC$ and $AE = EF$	Given
$\therefore BE // CF$	Midpoint thm.
$\therefore BE // CG$	
$BC = CD$	Given
$\therefore BE = 2CG$	Converse midpoint thm.
$= 2(30)$ $= 60$ units	
$CF = 2BE$	Midpoint thm.
$= 2(60)$ $= 120$	
$GF = CF - CG$	
$= 120 - 30$ $= 90$ units	

Complete Exercise 4 on pages 266 and 267.

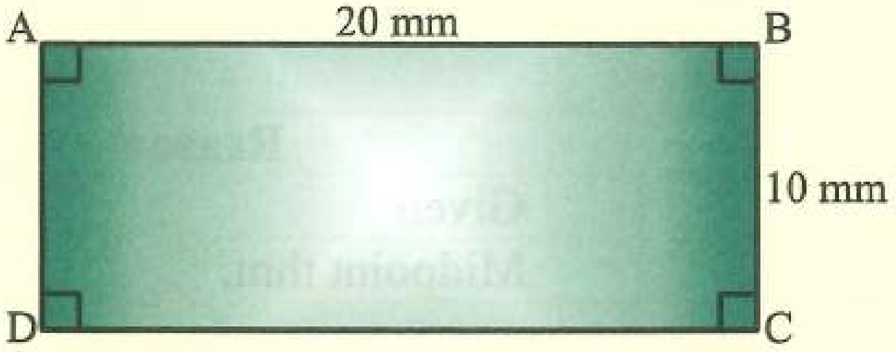
C. SIMILARITY

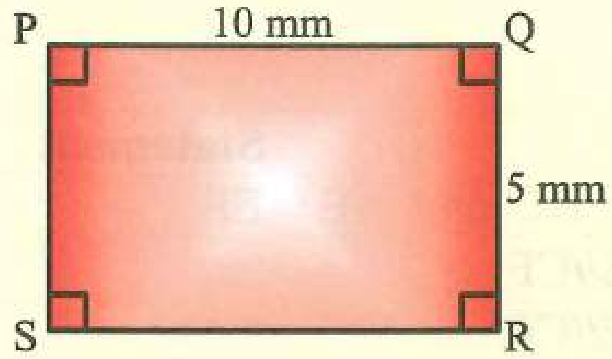
SIMILARITY OF POLYGONS

Two polygons (which have the same number of sides) are similar if they meet both of the following conditions:

1. **Their corresponding angles are equal.**
2. **All of the corresponding sides are in the same proportion.**

For example, consider rectangle ABCD and PQRS below:





$$\frac{AB}{PQ}=\frac{20}{10}=2 \quad \text{and} \quad \frac{DC}{SR}=\frac{20}{10}=2$$

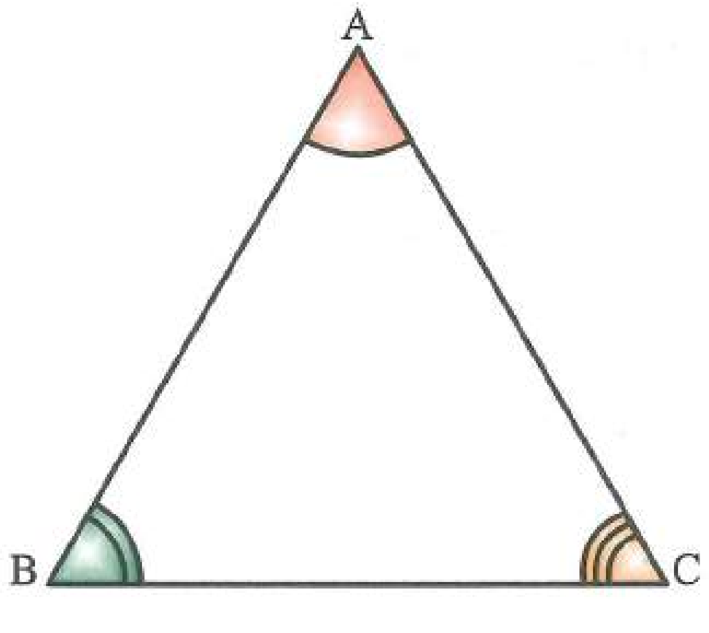
$$\frac{BC}{QR}=\frac{10}{5}=2 \quad \text{and} \quad \frac{AD}{PS}=\frac{10}{5}=2$$

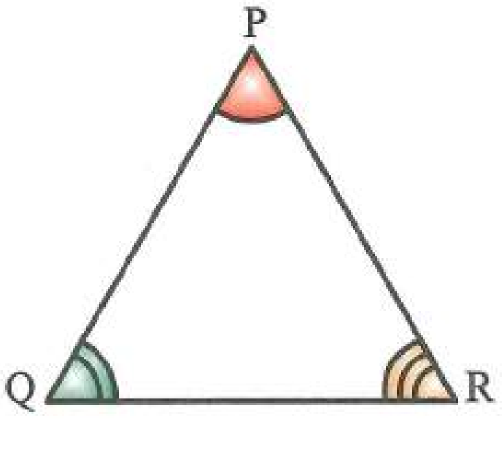
As all corresponding sides are in proportion and angles are equal in the above diagrams, it can be seen that: ABCD $|||$ PQRS → the notation used for similarity

THEOREM 2: EQUIANGULAR TRIANGLES ARE SIMILAR

If two triangles are equiangular then their corresponding sides are in proportion and the triangles are similar.

Proof:

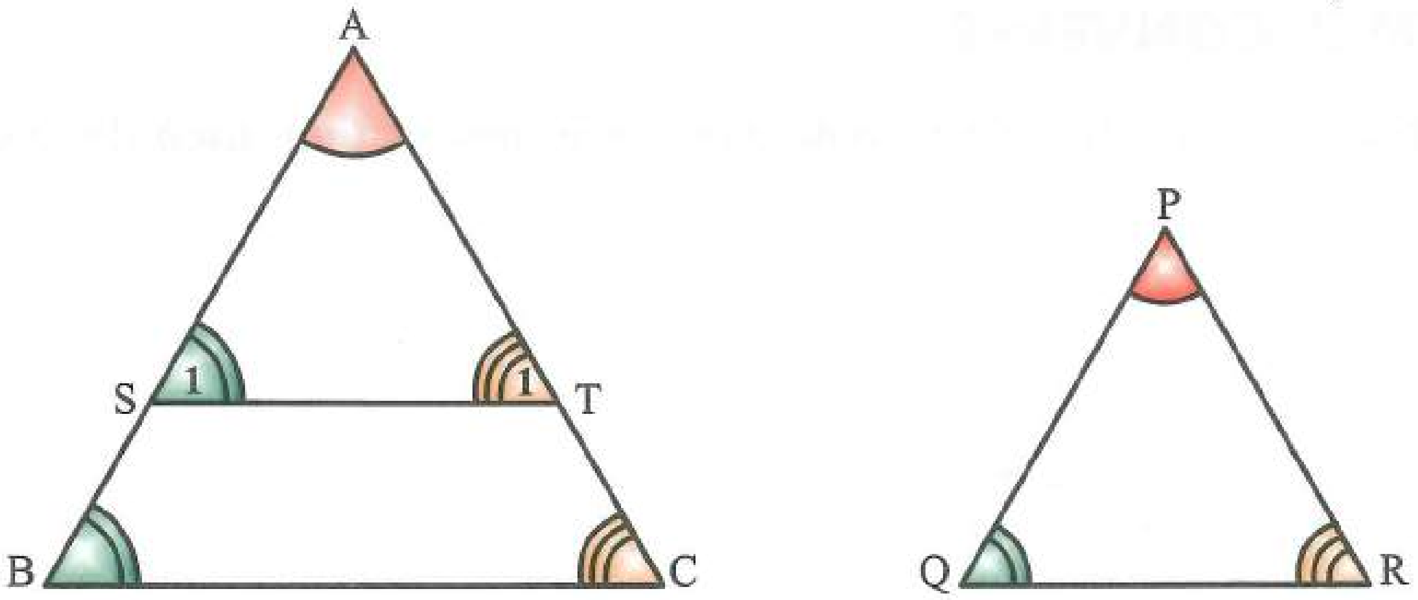




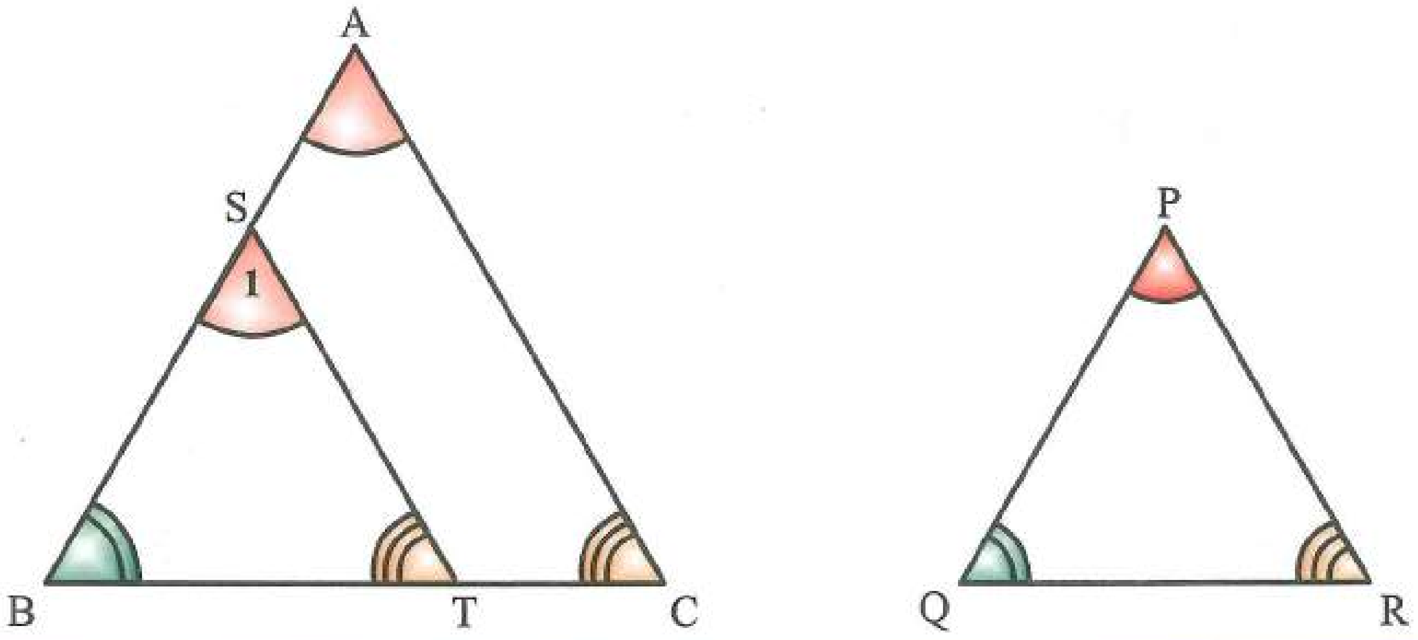
Given: $\hat{A}=\hat{P}$; $\hat{B}=\hat{Q}$ and $\hat{C}=\hat{R}$.

Required to prove (RTP): $\dfrac{AB}{PQ}=\dfrac{AC}{PR}=\dfrac{BC}{QR}$.

Construction: Draw S on AB such that $AS = PQ$. Draw T on AC such that $AT = PR$. Join line ST.



Statement	Reason
In ΔAST and ΔPQR:	
$AS = PQ$	By construction
$\hat{A} = \hat{P}$	Given
$AT = PR$	By construction
$\therefore \Delta AST \equiv \Delta PQR$	SAS
$\therefore \hat{S}_1 = \hat{Q}$	
But $\hat{B} = \hat{Q}$	Given
$\therefore \hat{S}_1 = \hat{B}$	
$\therefore ST \parallel BC$	Corresponding angles are equal
$\therefore \frac{AB}{AS} = \frac{AC}{AT}$	Line // to one side of Δ
$\therefore \frac{AB}{PQ} = \frac{AC}{PR}$	$AS = PQ$ and $AT = PR$
Similarly, as shown in the diagram below, by constructing BS and BT on AB and BC respectively, it can be shown that:	
$\frac{AB}{PQ} = \frac{BC}{QR}$ $\therefore \frac{AB}{PQ} = \frac{AC}{PR} = \frac{BC}{QR}$	

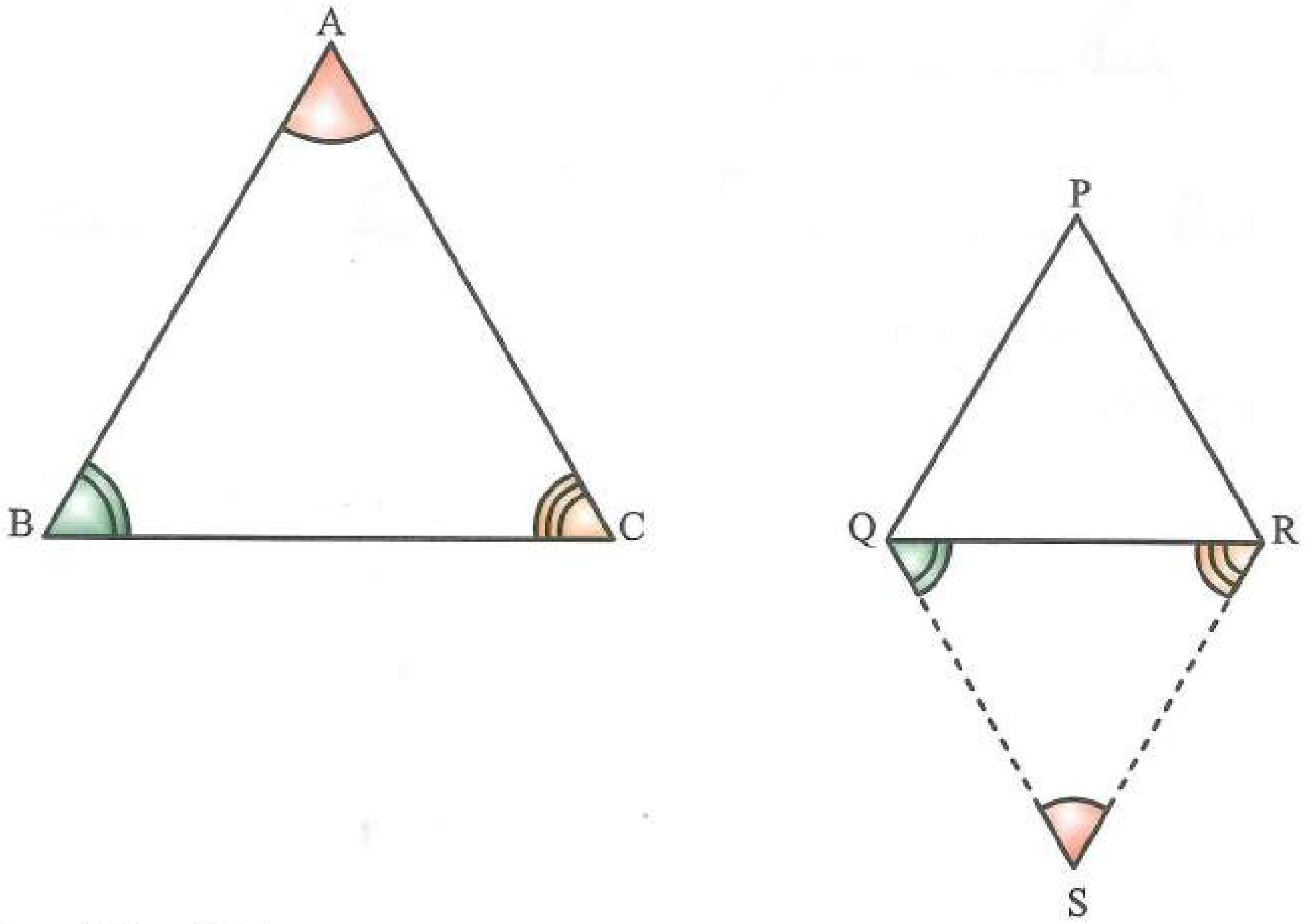


This proof shows that all equiangular triangles are similar.

THEOREM 2: CONVERSE

If the corresponding sides of two triangles are in proportion, then the triangles are similar.

Proof:



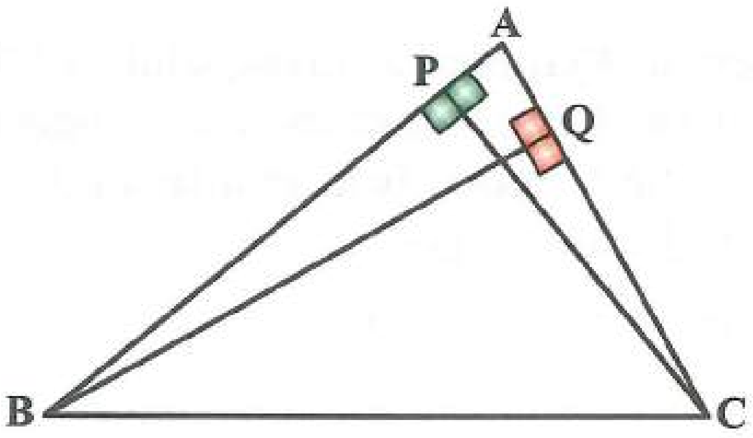
Given: $\frac{AB}{PQ}=\frac{AC}{PR}=\frac{BC}{QR}$.

Required to prove (RTP): $\triangle ABC\ \vert\vert\vert\ \triangle PQR$.

Construction: Draw ΔQRS on QR such that: $\hat{A}=\hat{S}$; $\hat{B}=\hat{Q}$ and $\hat{C}=\hat{R}$.

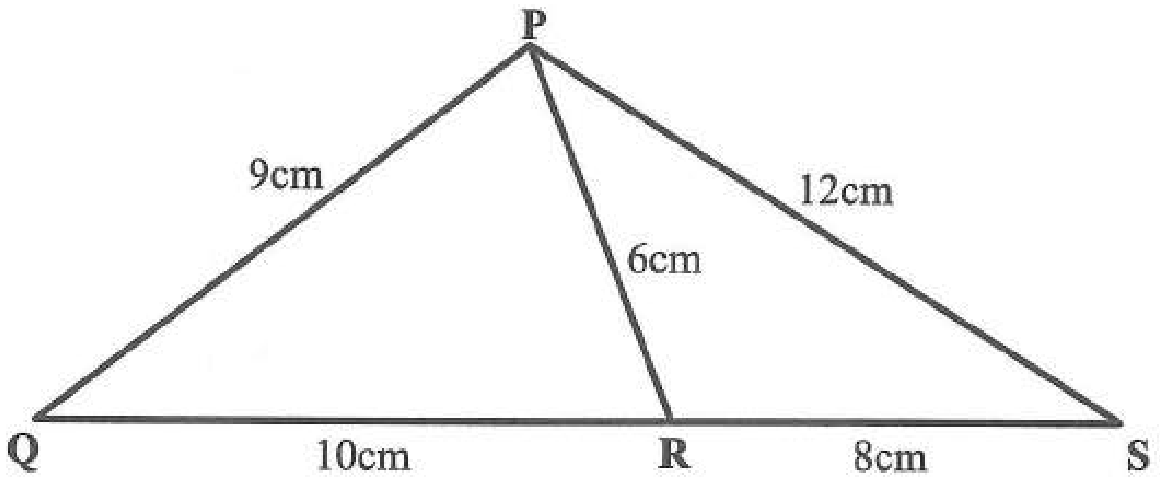
Statement		Reason
By construction $\triangle ABC\ \vert\vert\vert\ \triangle SQR$		A.A.A
$\therefore \frac{AB}{SQ}=\frac{BC}{QR}=\frac{AC}{SR}$		
But $\frac{AB}{PQ}=\frac{AC}{PR}=\frac{BC}{QR}$		Given
$\therefore \frac{AB}{SQ}=\frac{AB}{PQ} \rightarrow$ both are equal to $\frac{BC}{QR}$		
$\therefore SQ=PQ$		
Similarly $SR=PR$		
$\therefore$ In $\triangle PQR$ and $\triangle SQR$:		
$SQ=PQ$		Proved above
$SR=PR$		Proved above
$QR=QR$		Common side
$\therefore \triangle PQR \equiv \triangle SQR$		S.S.S
But $\triangle SQR\ \vert\vert\vert\ \triangle ABC$ $\therefore \triangle ABC\ \vert\vert\vert\ \triangle PQR$		

Example 1: In ΔABC, $CP \perp AB$ and $BQ \perp AC$. Prove that $CP \times AB = BQ \times AC$.



Statement	Reason
$\triangle ABQ$ and $\triangle ACP$	
$\hat{A} = \hat{A}$	Common
$\hat{Q} = \hat{P} = 90^\circ$	Given
$\therefore \hat{B} = \hat{C}$	3rd ∠ of a Δ
$\therefore \triangle ABQ \,\vert\vert\vert\, \triangle ACP$	A.A.A
$\therefore \frac{AB}{AC} = \frac{BQ}{CP} = \frac{AQ}{AP}$	
$\therefore \frac{AB}{AC} = \frac{BQ}{CP}$ $\therefore AB \times CP = BQ \times AC$	

Example 2: Given ΔSPR and ΔSQP, prove that $S\hat{P}R = \hat{Q}$.



Statement	Reason
In $\triangle SPR$ and $\triangle SQP$	
$\frac{SP}{SQ} = \frac{12}{18} = \frac{2}{3}$	
$\frac{PR}{QP} = \frac{6}{9} = \frac{2}{3}$	Given
$\frac{SR}{SP} = \frac{8}{12} = \frac{2}{3}$	3rd ∠ of a Δ
$\therefore \frac{SP}{SQ} = \frac{PR}{QP} = \frac{SR}{SP}$	
$\Delta SPR \,\vert\vert\vert\, \Delta SQP$	Sides in prop.
$\therefore S\hat{P}R = \hat{Q}$	

Complete Exercise 5 on pages 268 and 269.

THE THEOREM OF PYTHAGORAS

Before proving the theorem of Pythagoras using similar triangles, we have to prove that a perpendicular line from the vertex of the right angle of a right-angled triangle to the hypotenuse, divides the triangle into two triangles that are similar to each other and similar to the original triangle.

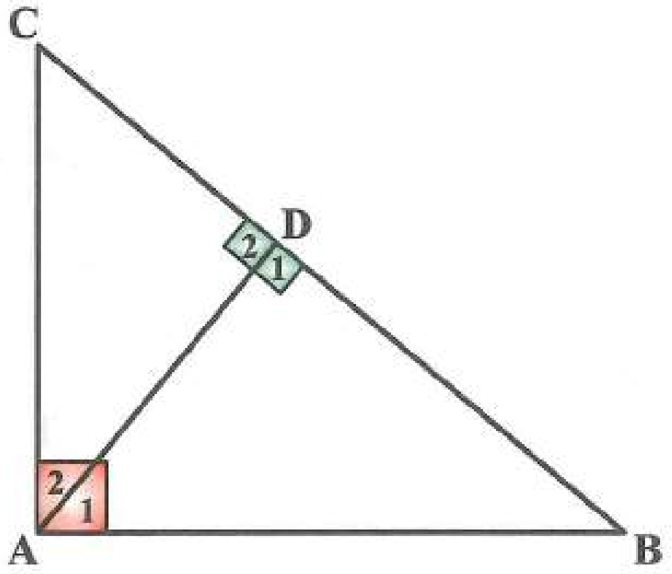
When using this theorem the reason used is:

⊥ from rt. ∠ vertex to hyp.

Proof:

Given: In the diagram below, $A_1 + A_2 = 90^\circ$ and AD⊥BC.

Required to prove (RTP): $\triangle ABC \,|||\, \triangle DBA \,|||\, \triangle DAC$.



Statement	Reason
In $\triangle ABC$ and $\triangle DBA$	
$\hat{A} = \hat{D}_1 = 90^\circ$	Given
$\hat{B} = \hat{B}$	Common ∠
$\hat{C} = \hat{A}_1$	3rd ∠ of a Δ
$\therefore \triangle ABC \,|||\, \triangle DBA$	A.A.A
In $\triangle ABC$ and $\triangle DAC$	
$\hat{A} = \hat{D}_2 = 90^\circ$	Given
$\hat{C} = \hat{C}$	Common ∠
$\hat{B} = \hat{A}_2$	3rd ∠ of a Δ
$\therefore \triangle ABC \,|||\, \triangle DAC$	A.A.A
$\therefore \triangle ABC \,|||\, \triangle DBA \,|||\, \triangle DAC$	

Corollaries:

1. $\triangle ABC \,|||\, \triangle DBA$ $\therefore \frac{AB}{DB} = \frac{BC}{BA}$ $\therefore AB^2 = BC.BD$	**2.** $\triangle ABC \,|||\, \triangle DAC$ $\therefore \frac{BC}{AC} = \frac{AC}{DC}$ $\therefore AC^2 = BC.DC$	**3.** $\triangle DBA \,|||\, \triangle DAC$ $\therefore \frac{DB}{DA} = \frac{DA}{DC}$ $\therefore DA^2 = DB.DC$

Using the results on the previous page you can now prove the theorem of Pythagoras, which states: $BC^2 = AC^2 + AB^2$.

When using this theorem the reason used is:
Pythag.

From the corollaries on the previous page:

$AB^2 = BC.BD$ and $AC^2 = BC.DC$

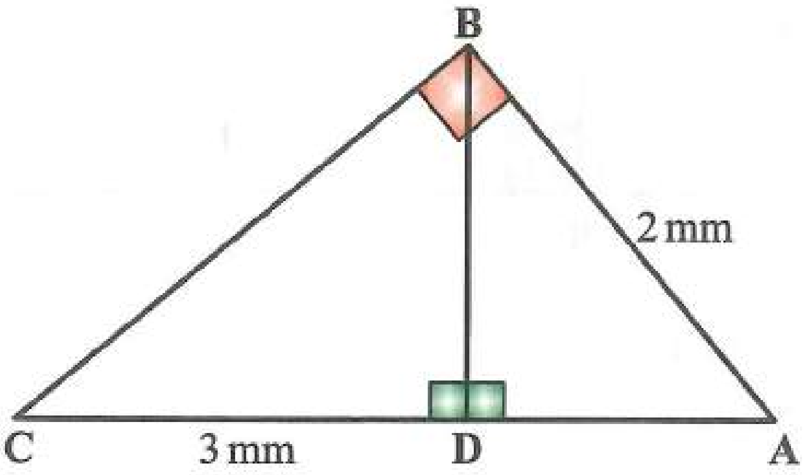
$\therefore AB^2 + AC^2 = BC.BD + BC.DC \rightarrow$ take out a HCF of BC

$\therefore AB^2 + AC^2 = BC(BD + DC) \rightarrow$ from the diagram $BD + DC = BC$

$\therefore AB^2 + AC^2 = BC(BC)$

$\therefore AB^2 + AC^2 = BC^2$

Example 1: **In ΔABC, $AB = 2mm$ and $DC = 3mm$. Determine the lengths of AD and BC. Leave your answers in simplest surd form where necessary.**



Statement	Reason
$\Delta ABC \,\vert\vert\vert\, \Delta ADB \,\vert\vert\vert\, \Delta BDC$	$\perp$ from rt. $\angle$ vertex to hyp.
$\therefore AB^2 = AC.AD$	
$(2)^2 = (AD+3)(AD) \rightarrow AC = AD + 3$ $\therefore 4 = AD^2 + 3AD$ $\therefore AD^2 + 3AD - 4 = 0$ $\therefore (AD-1)(AD+4) = 0$ $\therefore AD = 1mm$ or $AD = -4mm \rightarrow$ N/A	Common $\angle$
$BC^2 = AC^2 - AB^2$ $= (3+1)^2 - (2^2)$ $= 12$ $\therefore BC = \sqrt{12}$ $\therefore BC = 2\sqrt{3}$ mm	Pythag.

Remember that a length can never be negative.

Example 2: In $\triangle ADC$, $A\hat{D}C = 90^\circ$ and $DB \perp AC$ with B on AC. BK//AD with K on DC such that $DC \perp BK$. If $DK : KC = 1 : 2$ and $BC = x$:

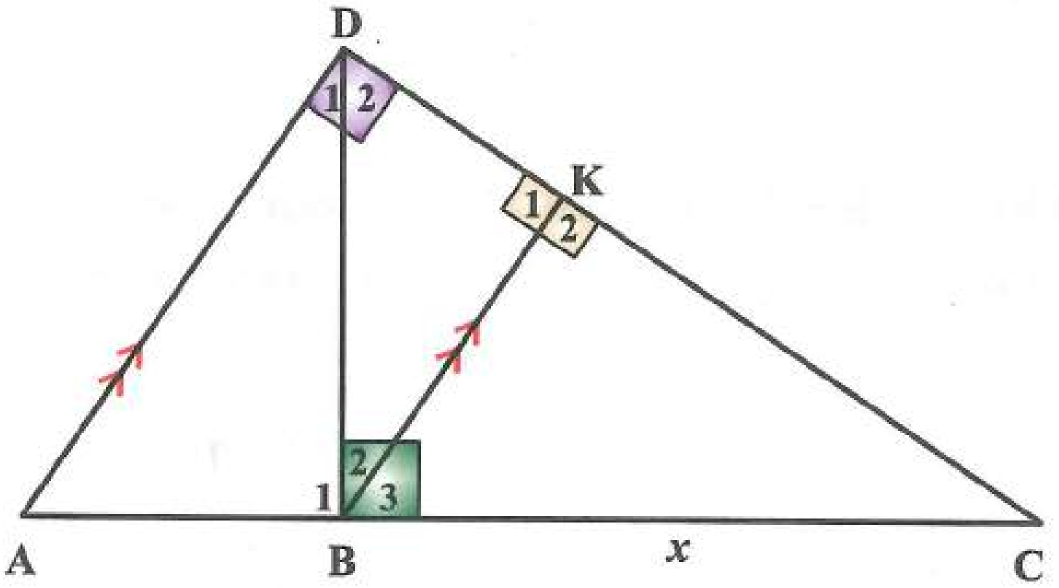
a) Prove that $DA^2 = AB.AC$

b) Prove that $BD^2 = AD.BK$

c) Express AB in terms of x

d) Express AC in terms of x

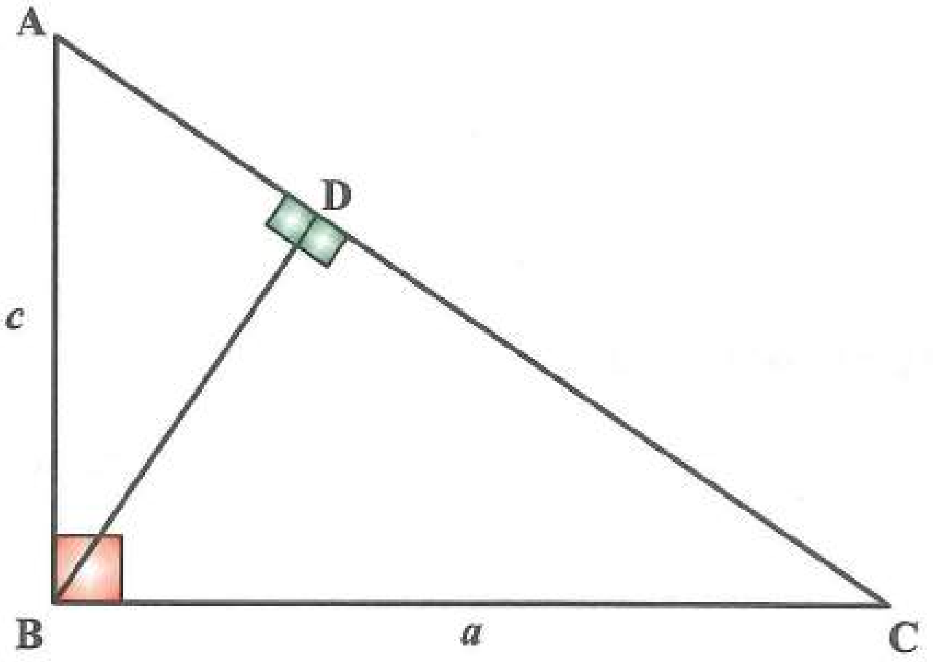
e) Hence, prove that $DA = \frac{x\sqrt{3}}{2}$



	Statement	Reason
a)	In $\triangle ABD$ and $\triangle ADC$	
	$\hat{A} = \hat{A}$	Common ∠
	$\hat{B}_1 = \hat{D} = 90^\circ$	Given
	$\hat{D}_1 = \hat{C}$	3rd ∠ of a Δ
	$\therefore \triangle ABD \,\vert\vert\vert\, \triangle ADC$	A.A.A
	$\therefore \frac{AB}{AD} = \frac{AD}{AC} = \frac{BD}{DC}$ $\therefore DA^2 = AB.AC$	
b)	In $\triangle BDA$ and $\triangle KBD$	
	$\hat{B}_1 = \hat{K}_1 = 90^\circ$	Given
	$\hat{D}_1 = \hat{B}_2$	Alt ∠'s (AD//BK)
	$\hat{A} = \hat{D}_2$	3rd ∠ of a Δ
	$\therefore \triangle BDA \,\vert\vert\vert\, \triangle KBD$	A.A.A
	$\therefore \frac{BD}{KB} = \frac{DA}{BD} = \frac{BA}{KD}$ $\therefore BD^2 = DA.KB$	

	Statement	Reason
c)	$\frac{AB}{BC}=\frac{DK}{KC}=\frac{1}{2}$	Line // to one side of Δ
	$\therefore \frac{AB}{x}=\frac{1}{2}$ $\therefore AB=\frac{x}{2}$	
d)	**Method 1: Using Proportionality**	
	$\frac{AC}{BC}=\frac{3}{2}$	Line // to one side of Δ
	$\therefore \frac{AC}{x}=\frac{3}{2}$ $\therefore AC=\frac{3x}{2}$	
	Method 2: Using Lengths of Lines	
	$\therefore AC=AB+BC=\frac{x}{2}+x=\frac{3x}{2}$	
e)	$\triangle ADC\ \vert\vert\vert\ \triangle ABD\ \vert\vert\vert\ \triangle DBC$	⊥ from rt. ∠ vertex to hyp.
	$\therefore AD^2=AC.AB$ $\therefore AD^2=\left(\frac{3x}{2}\right)\left(\frac{x}{2}\right)$ $\therefore AD^2=\frac{3x^2}{4}$ → square root both sides $\therefore AD=\frac{\sqrt{3}x}{2}$	Given

Example 3: In a right-angled triangle ABC, with $\hat{B} = 90^\circ$ let $BC = a$ and $AB = c$. Let D be on AC such that $BD \perp AC$. Determine the length of BD in terms of a and c.



Statement	Reason
$\Delta ABC \,\vert\vert\vert\, \Delta ADB \,\vert\vert\vert\, \Delta BDC$	$\perp$ from rt. $\angle$ vertex to hyp.
$\therefore AB^2 = AC.AD$ and $BC^2 = AC.DC$ and $DB^2 = AD.DC$ $\therefore c^2 = AC.AD$ and $a^2 = AC.DC$ $\therefore AD = \dfrac{c^2}{AC}$ and $DC = \dfrac{a^2}{AC}$ $\therefore DB^2 = \dfrac{c^2}{AC} \times \dfrac{a^2}{AC}$ $= \dfrac{a^2c^2}{AC^2}$ $\therefore DB = \dfrac{ac}{AC}$ You can now solve for AC using the theorem of Pythagoras.	
$AC^2 = BC^2 + AB^2$ $= a^2 + c^2$ $\therefore AC = \sqrt{a^2 + b^2}$ $\therefore DB = \dfrac{ac}{\sqrt{a^2 + b^2}}$	Pythag.

Complete Exercise 6 on page 269 and the Mixed Exercise on pages 270-273.

Ratios

Exercise 1: (answers on page 367)

1) If R divides line segment AB in a ratio of 5 : 3, determine the following ratios:

a) $\frac{AR}{RB}$ b) $\frac{AR}{AB}$ c) $\frac{AB}{RB}$

2) If A divides line segment PQ in a ratio of 1 : 3, determine the following ratios:

a) $\frac{PA}{AQ}$ b) $\frac{PA}{PQ}$ c) $\frac{PQ}{QA}$

3) If C is the midpoint of line segment DE, determine the following ratios:

a) $\frac{DC}{CE}$ b) $\frac{DC}{DE}$ c) $\frac{DE}{EC}$

4) The length of PQ is 252 units and it is divided by R in the ratio 7 : 2. Calculate the length of PR.

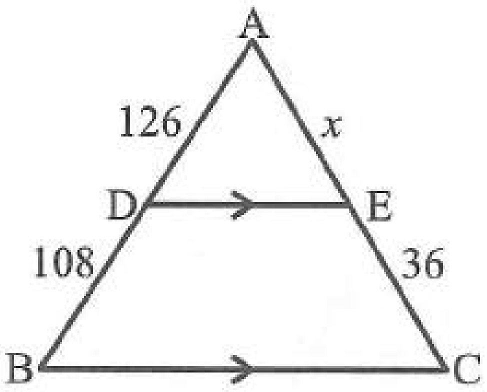
5) Line segment AB is divided into a ratio of 9 : 4 by C. Determine the length of AB if $AC = 90\,\text{mm}$.

Theorem 1

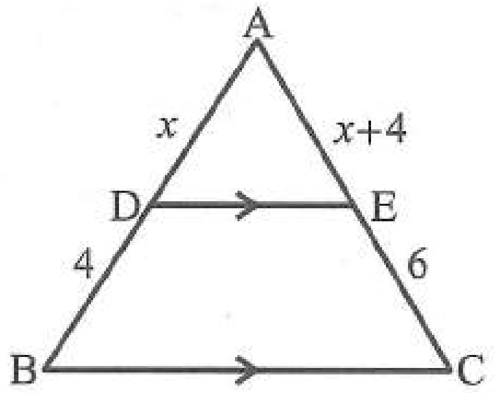
Exercise 2: (answers on page 367)

1) In each of the following diagrams, determine the value of x:

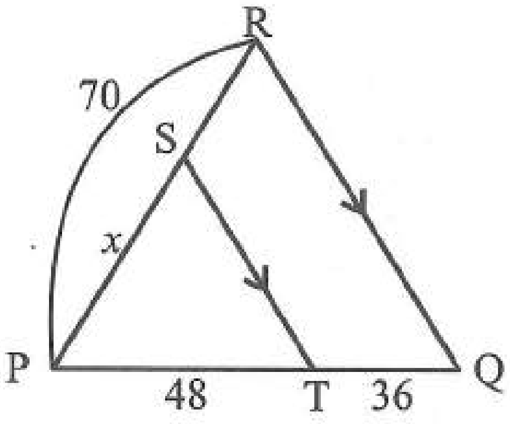
a)


A
126
x
D
E
108
36
B
C

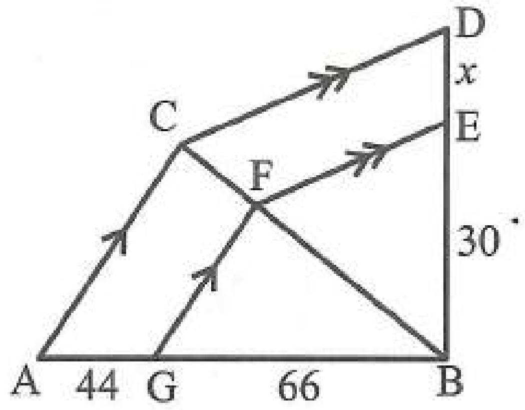
d)


A
x
x+4
D
E
4
6
B
C

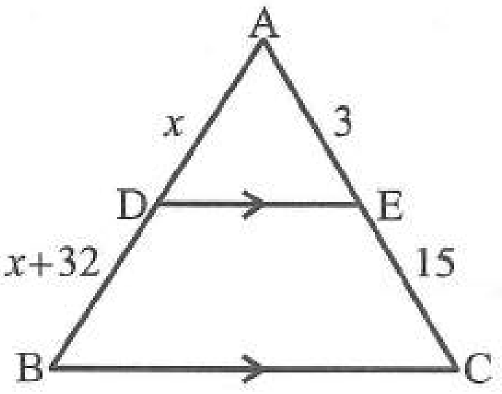
b)


R
70
S
x
P
48
T
36
Q

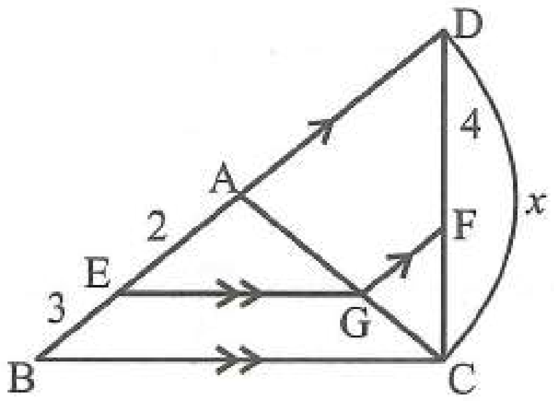
e)


D
x
C
E
F
30
A
44
G
66
B

c)

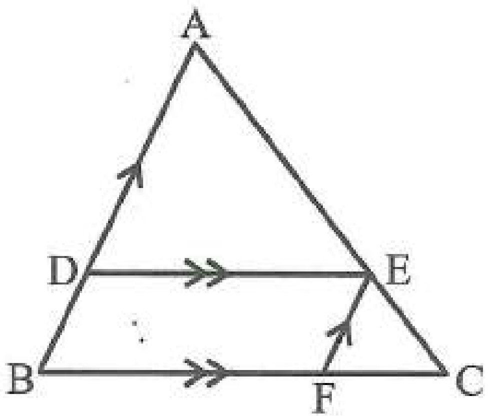

A
x
3
D
E
x+32
15
B
C

f)

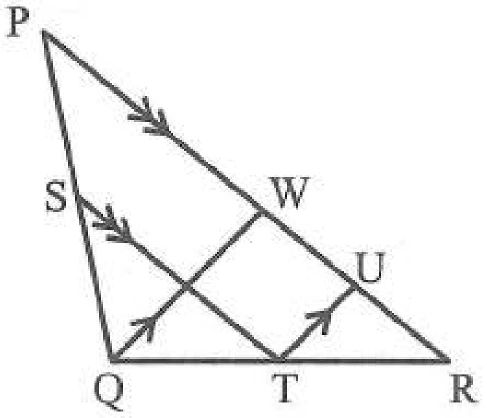

D
4
A
x
2
F
E
3
G
B
C

2) Answer the following questions:

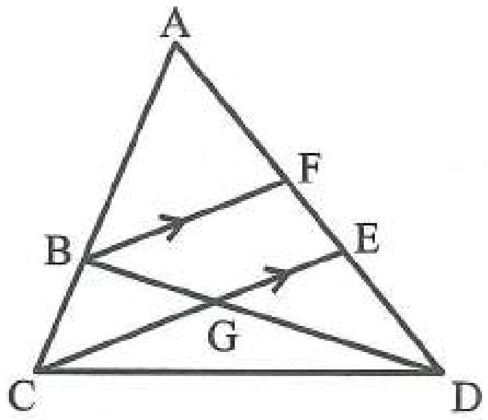
a) In the diagram below, prove that $\frac{AB}{AD} = \frac{BC}{BF}$.



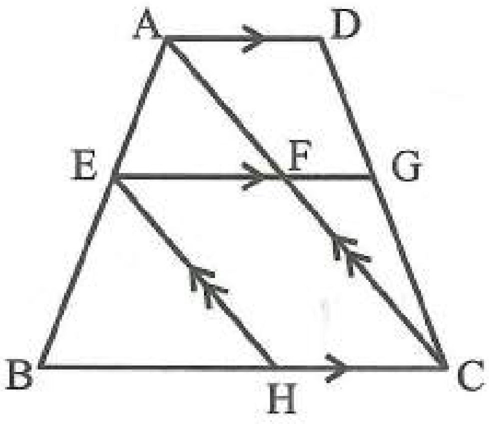
b) In the diagram below, prove that $\frac{PS}{SQ} = \frac{RU}{WU}$.



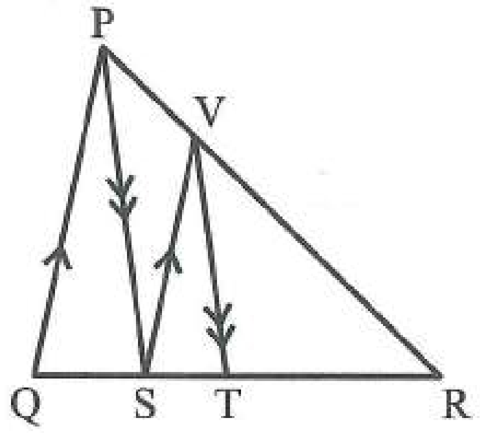
c) In the diagram below, BF//CE, $\frac{BC}{AB} = \frac{3}{5}$ and $AE : ED = 4 : 3$. Determine the ratio $DG : GB$.



d) In the diagram below, prove that $\frac{BH}{HC} = \frac{GC}{DG}$.

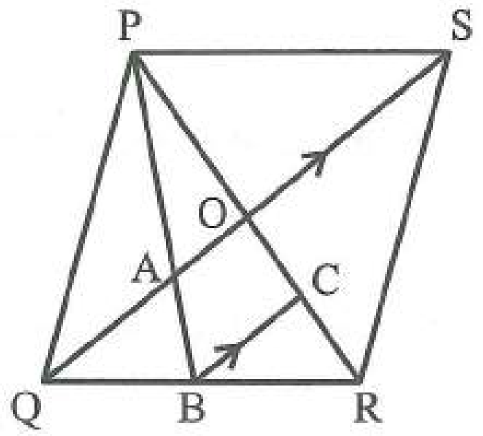


e) In the diagram below, PS//VT and PQ//SV. If $11PV = 12VR$, Determine the ratio $QS : QT$.



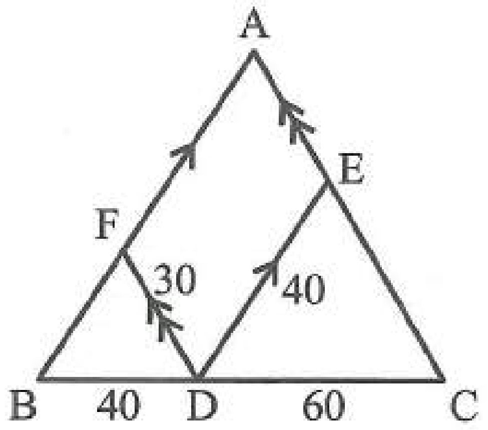
f) In parallelogram PQRS, the diagonals intersect at O. QS//BC and $QB : BR = 2 : 3$.

Calculate $PO : OC$.



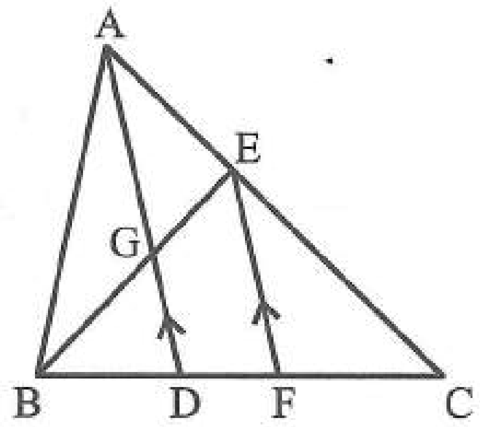
g) In the diagram below, DF//AC and AB//DE. $BD = DE = 40$ units, $DF = 30$ units and $DC = 60$ units.

Determine the length of BF.

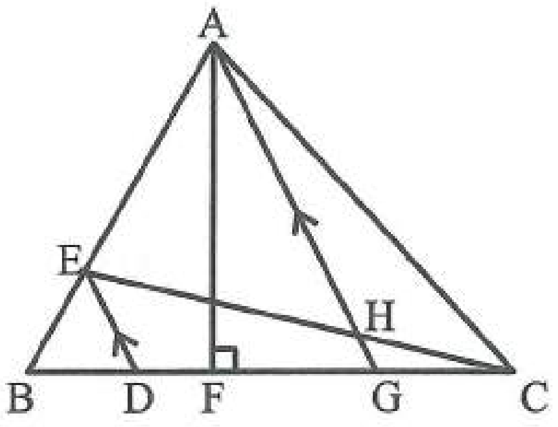

A
E
F
30
40
B
40
D
60
C

h) In the diagram below, AD//EF, $BD : DC = 2 : 3$ and $AE : EC = 2 : 7$.

Determine $BE : BG$.



3) In the diagram below $\frac{BG}{BC} = \frac{2}{3}$ and $\frac{BE}{EA} = \frac{1}{2}$. If AG//DE, determine:



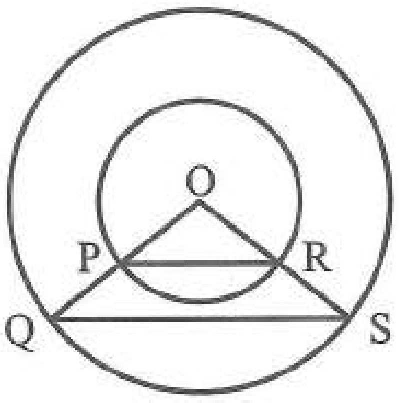
a) $\frac{EH}{HC}$.

b) $\frac{\text{Area } \triangle BAG}{\text{Area } \triangle CAG}$.

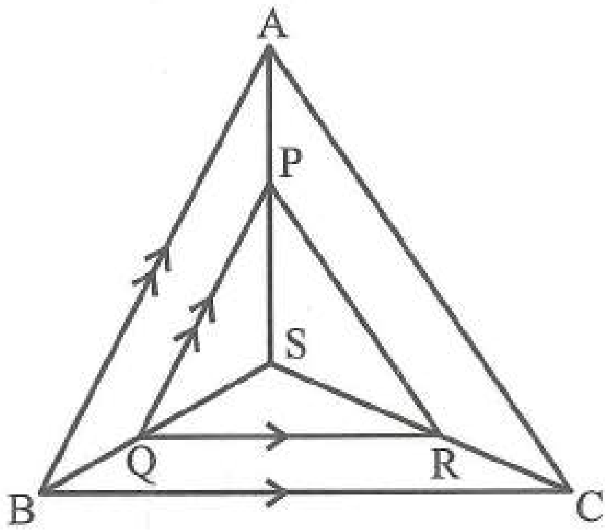
Theorem 1 (Converse)

Exercise 3: (answers on page 367)

1) **In the diagram below, the two circles drawn have the same centres at O. OPQ and ORS are straight lines. Prove that PR//QS.**



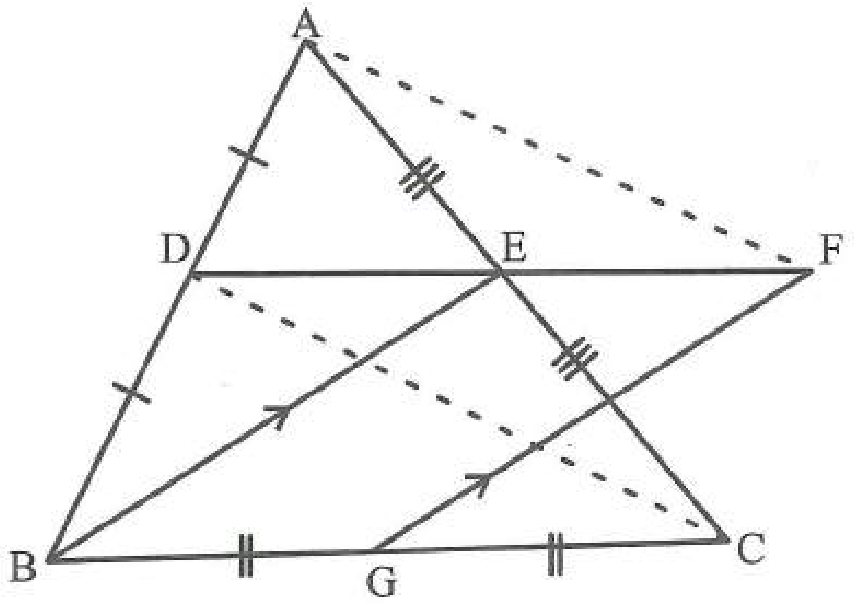
2) **In the diagram below, AB//PQ and QR//BC. Prove that AC//PR.**



Midpoint Theorem

Exercise 4: (answers on page 367)

1) **In the figure below D, G and E are the midpoints of AB, BC and AC respectively and BE//GF.**



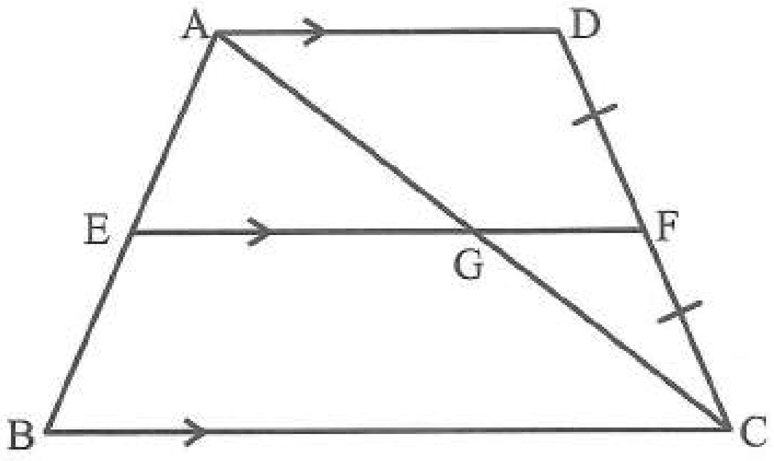
Prove that:

a) EBGF is a parallelogram.

b) DE = EF

c) AF//DC

2) **In the figure below, AD//EF//BC and F is the midpoint of DC.**

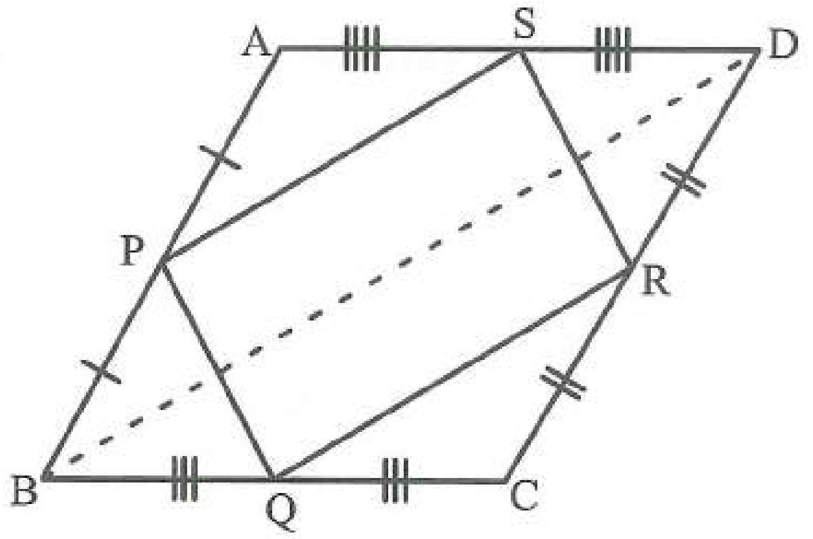


Prove that:

a) AE = EB

b) $EF = \frac{BC + AD}{2}$

3) **In the figure below, P, Q, R and S are the midpoints of AB, BC, CD and AD respectively.**

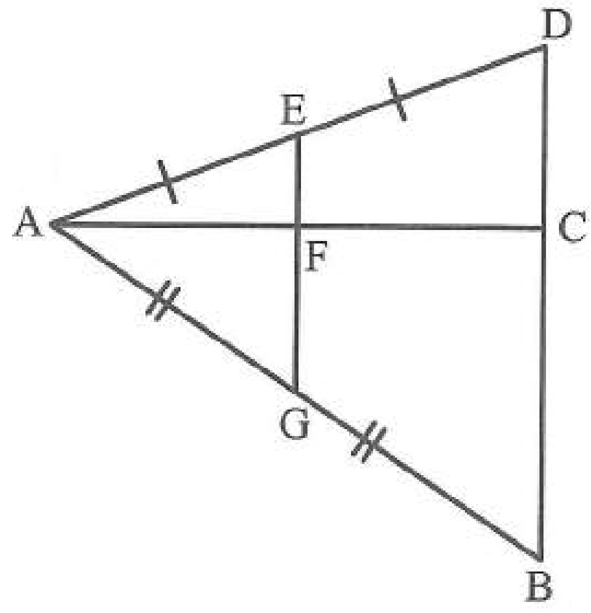


Prove that:

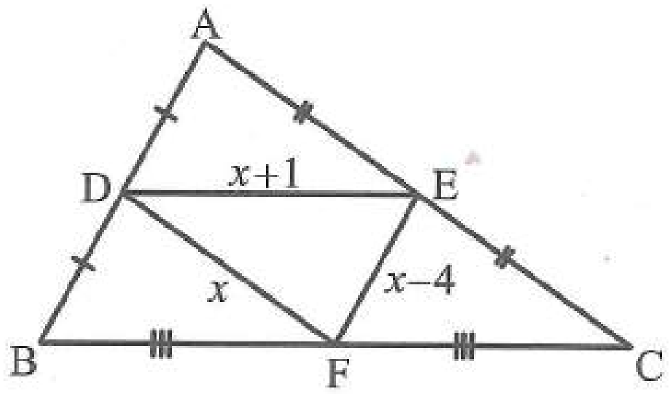
a) PQRS is a parallelogram.

b) $BD + AC = SP + RQ + PQ + SR$

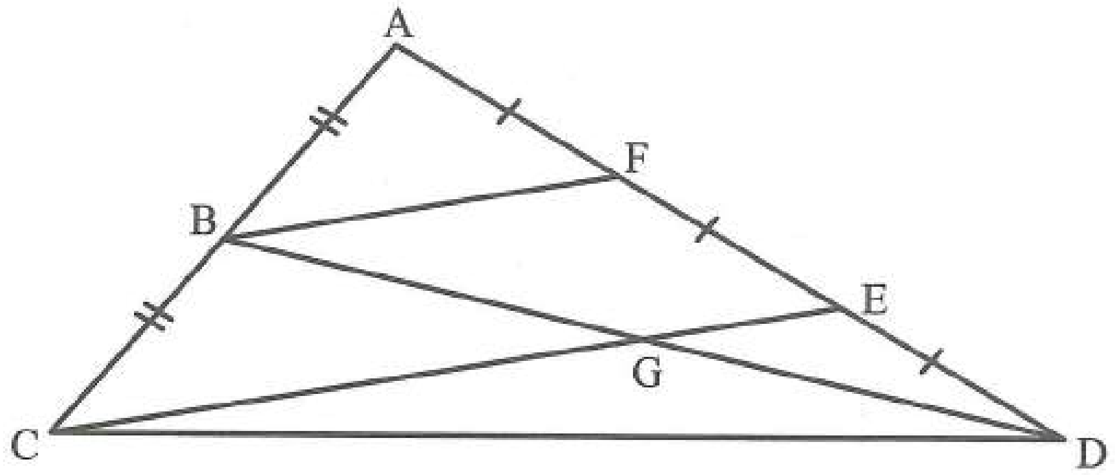
4) **In the figure below, the area of $\triangle ABD = 24\,cm^2$. Determine the area of $\triangle AEG$ if F is the midpoint of AC and $BD \perp AC$.**



5) **In the diagram below, D, E and F are the midpoints of AB, AC and BC respectively. Determine the perimeter of $\triangle ABC$ in terms of x.**

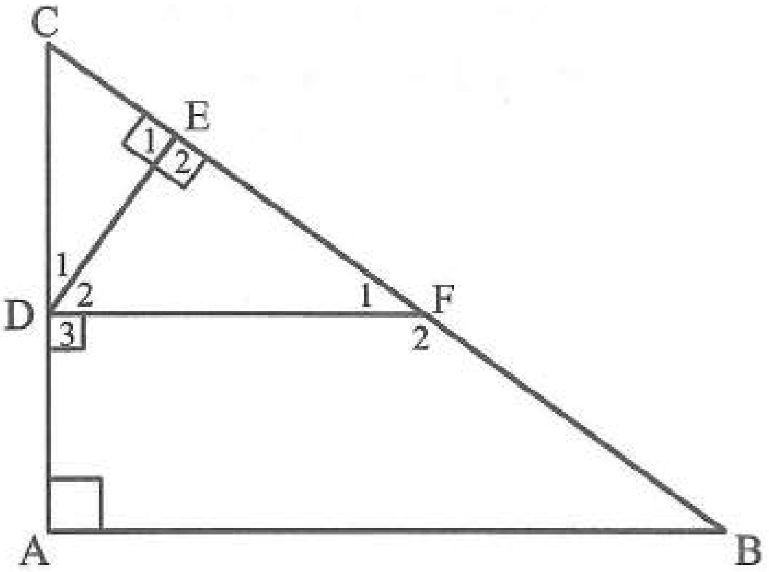


6) **In the diagram below, B is the midpoint of AC and $AF = FE = ED$. If $GE = 90\,cm$, determine the length of CG.**



Exercise 5: (answers on page 368)

1) **In the figure alongside, $\hat{A} = 90^\circ$, $DE \perp BC$ and $AC \perp DF$. $BC = 40\,m$, $DC = 18\,m$ and $AD = 12\,m$.**


C
E
1
2
1
2
3
D
1
F
2
A
B

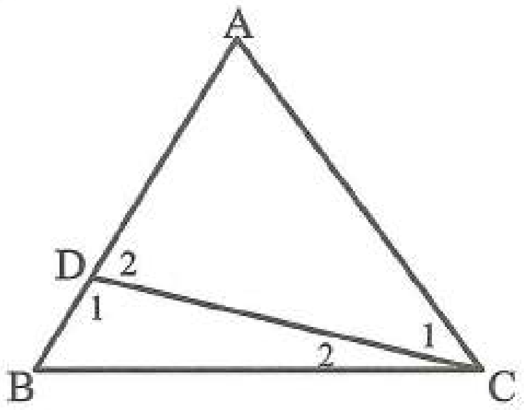
   a) Prove that $\triangle CDE\,|||\,\triangle CBA$.

   b) Determine the length of EC.

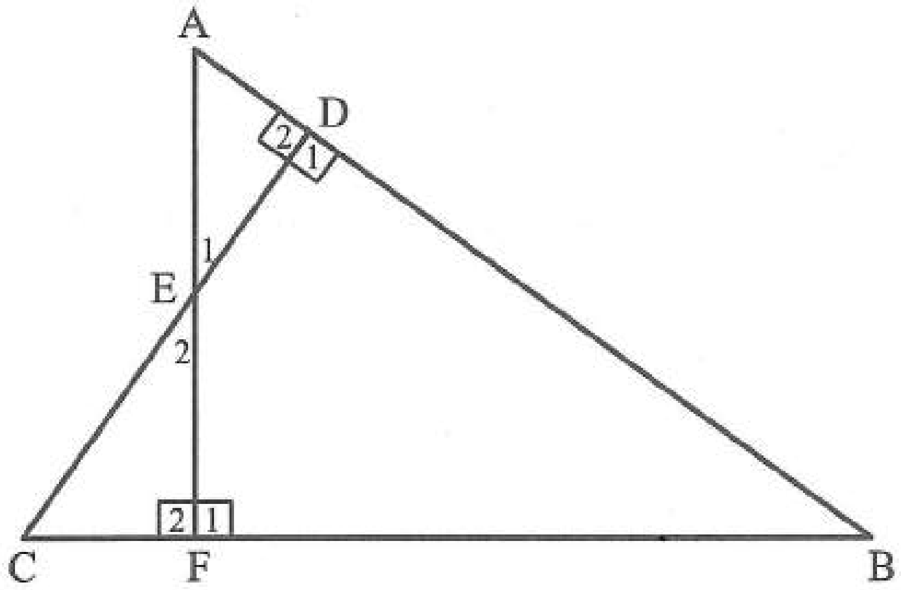
2) **In the figure alongside, $\hat{C}_1 = \hat{B}$.**

   a) Prove that $\triangle ABC\,|||\,\triangle ACD$.

   b) Hence, prove that $AC^2 = AB \times AD$.



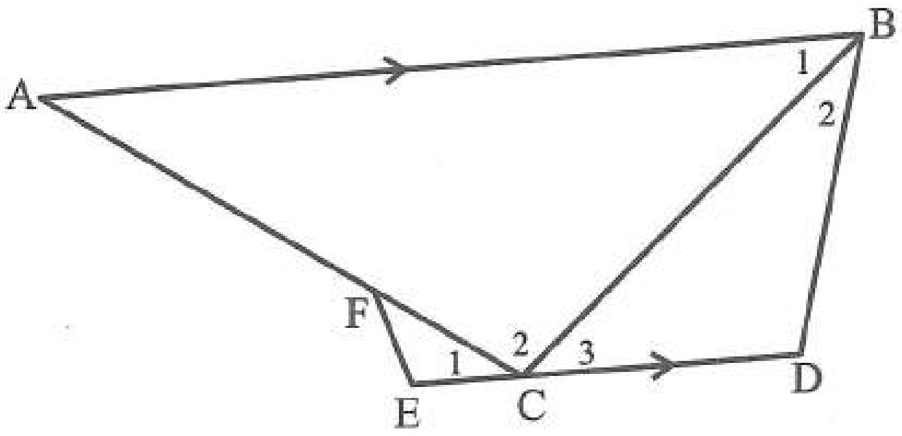
3) **In the diagram alongside, $CD \perp AB$ and $AF \perp BC$. E is the midpoint of AF. Prove that:**


A
D
2
1
1
E
2
2
1
C
F
B

   a) $\dfrac{BD}{CD} = \dfrac{EF}{CF}$.

   b) $EF^2 = DE \times CE$.

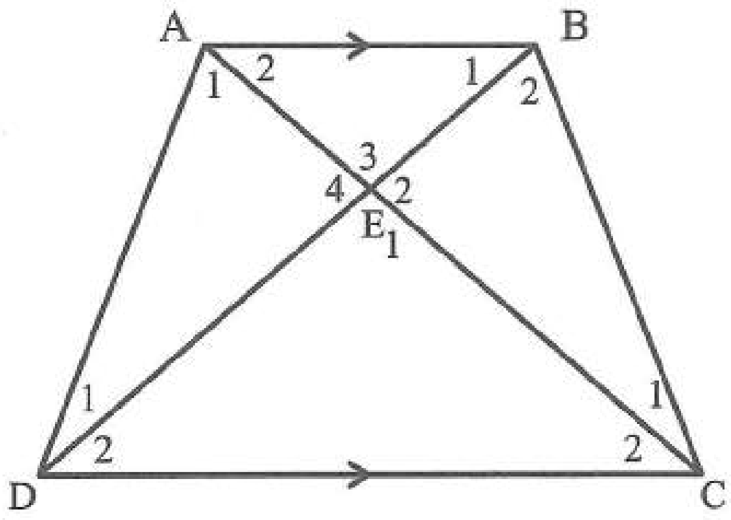
4) **In the diagram alongside, AB//ED and $\hat{C}_1 = \hat{B}_2$.**


B
1
2
A
F
1
2
3
D
E
C

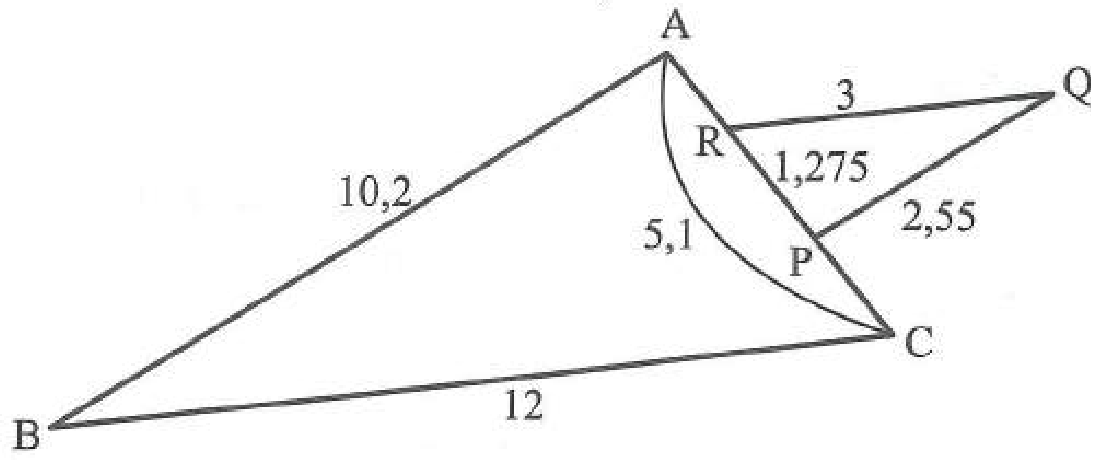
   a) Show that $\dfrac{AB}{BC} = \dfrac{AC}{BD}$.

   b) Show that $\dfrac{AC}{AB} = \dfrac{CE}{CF}$ if $\hat{E} = \hat{D}$.

5) **In the diagram alongside, AB//DC. Prove that $AE \times DE = CE \times BE$.**


A
B
1
2
1
2
3
4
2
E
1
1
2
1
2
D
C

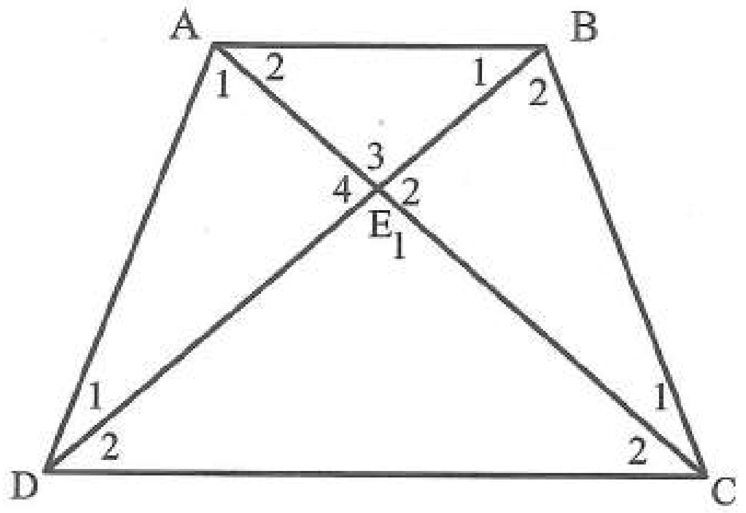
6) **In $\triangle ABC$, $AB = 10{,}2\text{m}$,**
$AC = 5{,}1\text{m}$ and $BC = 12\text{m}$.
In $\triangle PQR$, $PQ = 2{,}55\text{m}$,
$PR = 1{,}275\text{m}$ and $QR = 3\text{m}$.
Prove that $\triangle ABC\ |||\ \triangle PQR$.



7) **In the diagram alongside, ABCD is a quadrilateral.**
$AE.DE = CE.BE$ and $BE.CD = DE.AB$:

a) Prove that $\triangle AEB\ |||\ \triangle CED$.

b) Determine $\frac{EB}{ED}$ if $CD = 3AB$.



Pythagoras

Exercise 6: (answers on page 368)

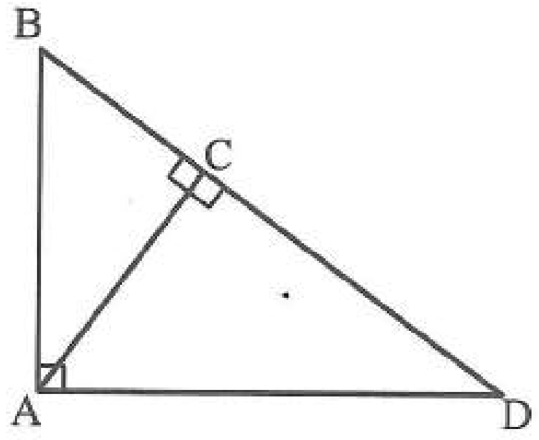
1) **In the diagram alongside, $BA \perp AD$ and $BD \perp AC$.**

a) Complete the following with reason:
$\triangle ABD\ |||\ \triangle ...\ |||\ \triangle ...$

b) Hence, complete the following:

$AD^2 = ...$	$AB^2 = ...$	$AC^2 = ...$

c) If $BC = 4\text{m}$ and $AB = 5\text{m}$, determine the length of AD.



2) **In the diagram alongside, ABCD is a kite with diagonals intersecting at P. $\hat{B} = \hat{D} = 90°$.**

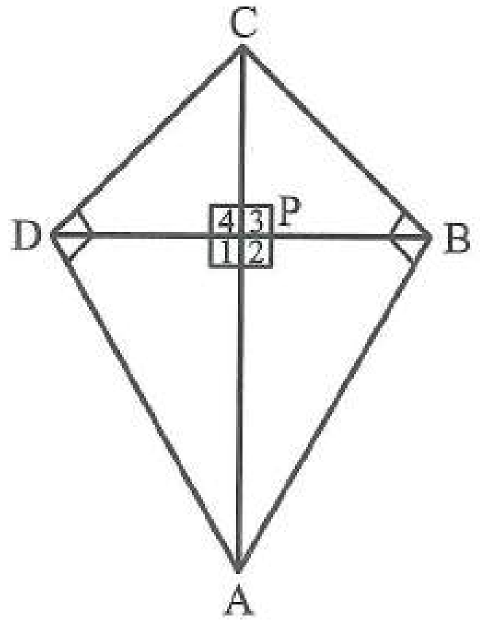
a) Give a reason why $\triangle CDP\ |||\ \triangle CAD$.

b) Hence, complete the following:

$CD^2 = ...$	$AD^2 = ...$	$DP^2 = ...$

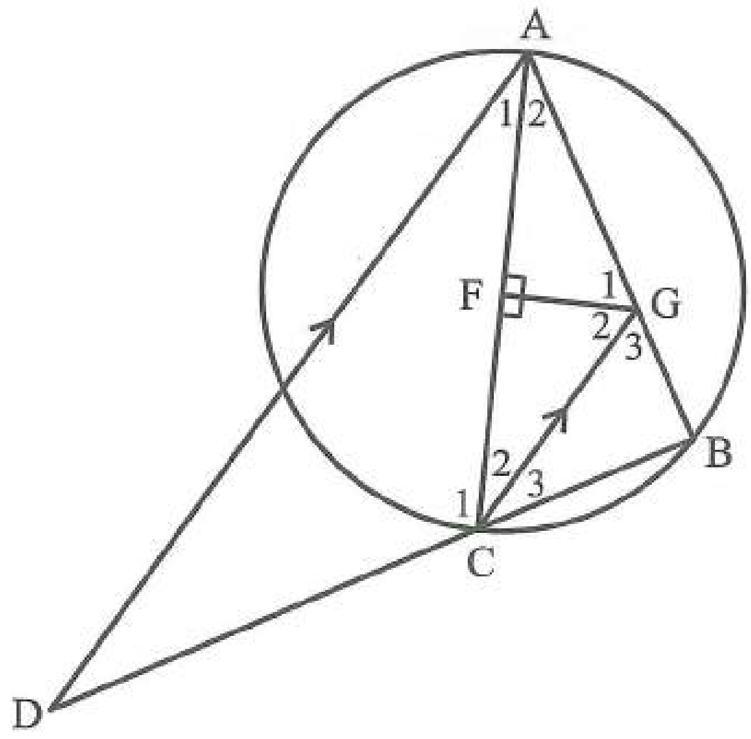
c) Prove that $\frac{BP^2}{CB^2} = \frac{AP}{AC}$.

d) Prove that $CD^2 - CP^2 = CP \times PA$.



Mixed Exercise: (answers on pages 368 and 369)

1) **AC is a diameter of circle ABC. $\hat{C}_2 = \hat{C}_3$, $AC \perp FG$, $AD // CG$, $BC = 6$ units and the diameter of the circle is 10 units.**



a) Prove that $\triangle CAD$ is isosceles.

b) Prove that $\frac{BC}{AC} = \frac{BG}{AG}$.

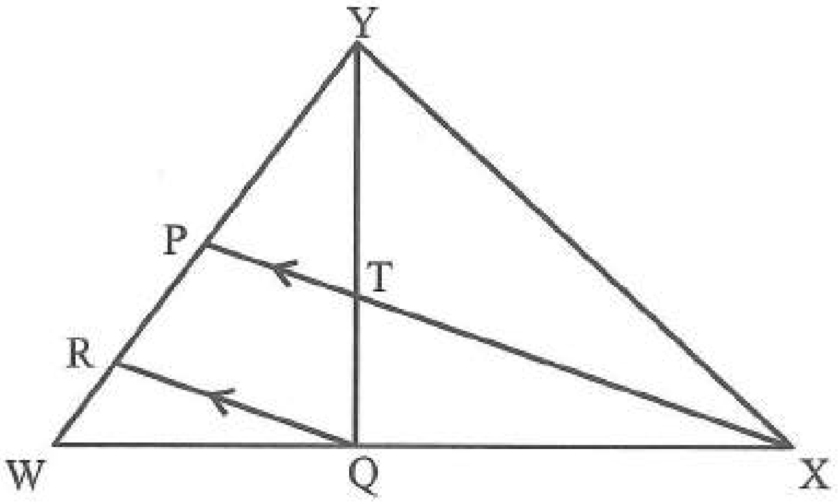
c) Prove that $\triangle AFG \,|||\, \triangle ABC$.

d) Determine the length of AB.

e) Determine the length of AG.

f) Determine the length of AF.

2) **In the diagram alongside, P is the midpoint of WY and $PX \perp WY$. Q is a point on WX such that $WQ : WX = 3 : 5$ and $QR // XP$. Determine:**

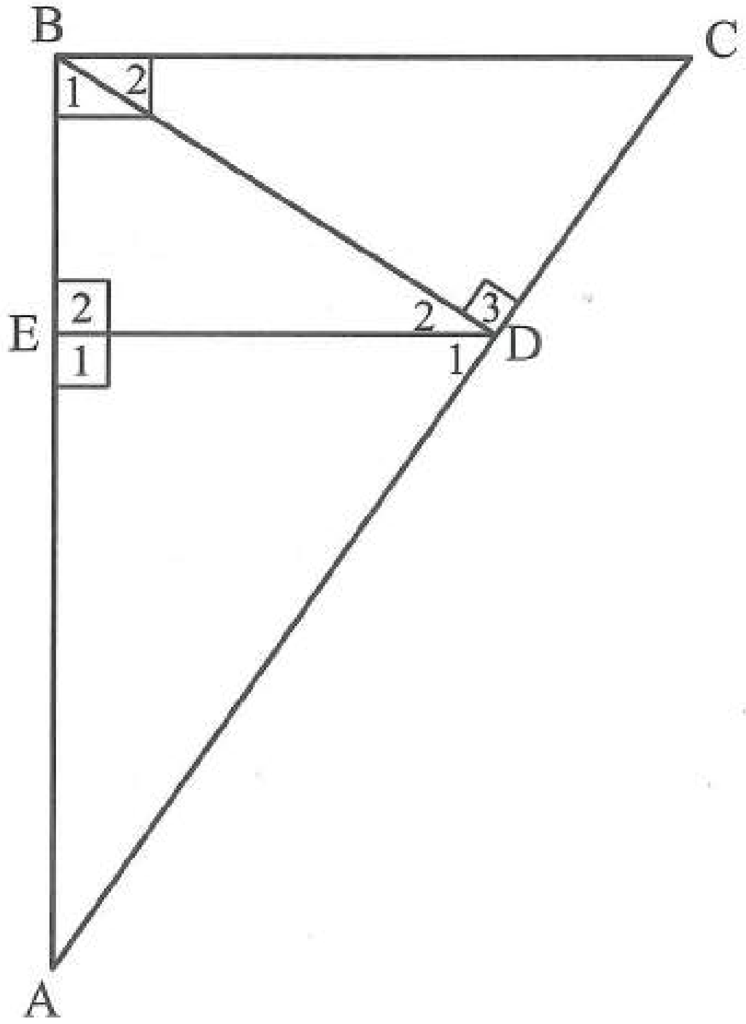


a) $\frac{YP}{YR}$.

b) $\frac{QR}{TP}$.

c) $\frac{\text{Area } \triangle TPY}{\text{Area } \triangle QRY}$.

3) **$\triangle ABC$ is right-angled triangle at B. D is a point on AC such that $BD \perp AC$ and E is a point on AB such that $DE \perp AB$. $AD : DC = 3 : 2$ and $AD = 30$ m.**



a) Prove that $\triangle BDA \,|||\, \triangle CDB$.

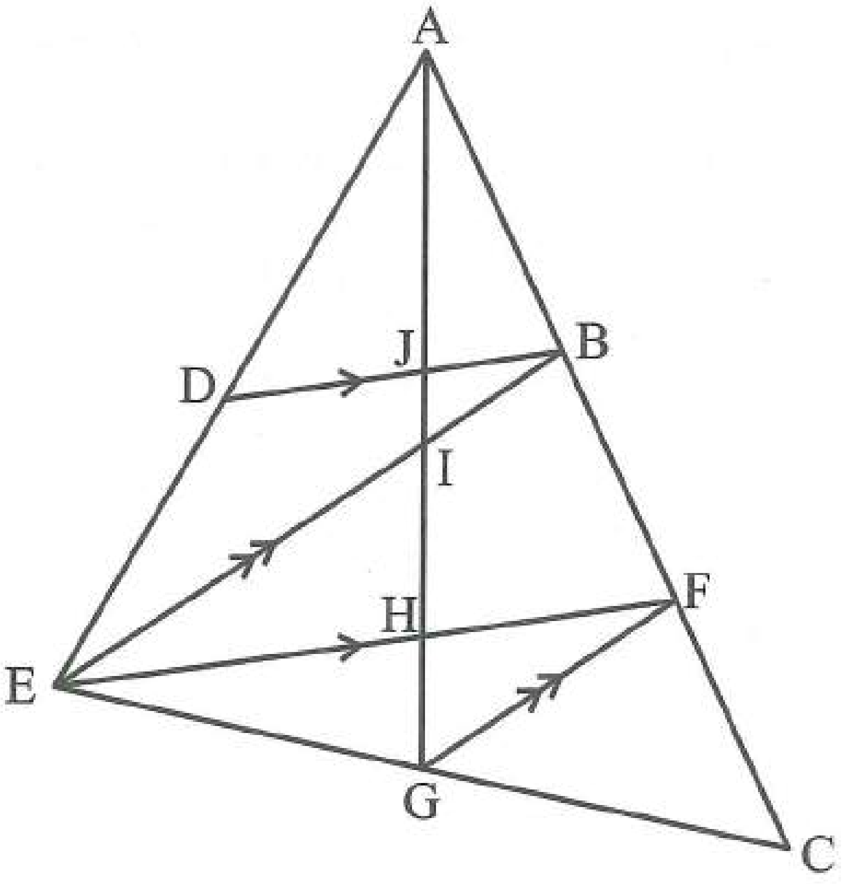
b) Calculate BD, leaving your answer in simplest surd form.

c) Calculate AE, leaving your answer in simplest surd form.

4) In the diagram alongside, $EG = GC$, $DB//EF$ and $EB//GF$. If the ratio of $AD : AE = 3 : 4$, determine:

a) The ratio of $BF : FC$.

b) The ratio of $AB : AC$.

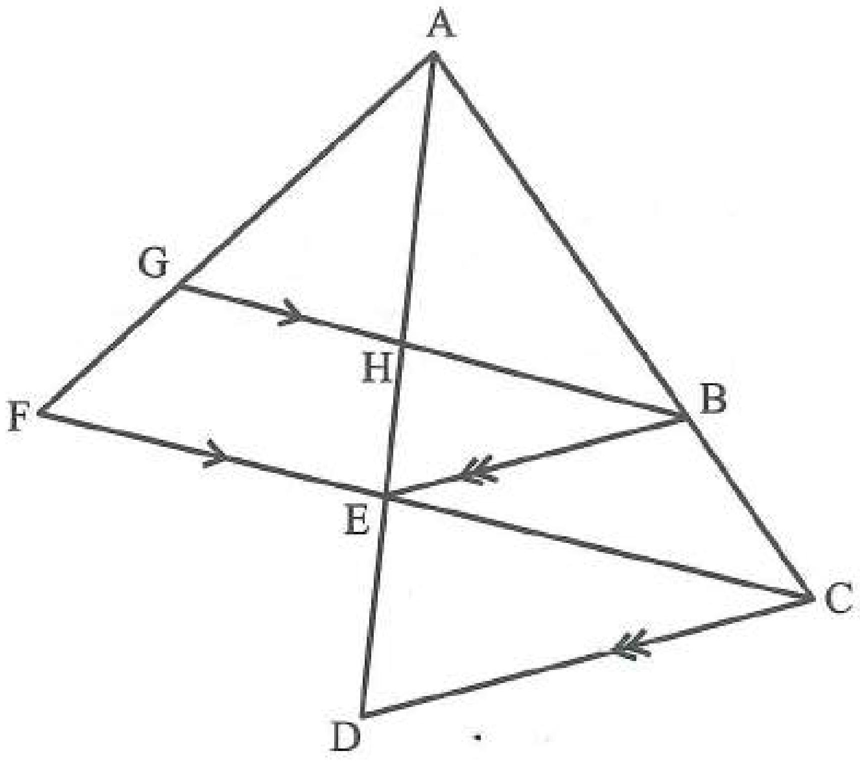


5) In the diagram alongside, $GB//FC$ and $BE//CD$. $AC = 12\text{units}$ and $AB = 2BC$.

a) Calculate $\frac{AH}{ED}$.

b) Calculate $\frac{BE}{CD}$.

c) If $HE = 4\text{cm}$, determine the value of $AD \times HE$.



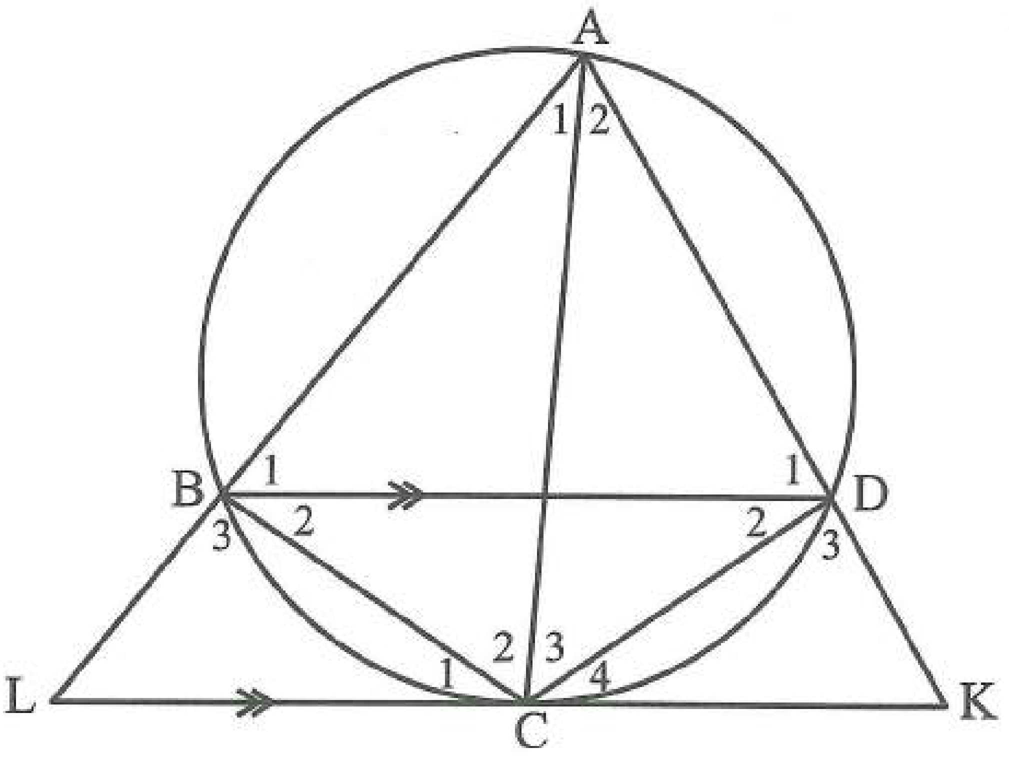
6) In the diagram alongside, KCL is a tangent to the circle. $LK//BD$ and ABL and ADK are straight lines.

a) Prove that $BC = DC$.

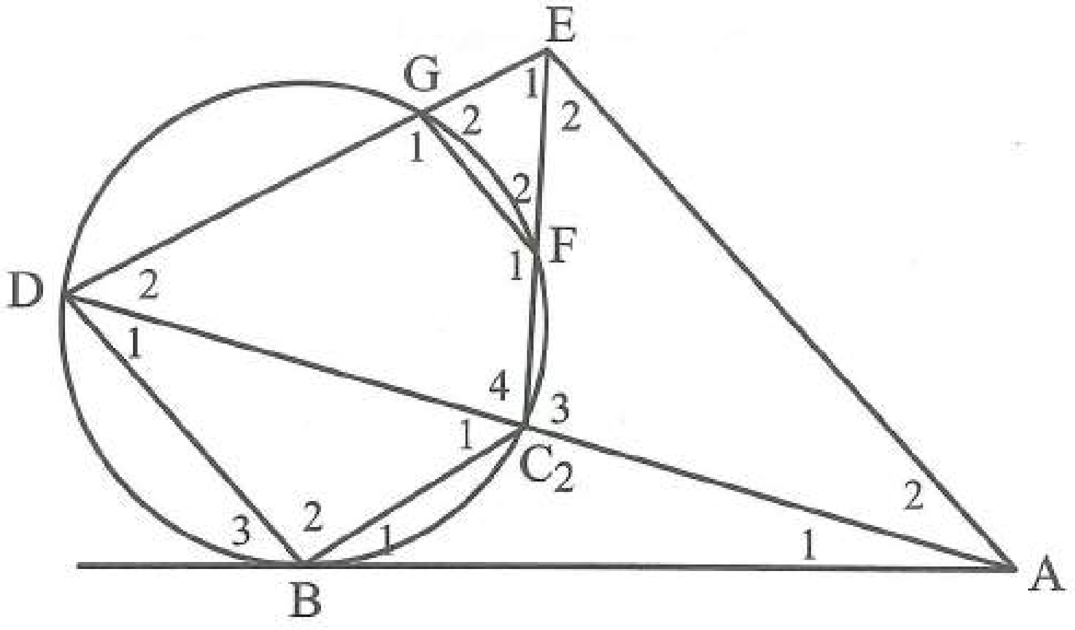
b) Prove that $\triangle CKD \,|||\, \triangle ACB$.

c) Show that $BC^2 = AB.KD$.

d) Determine the length of KD if $BC = 4\text{cm}$ and $AB = 6\text{cm}$.



7) In the diagram alongside, AB is a tangent to the circle at B and CFE and ACD are straight lines.



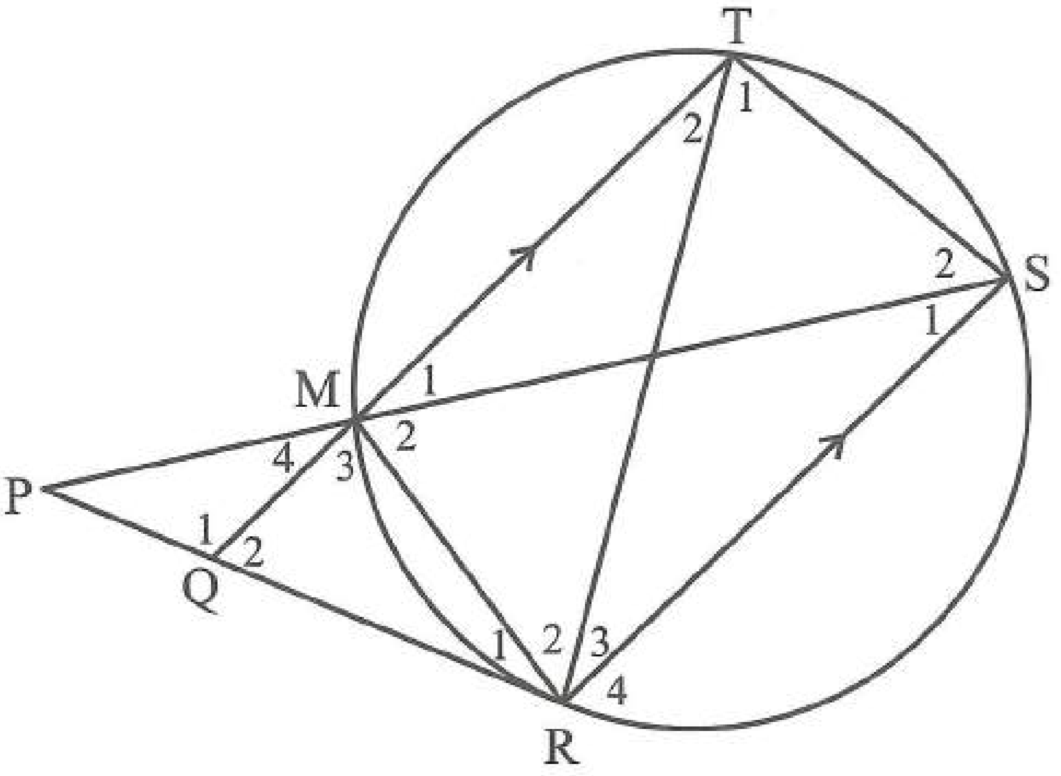
a) Prove that $AB = \sqrt{AC.AD}$.

b) If $AB = 5\sqrt{2}\,\text{cm}$ and $AD = 10$ cm, determine the length of AC.

c) If $\triangle ACE\ |||\ \triangle AED$, prove that $AE = AB$.

d) Prove that GF//EA.

8) In the diagram alongside, RS//QT and PR is a tangent to the circle at R. SMP is a straight line and $\hat{T}_2 = x$.



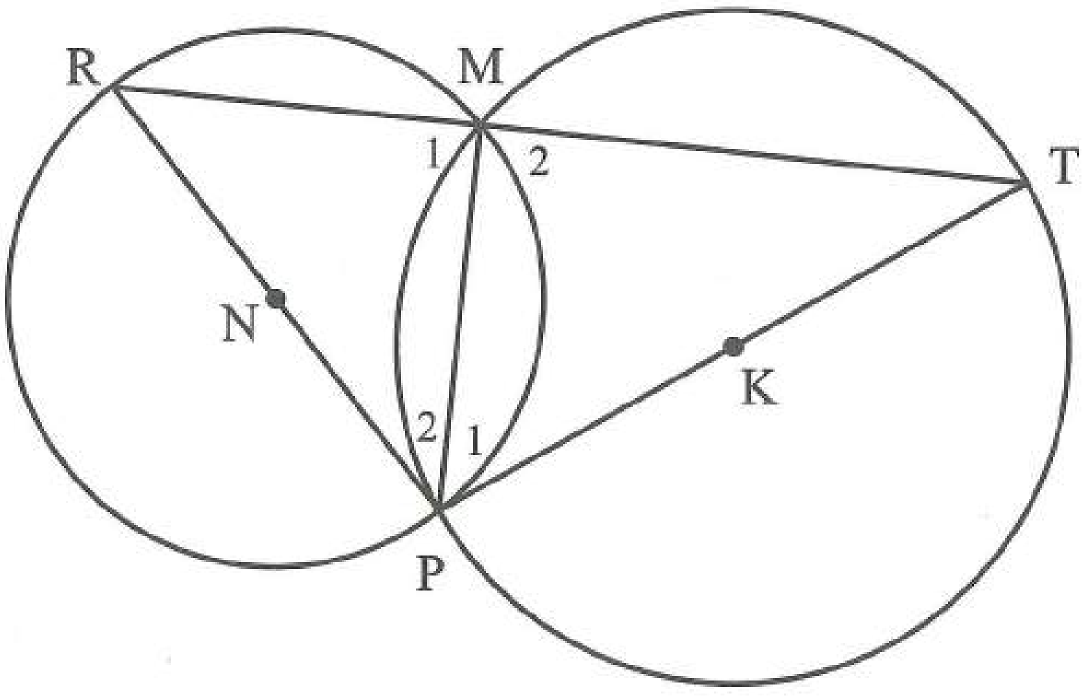
a) With reasons, state five other angles which are equal to x.

b) Prove that $\triangle RTS\ |||\ \triangle RQM$.

c) What can you conclude from the fact that $\hat{M}_4 = \hat{R}_1$?

d) If $RQ : RP = 2 : 3$ and $SM = 4\,\text{cm}$, calculate the length of PM.

9) Circles with centres N and K intersect at M and P. RP is a tangent to circle centre K at P.



a) Prove that TMR is a straight line.

b) Prove that $\triangle RPM\ |||\ \triangle PTM$.

c) If $TM = 12,5\,\text{cm}$ and $RM = 8\,\text{cm}$, determine the length of PR in simplest surd form.

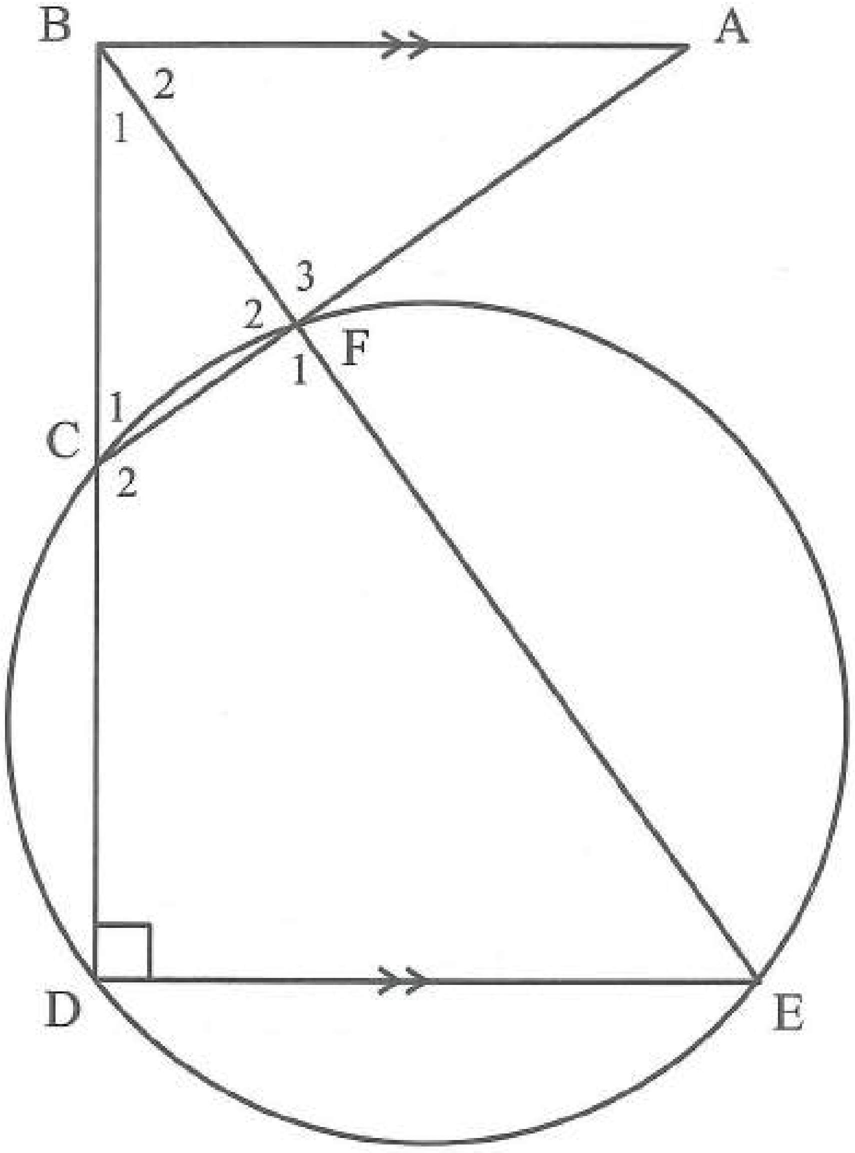
d) Prove that NK//RT and then determine the length of NK.

10) In the diagram alongside, AB//ED and $\hat{CDE} = 90°$.

a) Prove that $\triangle ABC \,|||\, \triangle BFC$.

b) If $BC = 6$ and $AB = 8$, calculate the length of AC and AF.

c) Name two other triangles that are similar to $\triangle ABC$.



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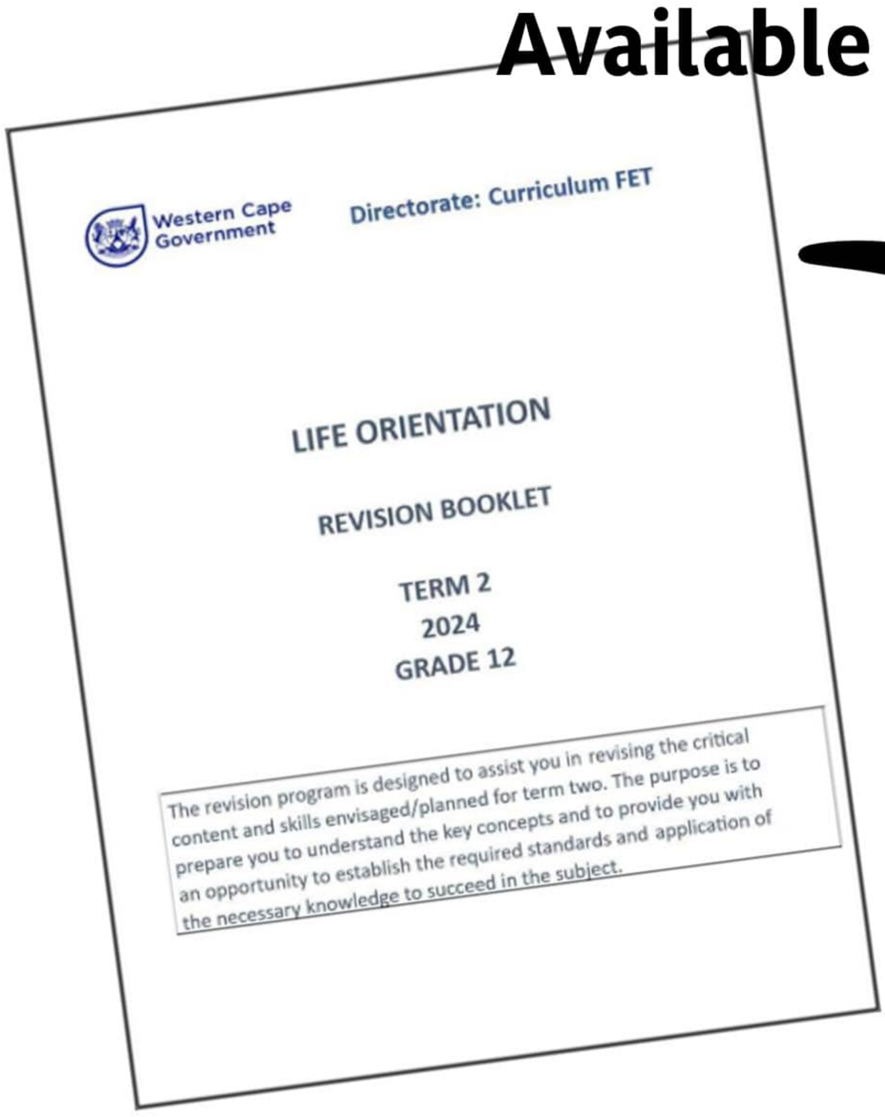

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REVISION BOOKLET

TERM 2
2024
GRADE 12

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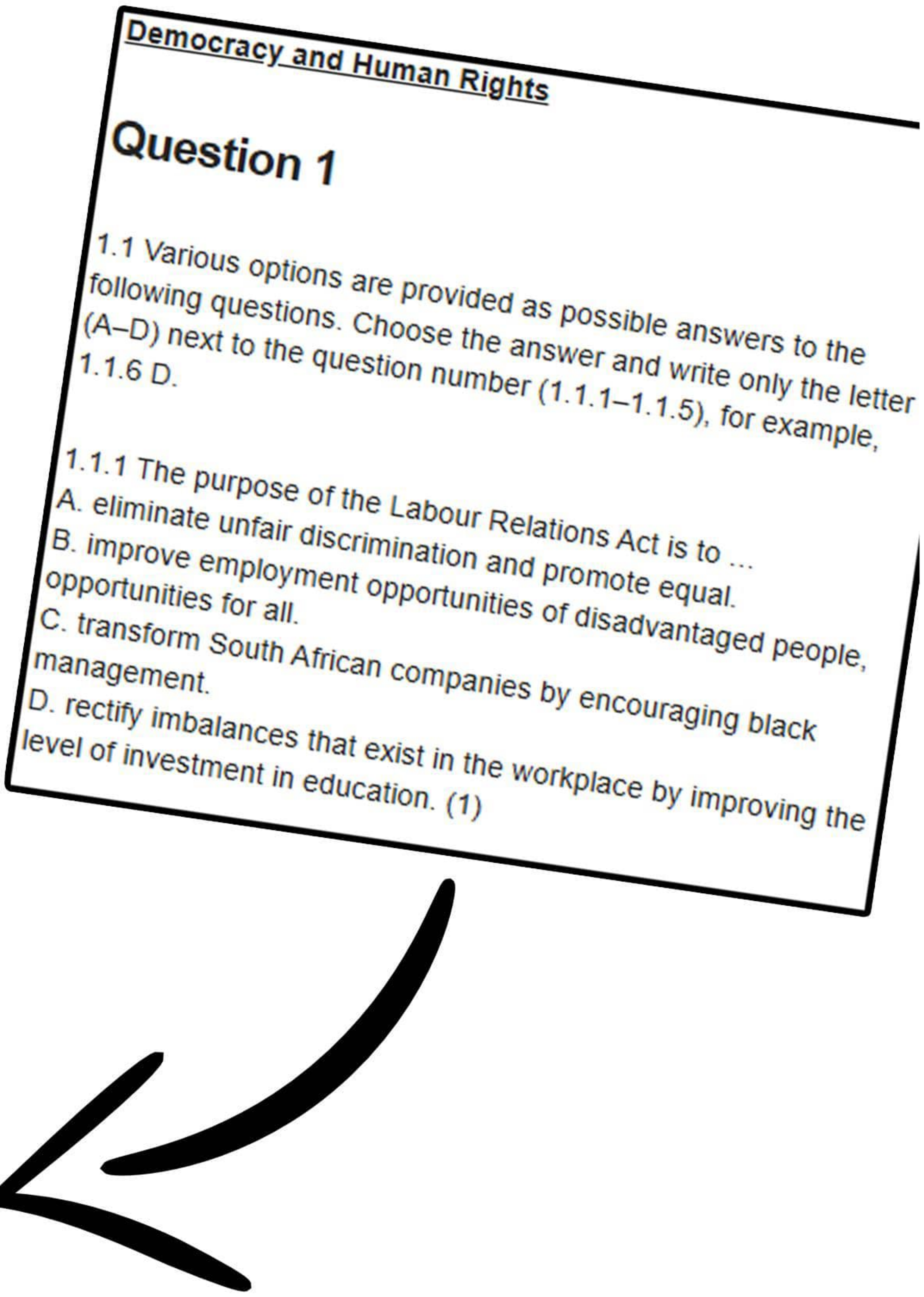
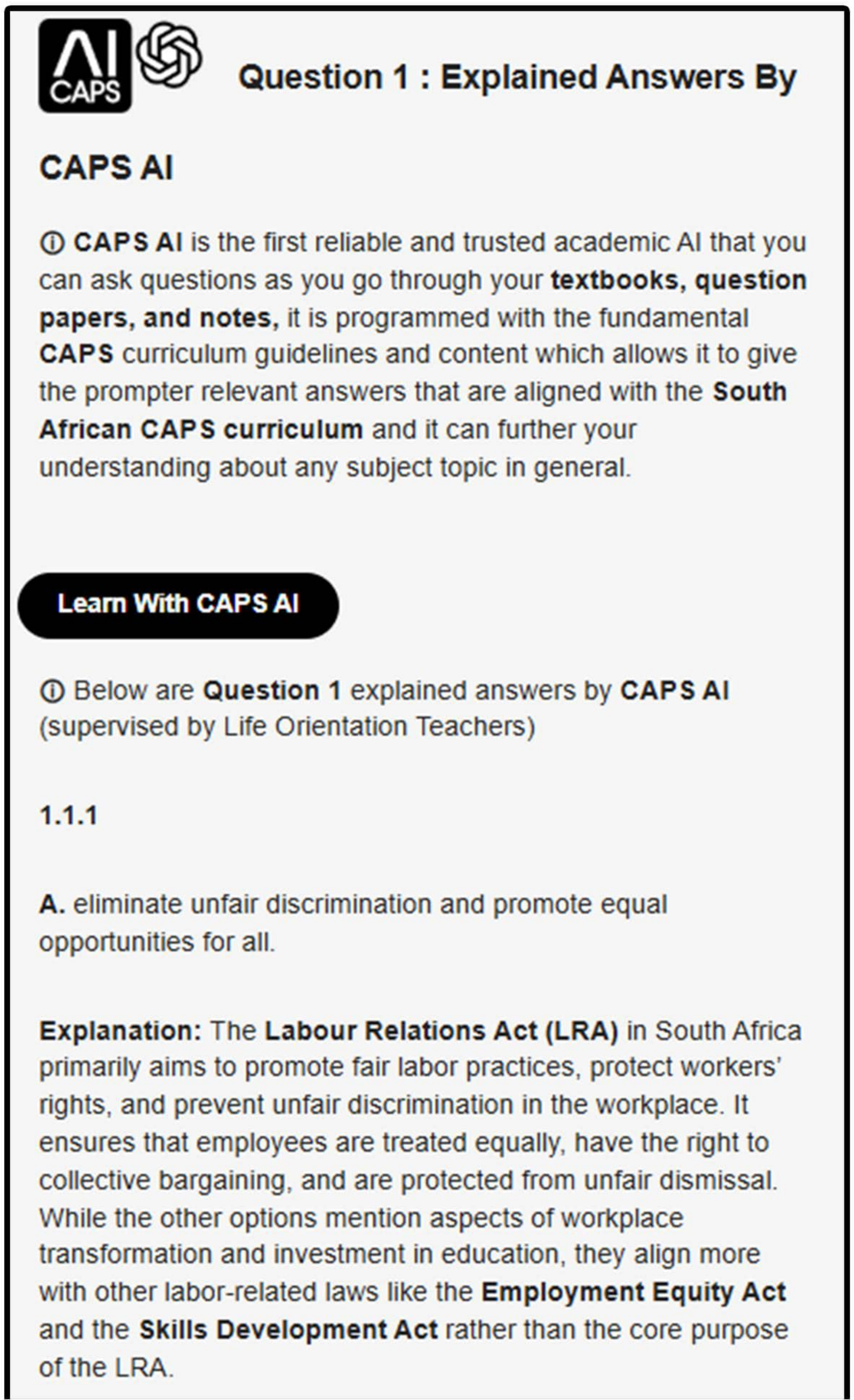


Democracy and Human Rights

Question 1

1.1 Various options are provided as possible answers to the following questions. Choose the answer and write only the letter (A–D) next to the question number (1.1.1–1.1.5), for example, 1.1.6 D.

1.1.1 The purpose of the Labour Relations Act is to …
A. eliminate unfair discrimination and promote equal opportunities for all.
B. improve employment opportunities of disadvantaged people.
C. transform South African companies by encouraging black management.
D. rectify imbalances that exist in the workplace by improving the level of investment in education. (1)



Question 1 : Explained Answers By CAPS AI

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ⓘ Below are **Question 1** explained answers by **CAPS AI** (supervised by Life Orientation Teachers)

1.1.1

A. eliminate unfair discrimination and promote equal opportunities for all.

Explanation: The **Labour Relations Act (LRA)** in South Africa primarily aims to promote fair labor practices, protect workers' rights, and prevent unfair discrimination in the workplace. It ensures that employees are treated equally, have the right to collective bargaining, and are protected from unfair dismissal. While the other options mention aspects of workplace transformation and investment in education, they align more with other labor-related laws like the **Employment Equity Act** and the **Skills Development Act** rather than the core purpose of the LRA.

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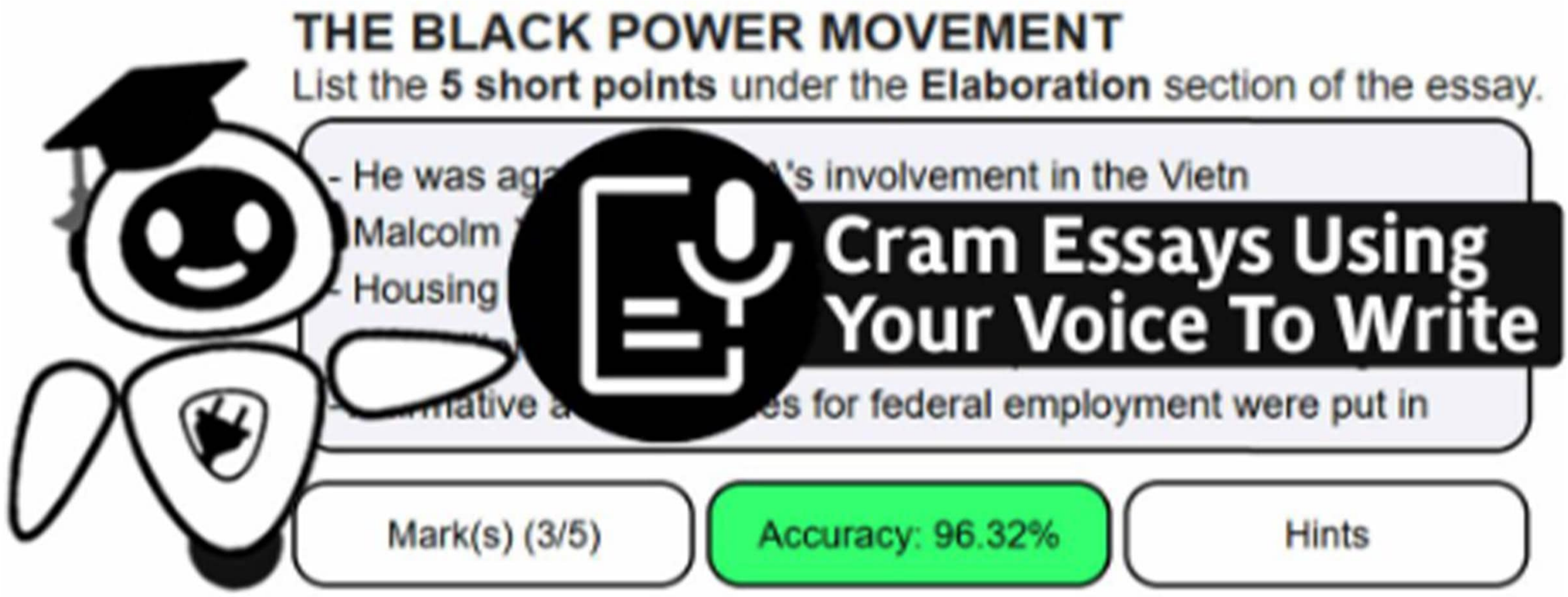

THE BLACK POWER MOVEMENT

List the **5 short points** under the **Elaboration** section of the essay.

- He was ag A's involvement in the Vietn
- Malcolm
- Housing
- mative a es for federal employment were put in

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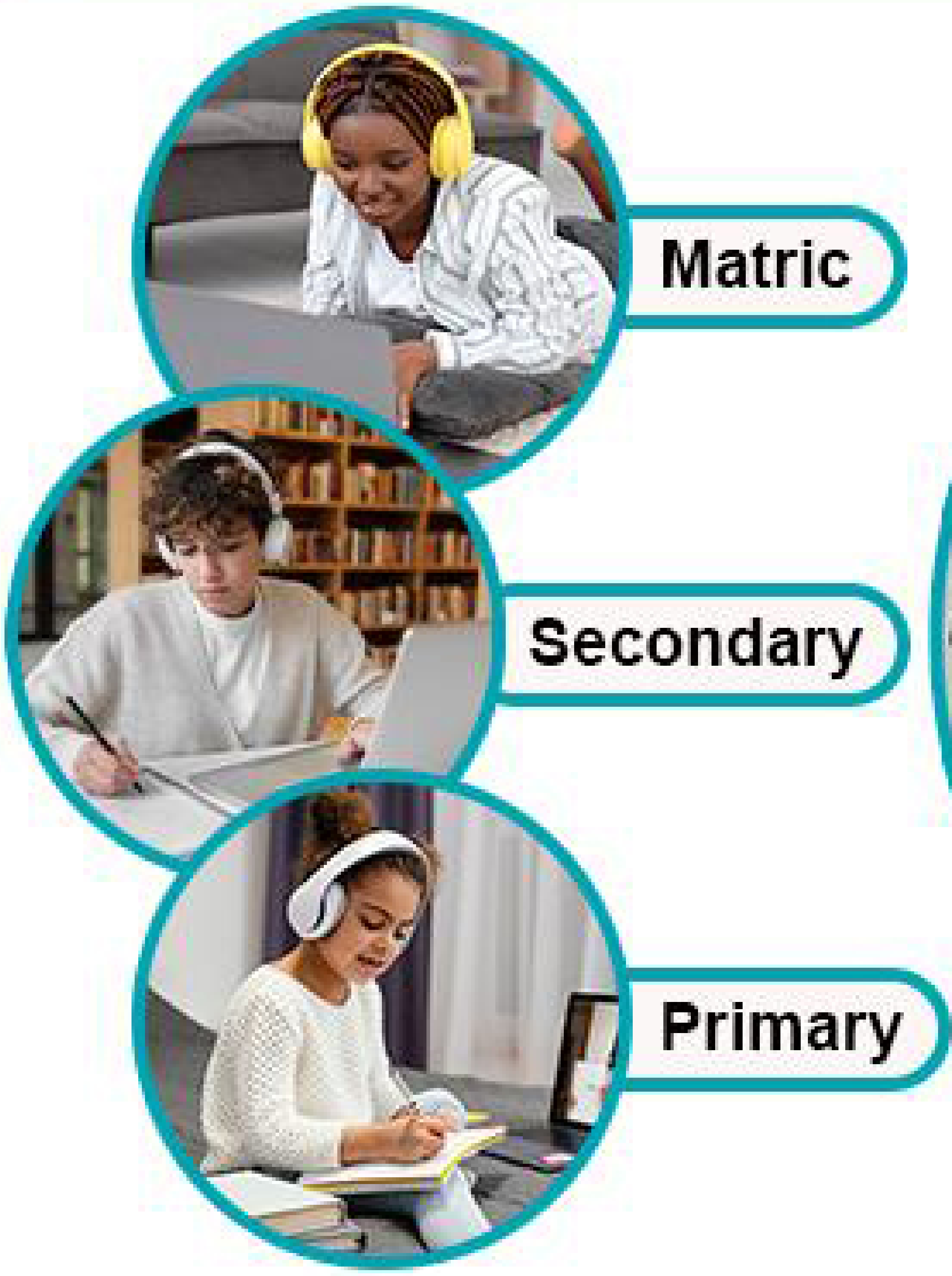

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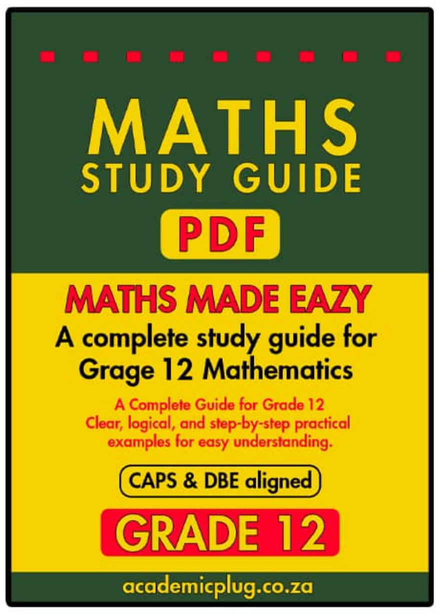
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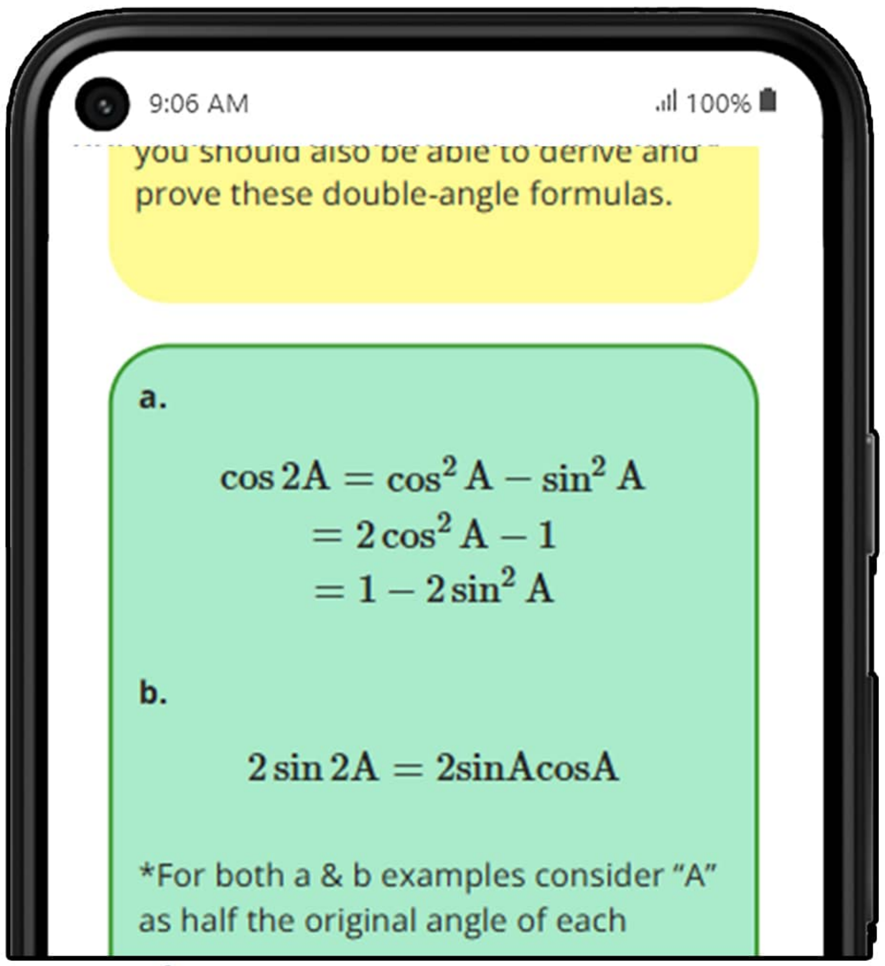
9:06 AM 100%

you should also be able to derive and prove these double-angle formulas.

a.

$$\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A \\ &= 2\cos^2 A - 1 \\ &= 1 - 2\sin^2 A\end{aligned}$$

b.

$$2\sin 2A = 2\sin A\cos A$$

*For both a & b examples consider "A" as half the original angle of each

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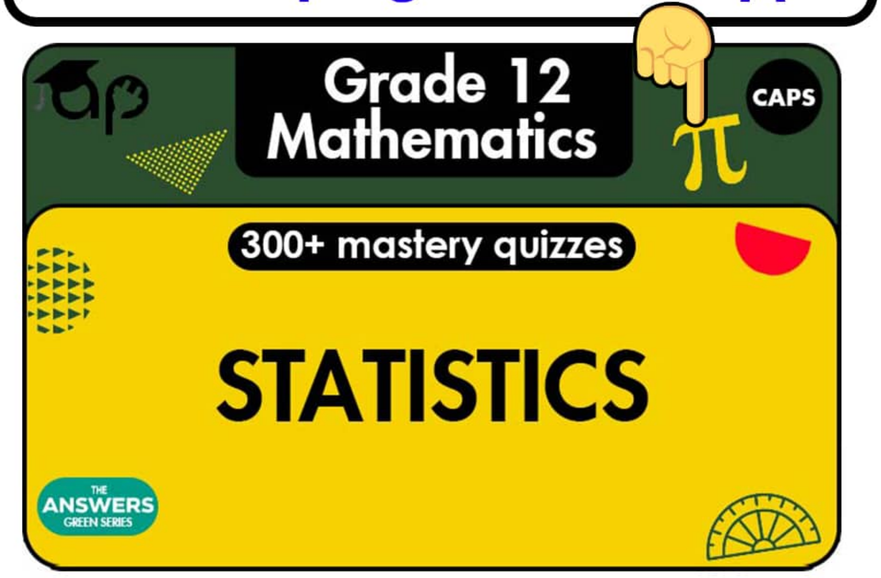
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CHAPTER 13
STATISTICS

WHAT YOU WILL LEARN IN THIS CHAPTER

PAGE	CONTENT	CAPS	ASSESSMENTS YOU WILL BE ABLE TO:
275	**A. Grade 11 Revision**	10	• Represent bivariate numerical data as a scatter plot and suggest intuitively and by a simple investigation whether a linear, quadratic or exponential function would best fit the data. • Use a calculator to calculate the linear regression line which best fits a given set of bivariate numerical data. • Use a calculator to calculate the correlation coefficient of a set of bivariate numerical data and make relevant deductions.
275	• Measures of Central Tendency		
275	• Five Number Summary		
275	• Measures of Dispersion		
275	• Calculation of the Mean		
275	• Calculation of Quartiles and Percentiles		
276	• Ogives		
279	• Variance and Standard Deviation		
281	• Standard Deviation with Grouped Data		
282	• Symmetrical and Skewed Data		
282	• Identification of Outliers		
283	**B. Scatter Plots**		
283	• Regression Lines (Lines of Best Fit)		
284	• Least Squares Regression Line		
284	• Using a Calculator to Determine Regression Lines and the Mean Point		
286	• The Correlation Coefficient		
288	• Notes on Regression Lines		

A. GRADE 11 REVISION

MEASURES OF CENTRAL TENDENCY

Mode: The most common value in a data set.
Median (Q_2): The median is the middle value in an ordered data set.
Mean ($\bar{x}$): The average of a data set.

FIVE NUMBER SUMMARY

Minimum Value in the set: The lowest value in a data set.
Lower Quartile (Q_1): The median of the lower half of an ordered data set.
Median (Q_2): The median is the middle value in an ordered data set.
Upper Quartile (Q_3): The median of the upper half of an ordered data set.
Maximum Value in the set: The highest value in a data set.
The Five Number Summary is represented visually on a box and whisker diagram.

MEASURES OF DISPERSION

Range: The difference between the largest value and smallest value in a data set.

Range = largest value – smallest value.

Inter-quartile range (IQR): The difference between the upper and lower quartiles.

$$\text{IQR} = Q_3 - Q_1.$$

CALCULATION OF THE MEAN

The mean is the most commonly used measure of central tendency. It is commonly known as the average. The mean is the sum of the data values in the data set, divided by the number of elements in the data set.

$$\textbf{Mean} = \bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{\textbf{sum of the values in the data set}}{\textbf{number of elements in the data set}}$$

CALCULATION OF QUARTILES AND PERCENTILES

Median (Q_2)

The median is the middle value of an ordered data set. When determining the median of a data set, the **data must always be ordered.**
Use the formula below to determine the position of the median.

$$\textbf{Position of the median} = \frac{n+1}{2}$$

Where n is the number of elements in the data set.

Lower (Q_1) and Upper Quartiles (Q_3)

The lower and upper quartiles are the median of the lower and upper half of the ordered data set respectively. You can therefore calculate the upper and lower quartile by determining the median of the lower and upper halves.

Percentiles

A percentile is the value below which a specific percentage of data elements lie.

$$\textbf{Position of percentile} = \frac{\textbf{percentile}}{100} \times \textbf{number of elements in the data set}$$

OGIVES

An ogive is a graph which represents the cumulative frequency of a set of data. The graph usually has an s-shape and is a smooth curve not joined by straight lines. Points are plotted using the upper limit of each interval while the graph is connected to the x-axis at the lower limit of the first class. In other words, the x-intercept is not necessarily always at the origin. Another name for an ogive is a cumulative frequency plot.

Example: The following table shows the amount of time (in minutes) children aged between 6 and 10 spend watching television (TV) each day:

Daily time (T) spent watching TV (minutes)	Number of children
$0 \leq T < 20$	21
$20 \leq T < 40$	48
$40 \leq T < 60$	56
$60 \leq T < 80$	32
$80 \leq T < 100$	10
$100 \leq T < 120$	5

a) Draw a histogram to represent the data given in the table above.

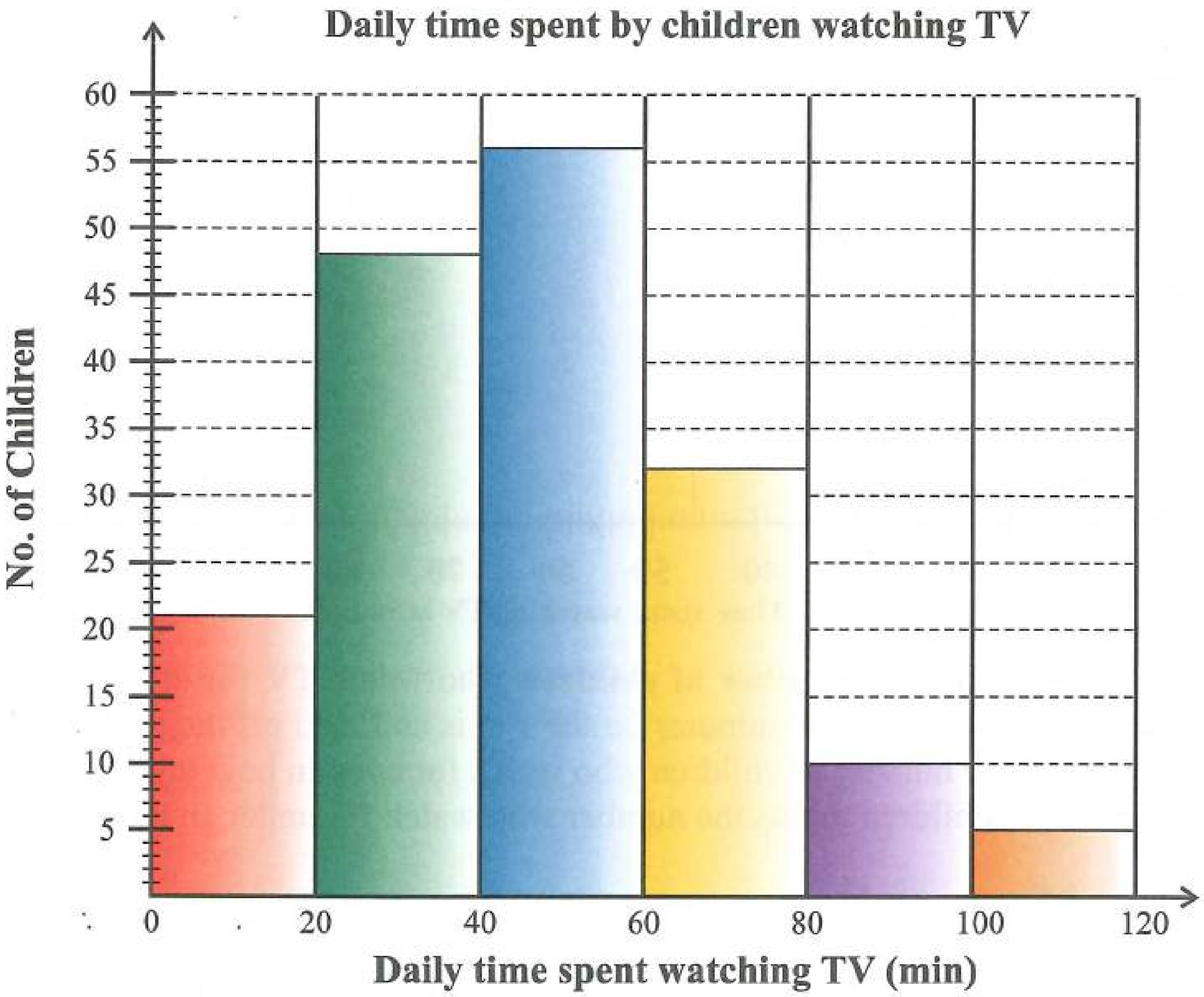
b) Draw an ogive to represent the above data.

c) Using the ogive, estimate the number of children who watch TV for an hour and half or less each day, as well as the number of children who watch TV for over an hour and a half each day.

d) Using your graph, determine the lower quartile, median and upper quartile.

e) Determine the estimated daily mean (average) amount of time that these children spend watching TV .

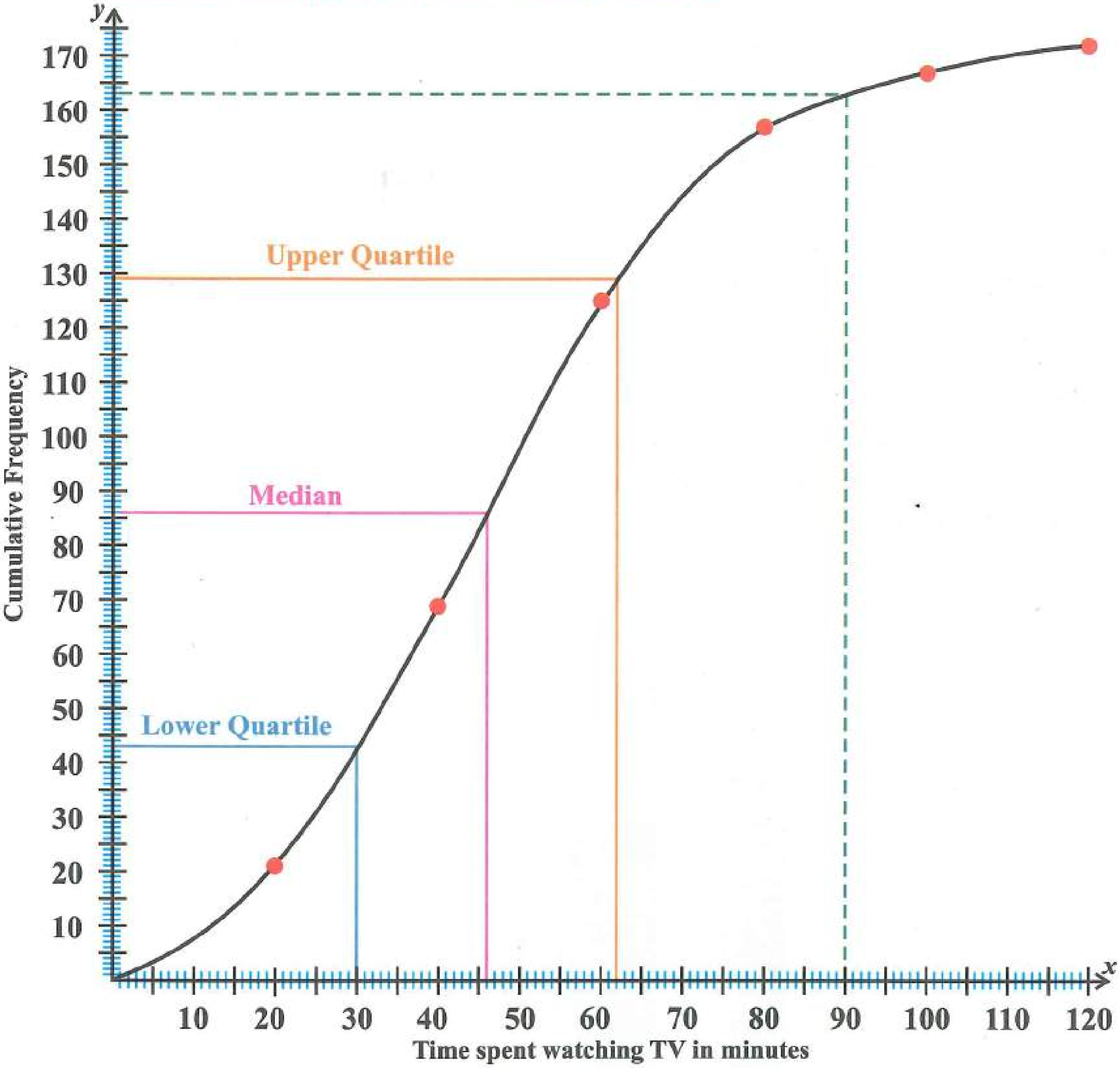
a)



b) To draw an ogive you have to add a cumulative frequency column to the table:

Time	Frequency	Cumulative Frequency
$0 \le T < 20$	21	21
$20 \le T < 40$	48	$69 \rightarrow 21+48$
$40 \le T < 60$	56	$125 \rightarrow 69+56$
$60 \le T < 80$	32	$157 \rightarrow 125+32$
$80 \le T < 100$	10	$167 \rightarrow 157+10$
$100 \le T < 120$	5	$172 \rightarrow 167+5$

Now plot the **cumulative frequency on the y-axis** and the **class intervals on the x-axis**. The x-intercept is at the lowest value of the smallest class (in this case 0) and you plot the **cumulative frequency against the upper limits** of the class intervals.



c) In order to estimate the **number of children** who watch TV for under an hour and a half, draw a dotted line at 90 minutes on the x-axis and read off the number of children on the y-axis. The number of children who watch for over an hour and a half will be the total number of children minus the number who watch for under an hour and a half.

$\therefore$ No. of children who watch for under an hour and a half ≈ 163

$\therefore$ No. of children who watch for over an hour and a half $\approx 172 - 163 = 9$

d) The person who has the **lower quartile** time will be in the position which is a quarter of the cumulative frequency (total number of people). The **median position** will lie at half of the cumulative frequency and **upper quartile** will lie at three quarters of the cumulative frequency. Once these values have been found, draw a line from the y-axis to the graph and then read the x-values at these points.
The x-values are the actual lower quartile, median, and upper quartile times.

$\therefore$ Calculate the y-value of the lower quartile, median and upper quartile

Lower quartile position: $172 \times \frac{1}{4} = 43$

$\therefore$ Lower quartile $= 30$ (read off the graph)

Median position: $172 \times \frac{1}{2} = 86$

$\therefore$ Median $= 46$ (read off the graph)

Upper quartile position: $172 \times \frac{3}{4} = 129$

$\therefore$ Upper quartile $= 62$ (read off the graph)

e) In order to determine the **estimated mean,** calculate the midpoint of each interval and multiply it by the number of students in that interval (frequency). The sum of these values is then divided by the total of the cumulative frequency to obtain the estimated mean.

Time	Frequency (f)	Midpoint (x)	Frequency x midpoint ($f.x$)
$0 \leq T < 20$	21	10	210
$20 \leq T < 40$	48	30	1440
$40 \leq T < 60$	56	50	2800
$60 \leq T < 80$	32	70	2240
$80 \leq T < 100$	10	90	900
$100 \leq T < 120$	5	110	550
Sum	$\sum f = 172$		$\sum f.x = 8140$

$\therefore$ Estimated mean $= \frac{8140}{172} \rightarrow \frac{\sum f.x}{\sum f}$

$\therefore \bar{x} = 47$ minutes

Do not get confused between the MEAN and the MEDIAN. The mean is the average value of a data set, while the median is the middle value in an ordered set of data.

In order to determine the exact position of the median, the formula shown below should be used. However, when working with ogives it is considered accurate enough to divide the cumulative frequency by 2.

$$\textbf{Position of median} = \frac{n+1}{2}$$

VARIANCE AND STANDARD DEVIATION

Variance is a useful measure which is used to determine the standard deviation. The standard deviation is a representation of how much variation or dispersion there is around the mean. A low standard deviation shows that the data points lie close to the mean and the data set is generally consistent. A high standard deviation shows that the values are spread out over a large range and the data is not consistent.

Variance: $\sigma^2 = \dfrac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n}$ **Standard deviation:** $\sigma = \sqrt{\dfrac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n}}$

Where : $x_i \rightarrow$ Data value given to you

$\overline{x} \rightarrow$ Mean

n $\rightarrow$ Sample size (number of data points in the sample)

Example 1: The table below shows the results of a Xhosa test of a Grade 11 class:

Mark obtained (%)	$(x-\overline{x})$	$(x-\overline{x})^2$
52		
44		
62		
66		
60		

a) Calculate the mean mark for the class.
b) Complete the table and use it to calculate the standard deviation of the marks.

a) Mean: $\overline{x} = \dfrac{52+44+62+66+60}{5} = 56,8$

b)

Mark obtained (%)	$(x-\overline{x})$	$(x-\overline{x})^2$
52	–4,8	23,04
44	–12,8	163,84
62	5,2	27,04
66	9,2	84,64
60	3,2	10,24
		308,8

$$\therefore \sigma = \sqrt{\frac{\sum(x-\overline{x})^2}{n}}$$

$$= \sqrt{\frac{308,8}{5}}$$

$$= 7,86\,\%$$

Example 2: The table below shows the results of a Grade 11 class Maths test:

52	44	62	66	60
57	95	78	71	62
100	69	62	72	73
55	32	83	78	80

a) Determine the mean mark for the class and the standard deviation.

b) Determine the percentage of learners whose marks lie within one standard deviation of the mean.

If you are not given a table to fill out as in the previous example, use your calculator to determine the mean and standard deviation.

a) The table below shows how to calculate the standard deviation on a common scientific calculator. If you are using a different model, follow the same table but press the buttons that correspond to the word next to the number shown or consult your calculator manual for the correct sequence of numbers.

Step	Button to press/method	Calculator display
1	**Mode**	**A list of various modes**
2	**2 (STAT)**	**A list of options in STAT mode**
3	**1 (1–VAR)**	**A table to input values**
4	Input each data value one at a time, pressing the = button after each entry.	
5	Once all the data is entered press **AC**	**0**
6	**Shift 1 (STAT)**	**A list of options to choose**
7	**4 (Var)**	1: n 2: $\overline{x}$ 3: σx 4: sx
8	Pressing 1 will display the number of data entries. Pressing 2 will display the mean and pressing 3 will display the standard deviation. (option 4 will not be used)	
9	**2 ($\overline{x}$) and then =**	**67,55**
10	To determine the standard deviation repeat steps 6,7 and 8 pressing 3 instead of 2	
11	**3 (σx) and then =**	**15,75**

$\therefore$ The mean is 67,55 and the standard deviation is 15,75

b) One standard deviation lies between $\overline{x}-15{,}75$ and $\overline{x}+15{,}75$.

$\therefore$ One standard deviation lies between 51,8 and 83,3. By counting the number of marks that lie within this interval it can be seen that there are 16 learners within one standard deviation of the mean.

$\therefore \frac{16}{20}$ learners lie within one standard deviation of the mean.

$\therefore$ 80% of learners lie within one standard deviation.

STANDARD DEVIATION WITH GROUPED DATA

In most cases, calculating the standard deviation of grouped data will be performed using your calculator. In order to determine the standard deviation and mean of grouped data you have to add a frequency column to your calculator.

To add a frequency column follow the steps below:

1. Press setup (press **shift** and then **mode**)
2. Using the middle arrow keys scroll down to and press the number corresponding to STAT.
3. Press **1** (on).

You can now follow the same steps on the previous page except you will now have a frequency column in which you can enter the frequency of each data point.

You can leave your frequency column on for all your calculations.

Example: **For the grouped data shown below, determine the mean and standard deviation.**

Data point	Frequency
1	8
2	22
3	5
4	8
5	15
6	25
7	10

Using your calculator follow the steps on the previous page, entering the frequency of each point into the frequency column.

The mean of the above data set is: 4,24

The standard deviation of the above data set is: 1,98

Important notes on mean and standard deviation:

- **Adding or subtracting the same value to or from each point will not affect or change the value of the standard deviation of a data set.**
- **Adding or subtracting the same value to or from each point will increase or decrease the value of the mean by the amount added or subtracted.**

SYMMETRICAL AND SKEWED DATA

It is important for you to be able to identify symmetrical and skewed data. If there are items with very large or very small values compared to the rest of the data set, they will cause the data set to be skewed.

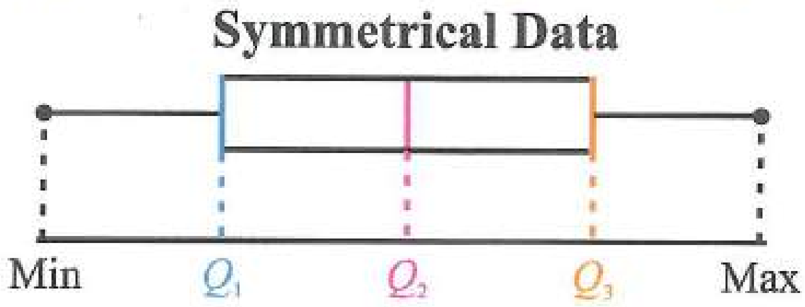
There are three general rules that can be used to test if a data set is skewed:

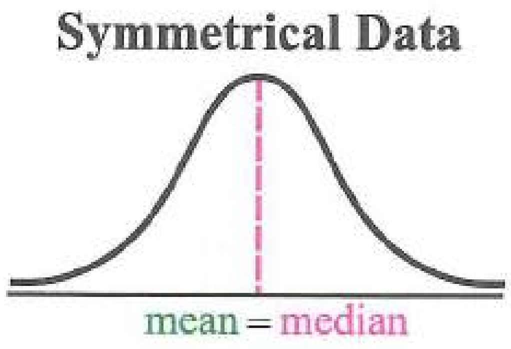
- If the **mean is the same as the median** the data is **normally distributed**.
- If the **mean is larger than the median** the data is **skewed to the right**.
- If the **mean is smaller than the median** the data is **skewed to the left**.

If the data is normally distributed it means that the data is symmetrical.

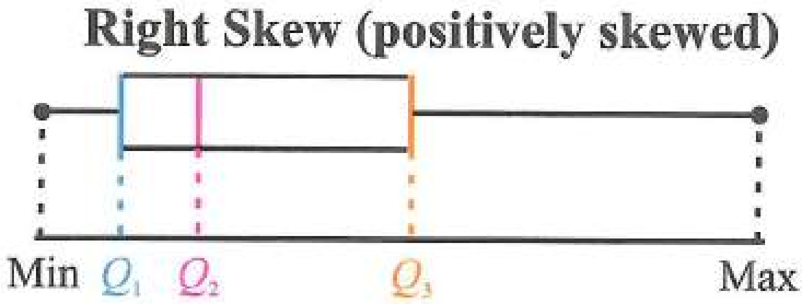
The diagrams below show symmetrical and skewed data. If the data has a tail from the median to the endpoint it is said to be skewed in the direction of the tail:

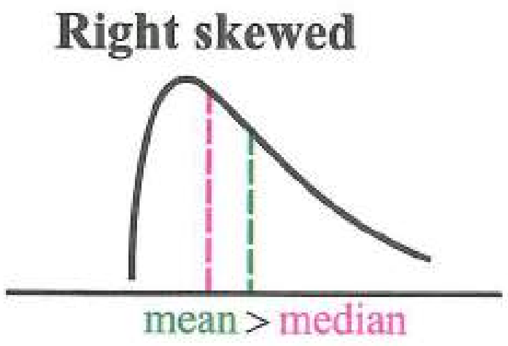
Symmetrical Data (normally distributed):


Symmetrical Data

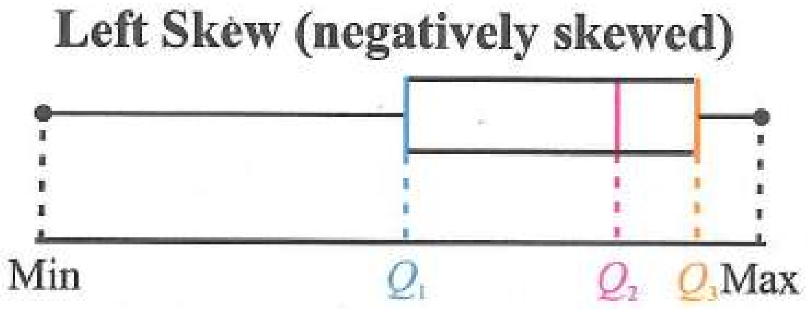

Symmetrical Data

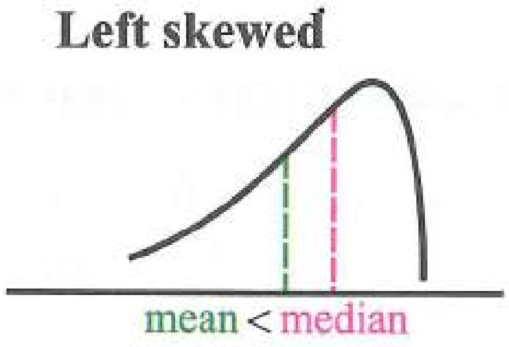
Right skew (positively skewed):


Right Skew (positively skewed)


Right skewed

Left skew (negatively skewed):


Left Skew (negatively skewed)


Left skewed

If a data set is normally distributed:

- **Approximately 68% of the data falls within 1 standard deviation of the mean.**
- **Approximately 95% of the data falls within 2 standard deviations of the mean.**

IDENTIFICATION OF OUTLIERS

An outlier is a data value which is much larger or smaller than the rest of the values in the data set. This value will not follow the general trend of the data.

A value is defined to be an outlier if it fulfils one of the conditions below:

- **The value is less than $Q_1 - (1{,}5 \times \text{IQR})$**
- **The value is greater than $Q_3 + (1{,}5 \times \text{IQR})$**

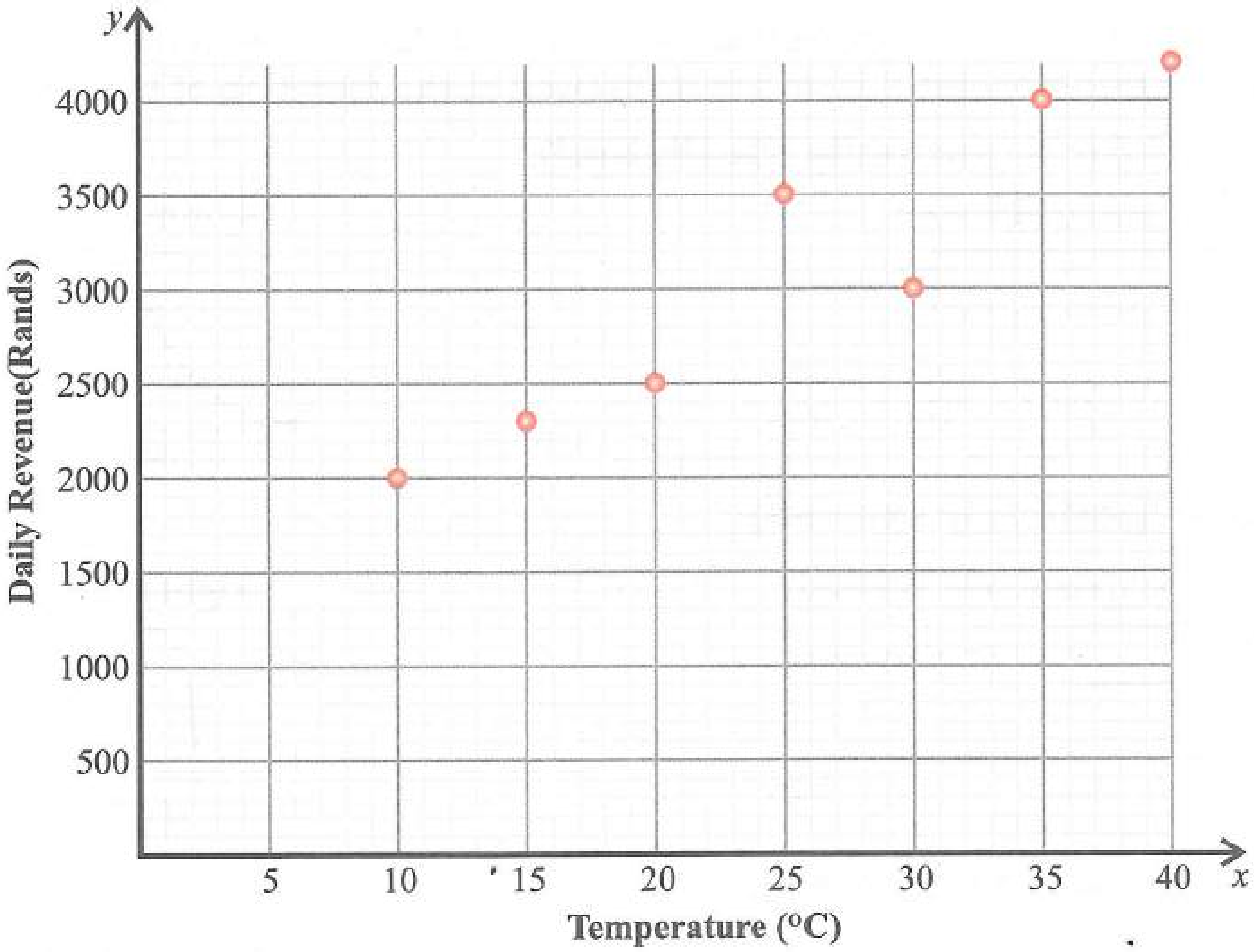
Complete Exercise 1 on pages 289 and 290.

B. SCATTER PLOTS

A scatter plot is a plot of bivariate data (data that has two variables) which shows a relationship between the two sets of data.

Example: **A local ice-cream shop keeps track of its daily revenue versus the temperature on each day. Its records for 7 days are shown below. Draw a scatter plot to represent the data below.**

Temp (°C)	10	15	20	25	30	35	40
Sales (R)	2000	2300	2500	3500	3000	4000	4200

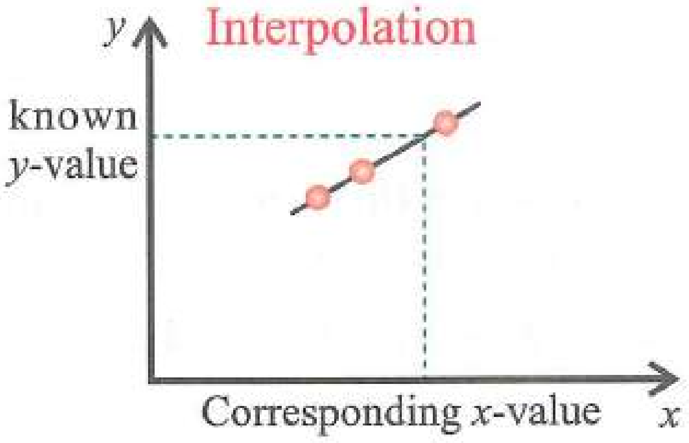


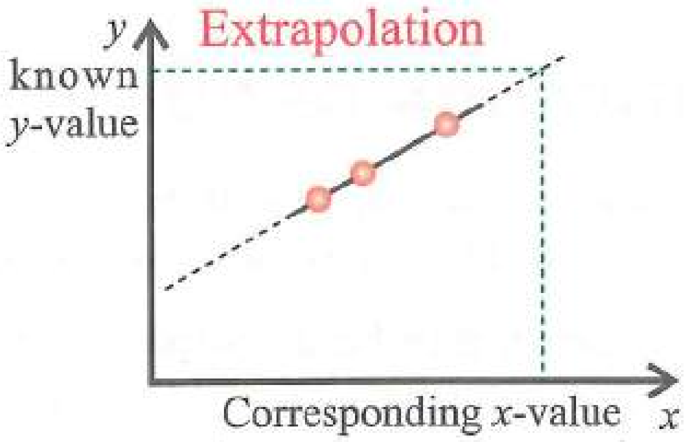
REGRESSION LINES (LINES OF BEST FIT)

A regression line (line of best fit) is used to show the general trend which a set of data follows. If a value of one of the variables not in the data set is found, say *x*, the corresponding value (*y*), can be predicted by reading off the line of best fit or from the equation of the line of best fit. This is known as interpolation and extrapolation.

Interpolation is the term used to predict a value which lies in the domain and range of the given data set.

Extrapolation is the term used to predict a value which lies outside the domain and range of the given data set.





Never join the points on a scatter plot. A line of best fit (regression line) is used on a scatter plot.

LEAST SQUARES REGRESSION LINE

In Grade 12 we will concentrate on the least squares regression line. This is a straight line in the form $y = mx + c$. This line will always pass through the mean point of the data $(\bar{x};\bar{y})$. The mean point is found by determining the mean of the x-values ($\bar{x}$) and the mean of the y-values ($\bar{y}$).

USING A CALCULATOR TO DETERMINE REGRESSION LINES AND THE MEAN POINT

Illustrative Example: **For the example on the previous page, determine the equation of the line of best fit and the mean point of the data.**

The table below shows how to determine the equation for the least squares of regression line and the mean point on a common scientific calculator. If you are using a different model, follow the same table but press the buttons that correspond to the word next to the number shown or consult your calculator manual. It is important to note that the line $y = mx + c$ is represented as $y = A + Bx$. This means B is the gradient and A is the y-intercept.

Step	Button to press/method	Calculator display
1	**Mode**	**A list of various modes**
2	2 (STAT)	**A list of options in STAT mode**
3	2 (A+BX)	**A table to input x and y-values**
4	Input each data value for x (Temp) and y (Sales) one at a time, pressing the = button after each entry.	
5	Once all the data is entered press AC	**0**
6	**Shift 1 (STAT)**	**A list of options to choose**
7	**5 (Reg)**	1: A 2: B 3: r 4: $\hat{x}$ 5: $\hat{y}$
8	Pressing 1 will display the value of A. Pressing 2 will display the value of B. Pressing 3 will display correlation coefficient. The correlation coefficient will be discussed in the next section.	
9	**1 (A) and then =**	**1 196,428571**
10	To determine the value of B repeat steps 6,7 and 8 pressing 2(B) instead of 1	
11	**1 (B) and then =**	**75**
12	To determine the mean $(\bar{x};\bar{y})$ point of the data repeat step 6.	
13	**4 (Var)**	1: n 2: $\bar{x}$ 3: δx 4: sx 5: $\bar{y}$ 6: δy 7: sy
14	Pressing 2 will display the x-value of the mean point (25) and the pressing 5 will display the y-value of the mean point (3071,43).	

$\therefore$ The equation of the line of best fit is: $y = 75x + 1196{,}43$ and

the mean point is $(25; 3071{,}43)$

Example: The Table below shows the results for five Grade 12 learners in Mathematics and Science.

Science (%)	64	88	52	46	72
Mathematics (%)	54	80	46	48	64

a) Draw a scatter plot of the above data.

b) Determine the equation of the line of best fit for the data.

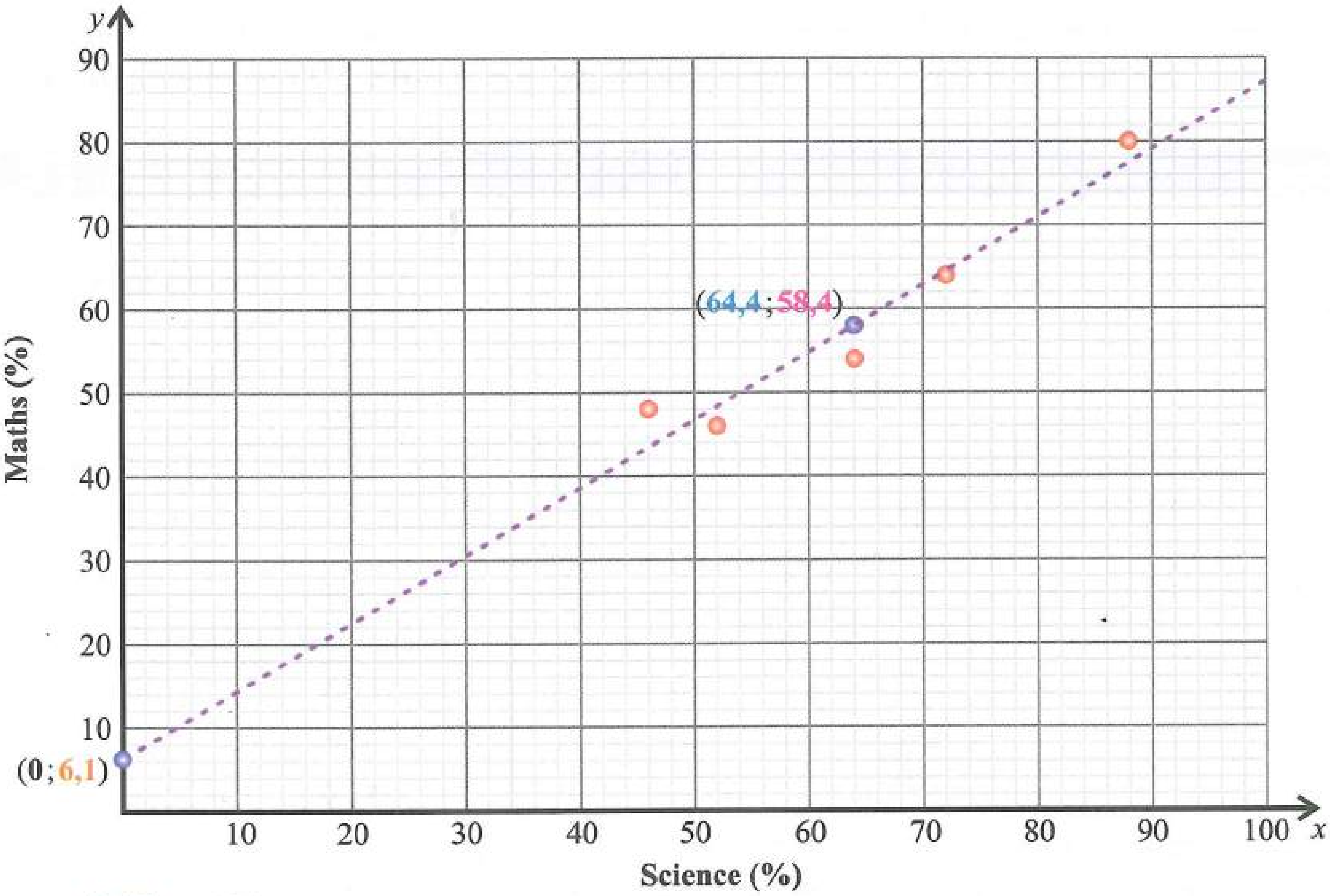
c) Determine the mean point of the data.

d) Draw the line of best fit on the scatter plot.

e) If a learner obtained 38% in Mathematics, predict the mark that they will get for Science,

f) If a learner obtains 70% for Science, predict the mark that they will get for Mathematics.

a)



b) $y = 0{,}81x + 6{,}1$

c) $(64{,}4; 58{,}4)$

d) The **purple** dotted line on the above plot represents the line of best fit. The easiest way to plot the line of best fit is to draw a line through the y-intercept and the mean point as shown on the plot above.

e) Using the equation for the line of best fit substitute 38 in for y and determine the corresponding x-value:

$38 = 0{,}81x + 6{,}1$

$\therefore x = 39{,}38$

$\therefore$ A learner who obtains 38% in Mathematics would get 39,38% in Science.

f) $y = 0{,}81(70) + 6{,}1 = 62{,}8$

$\therefore$ A learner who obtains 70% in Science would get 62,8% in Mathematics.

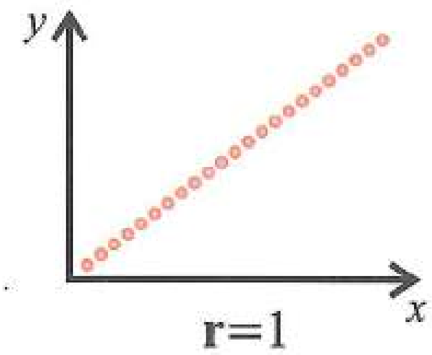
Complete Exercise 2 on page 291.

THE CORRELATION COEFFICIENT

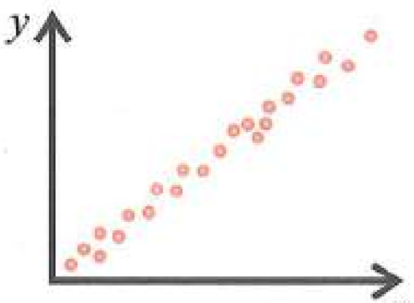
The correlation coefficient tells us about the strength of the relationship between the variables. In other words, it tells us tells us how well the data 'fits' the line of best fit. The closer the points lie to the line of best fit the stronger the relationship and correlation. The further the points lie from the line of best fit the weaker the correlation. If the line of best fit has a positive gradient, it is said to have a positive association and hence the correlation coefficient will be positive. If the line of best fit has a negative gradient, it is said to have a negative association and hence the correlation coefficient will be negative.

It is important to note that the correlation coefficient (r) always lies between –1 and 1. The closer the value of r lies to –1 or 1 the stronger the relationship between the variables. The closer the value of r is to 0 the weaker the relationship between the variables and the weaker the correlation.
The table and graphs below summarise the correlation for different values of r.

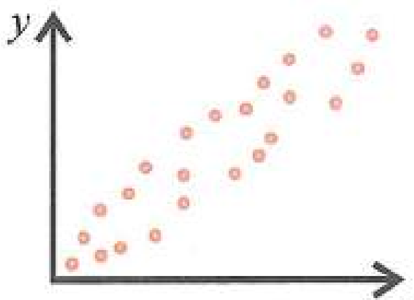
r ≈	Correlation
1	Perfect positive association
0,9	Strong positive association
0,5	Moderate positive association
0,2	Weak positive association
0	No correlation
– 0,2	Weak negative association
– 0,5	Moderate negative association
– 0,9	Strong negative association
–1	Perfect negative association



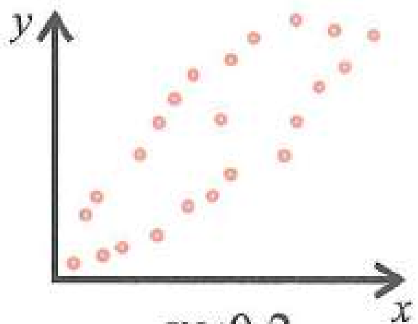
r=1



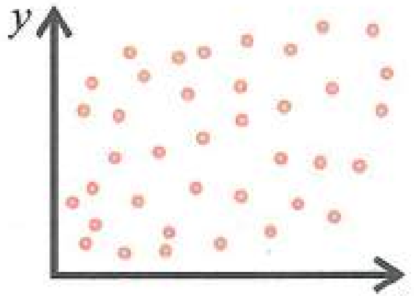
r≈0,9



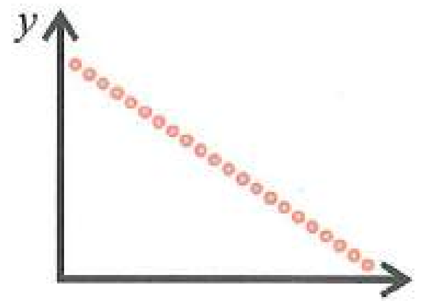
r≈0,5



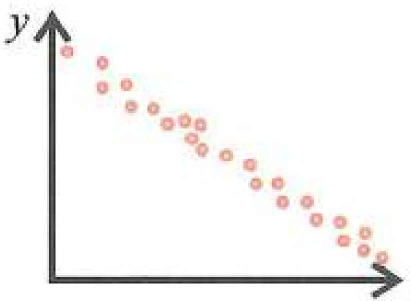
r≈0,2



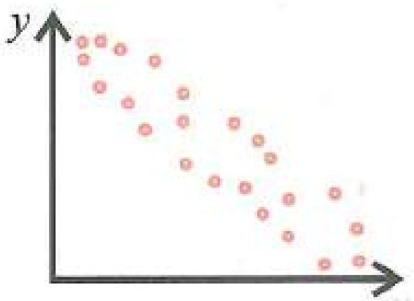
r≈0



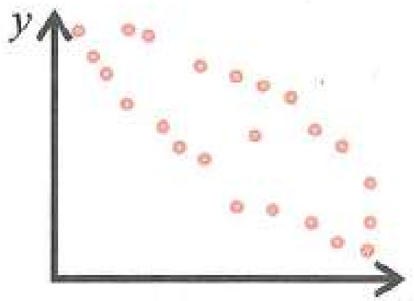
r=–1



r≈–0,9



r≈–0,5



r≈–0,2

Example 1: Given the data in the table below:

x	10	20	30	40	50	60	70	80	90	100
y	2	4	3	5	5	6	8	7	9	10

a) **Draw a scatter plot for the above data.**

b) **Determine the equation for the line of best fit for the data.**

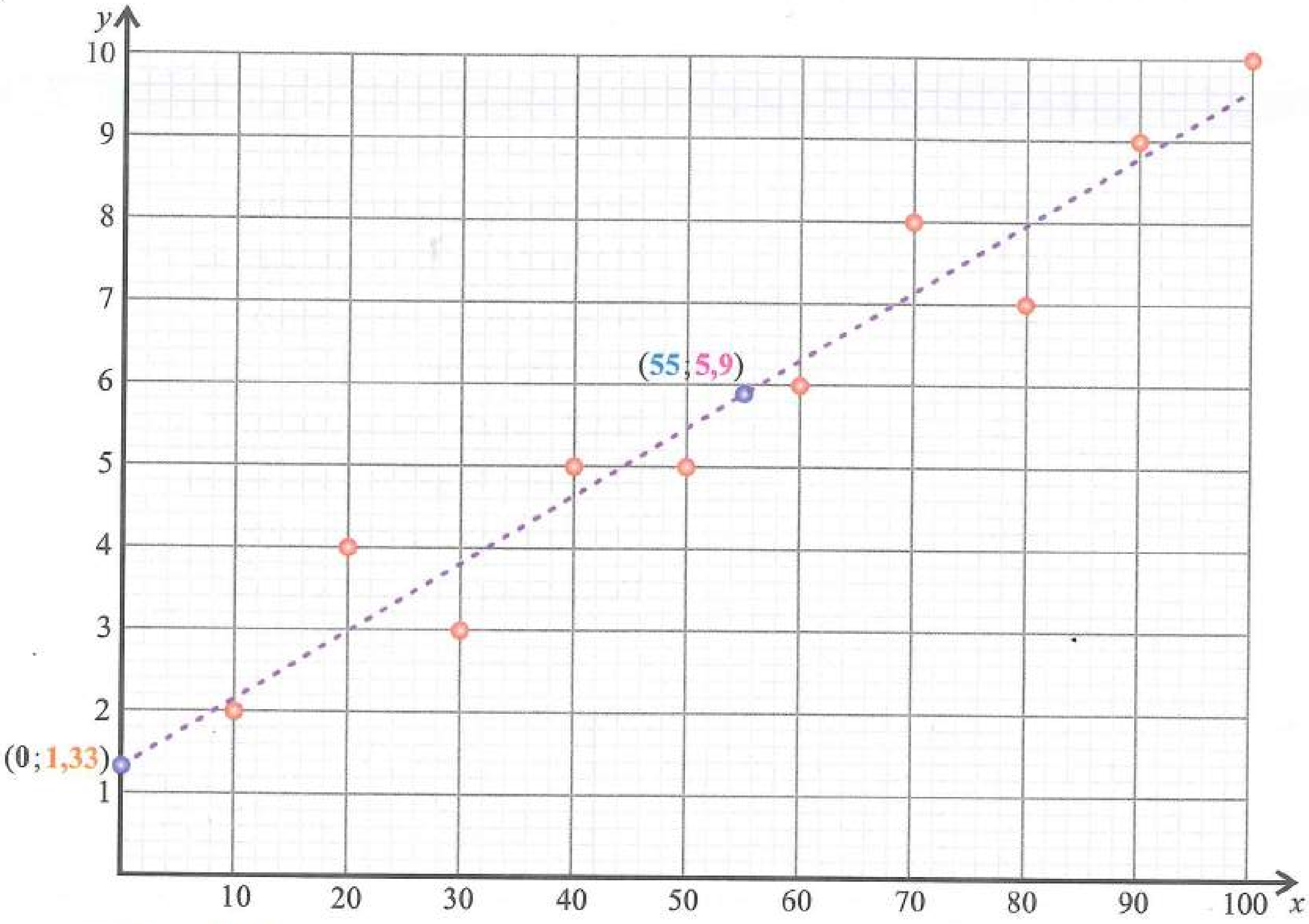
c) **Determine the mean point of the data.**

d) **Plot the line of best fit on your scatter plot.**

e) **Determine the correlation coefficient of the data.**

f) **A grade 12 learner analyses the data and predicts that the corresponding y-value for an x-value of 35 is 4,13. Is the learner correct and if so is the value reliable?**

a)



b) $y = 0{,}08x + 1{,}33$

c) $(55;5{,}9)$

d) The **purple** dotted line on the above plot represents the line of best fit. The easiest way to plot the line of best fit is to draw a line through the y-intercept and the mean point as shown on the plot above.

e) Using a calculator and the steps shown on page 284: $r = 0{,}966$.

f) $y = 0{,}08(35) + 1{,}33 = 4{,}13$

Therefore, the learner is correct. The correlation coefficient suggests a strong positive association between the variables. Therefore, predictions made using the line of best fit are reliable and generally accurate.

Example 2: **The following table summarises the number of revolutions per minute (RPM), x and the corresponding power output y (horse power of an engine:**

x	400	500	600	700	750
y	580	1030	1420	1880	2100

a) Determine the least squares regression line in the form $y = mx + c$.
b) Use the line to estimate the power output when the engine runs at 800 RPM.
c) Predict the speed of the engine when it has an output of 1200 horse power.
d) Determine the coefficient of correlation for the data. Hence, comment on the reliability of using engine speed to determine the power output of the engine.

a) $y = 4,32x - 1145,79$

b) $y = 4,32(800) - 1145,79$

$\therefore y = 2310,21$ horse power

c) $y = 4,32x - 1145,79$

$\therefore 1200 = 4,32x - 1145,79$

$\therefore x = 543$ RPM

d) $\text{r} = 0,9996821357$

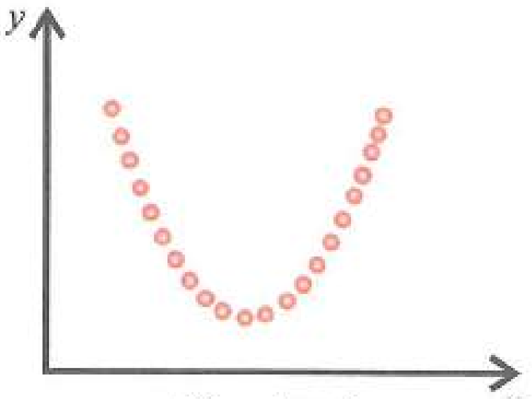
≈ 1

The correlation coefficient suggests an almost perfect positive association between the variables. This means there is a significant relationship between the speed of the engine and power output. Therefore, predictions made using the line of best fit would be reliable.

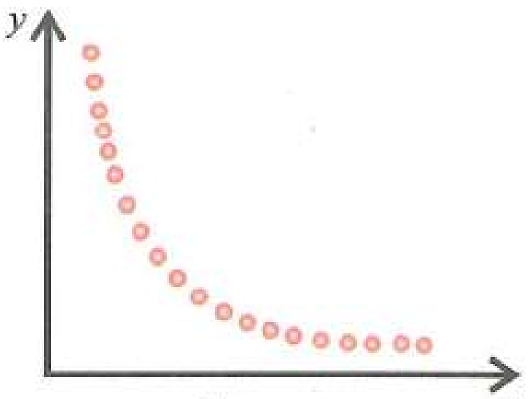
Complete the Mixed Exercise on page 292.

NOTES ON REGRESSION LINES

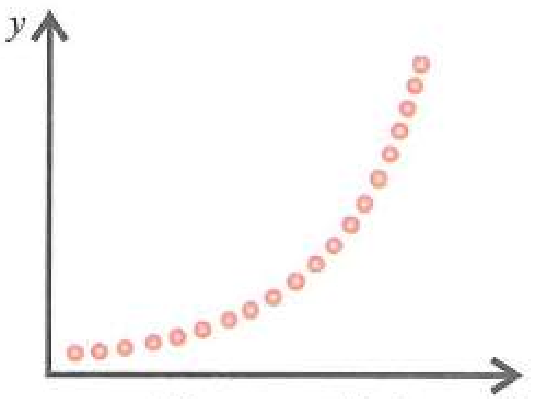
It is important to understand that you can fit a linear regression line to any data set. However, a linear line of best fit is not always the ideal line to use. The graphs below show different associations between bivariate data.



Quadratic association



Inverse association



Exponential association

Grade 11 Revision

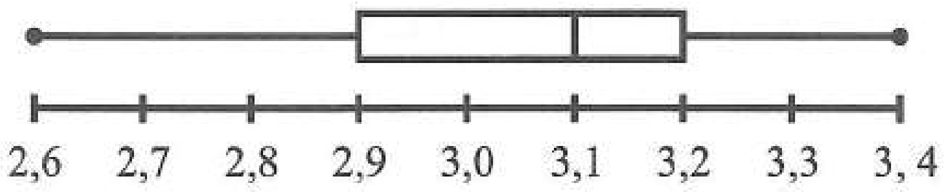
Exercise 1: (answers on page 370)

1) **A group of 50 Grade 12 learners were surveyed on the amount of money they spend on entertainment per month. The results are tabulated below.**

Expenditure (Rands)	Number of Matrics	Cumulative Frequency
$0 < x \leq 50$	2	d
$50 < x \leq 100$	a	5
$100 < x \leq 150$	8	13
$150 < x \leq 200$	b	26
$200 < x \leq 250$	c	e
$250 < x \leq 300$	8	f

- **a)** Determine the values of a, b, c, d, e and f.
- **b)** Plot an ogive for the data in the table.
- **c)** Using your ogive, determine the number of learners who spent more than R190 on entertainment in a month.

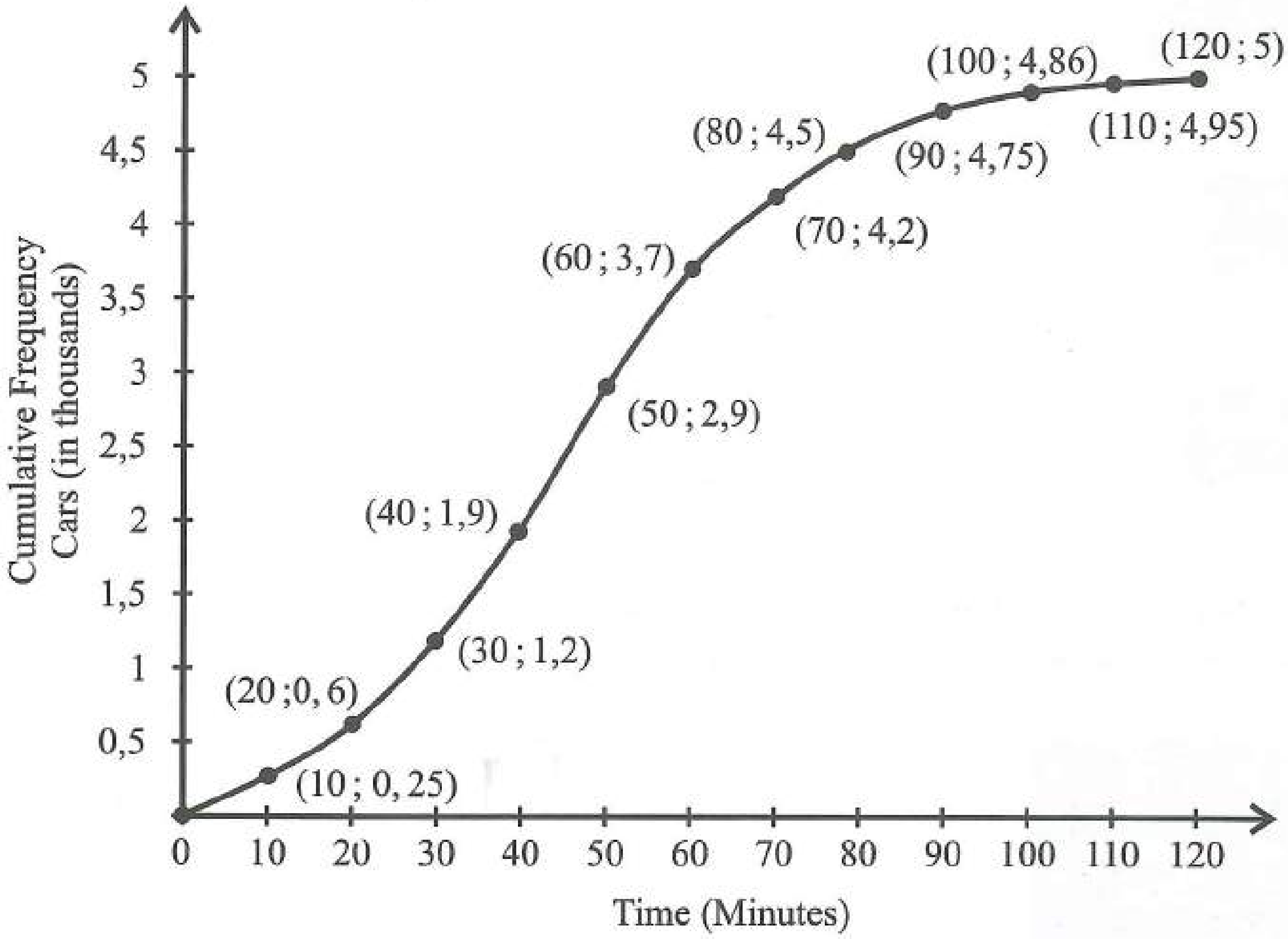
2) **The following box and whisker plot represents the mass of 60 babies born at the Johannesburg Hospital. Solve for w, x, y and z in the frequency table using the box and whisker diagram.**



Mass interval (Kg)	Cumulative Frequency
$2,6 \leq x < 2,7$	w
$2,7 \leq x < 2,8$	6
$2,8 \leq x < 2,9$	7
$2,9 \leq x < 3,0$	7
$3,0 \leq x < 3,1$	x
$3,1 \leq x < 3,2$	y
$3,2 \leq x < 3,3$	z
$3,3 \leq x < 3,4$	2

3) **The ogive below represents the number of cars passing through a toll gate over a 2 hour peak period during a long weekend. If the counting started at 13:00, determine:**

a) The number of cars that passed through the toll gate in the first hour.
b) The number of cars that passed through the toll gate between 13:30 and 13:40.
c) The number of cars that passed through the toll gate between 14:30 and 15:00
d) The median time in the 2 hour period?



4) **The number of points that each player in a basket ball team scored in a season is shown below:**

21	32	37	38	42	51	55	62	68	74

a) Determine the mean number of points scored per player.
b) Using an appropriate table, determine the standard deviation of points scored and verify your answer using a calculator.
c) If the data is to be normally distributed what percentage of players have to have scores that fall within 1 standard deviation of the mean? Using this answer comment on the performance of the team.

5) **The following data values are given in the table below:**

40	83	32	54	*d*

a) Determine the range of values of *d* so that *d* is the median of the data set.
b) What is the value of *d* if the mean for the data set is 60?

6) **Residents are petitioning their local council to erect speed humps in their area due to speeding cars. The local council conducts a study along a busy road in the residential area where the speed limit is 60km/h. The results of the study are recorded in the table below:**

a) Determine the approximate mean speed.
b) Determine the approximate median speed.
c) Determine the approximate standard deviation.
d) Should the council erect speed humps in the area? Justify your answer.

Speed (km/h)	Number of cars
$30 < x \leq 40$	40
$40 < x \leq 50$	54
$50 < x \leq 60$	50
$60 < x \leq 70$	108
$70 < x \leq 80$	42
$80 < x \leq 90$	30
$90 < x \leq 100$	16
$100 < x \leq 110$	8

Exercise 2: (answers on page 371)

1) **For each of the data sets below, draw a scatter plot and determine the equation of the line of best fit for the data as well as the mean point. Plot these on your graph.**

a)

x	1	3	4	5	6	7	8
y	5	11	15	14	17	23	22

b)

x	1	2	4	5	6	7
y	3	8	10	15	13	16

c)

x	2	4	5	6	8	11	14	15	17
y	1	3	6	9	8	11	13	12	15

2) **Lerato and Jane are Grade 12 learners who both take IT as a subject. In order to earn extra money they decide to write an Android App. The table below shows the number of sales, in thousands, which similar applications have made at various selling prices.**

Price (R)	1	4	5	7	10	11	15	18
Number of sales (in thousands)	20	16	12	11	8	4	2	1

a) Determine the equation of the line of best fit for the data.
b) Determine the mean point
c) If they set the selling price to be R8,24, estimate the number of sales they will make. Give your answer correct to the nearest thousand.
d) At what approximate price will no sales be made?
e) Draw a scatter plot for the above data. Show the line of best fit and mean point on your graph.

3) **A turkey is perfectly cooked when the temperature of the meat in the thigh is 82°C. The table below shows the temperature of the thigh at various times for a 3kg turkey in an oven at 180°C.**

Time (minutes)	10	40	80	120	160
Temperature (°C)	30	35	43	60	70

a) Determine the equation of the line of best fit for the data.
b) Determine the mean point
c) Predict the temperature of the turkey when it was placed in the oven.
d) Predict how long the turkey will take to be perfectly cooked.
e) Draw a scatter plot for the above data. Show the line of best fit and mean point on your graph.

Mixed Exercise: (answers on page 372)

1) **The table below shows the amount of money spent on advertising and the income of the company, in thousands of Rand, over a 6 month period.**

Month	**1**	**2**	**3**	**4**	**5**	**6**
Advertising	3	4,5	1	5	7	2,4
Income	32	56	18	48	60	25

- **a)** Draw a scatter plot for the data.
- **b)** Determine the mean point of the data and show it on your scatter plot.
- **c)** Determine the equation of the least squares regression line and show it on your scatter plot.
- **d)** Determine the correlation coefficient of the data.
- **e)** The owner of the company states that there is a strong relationship between the advertising expenditure and company's income. Explain this statement with reference to the correlation coefficient.
- **f)** Predict the company's income in a month where R3 500 is spent on advertising.

2) **An accounting firm wishes to assess their training program. They give 10 of their employees a test out of 70 and record their results against the number of hours they spent in training. The results are shown below:**

Employee	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Hours	16	36	20	38	40	30	35	22	40	24
Mark	45	69	44	56	60	48	70	60	63	38

- **a)** Draw a scatter plot for the data.
- **b)** Determine the mean point of the data and show it on your scatter plot.
- **c)** Determine the equation of the least squares regression line and show it on your scatter plot.
- **d)** Determine the correlation coefficient of the data.
- **e)** Predict the score of an employee who has received 22 hours of training.
- **f)** Is your prediction reliable? What can you say about the company's training program?

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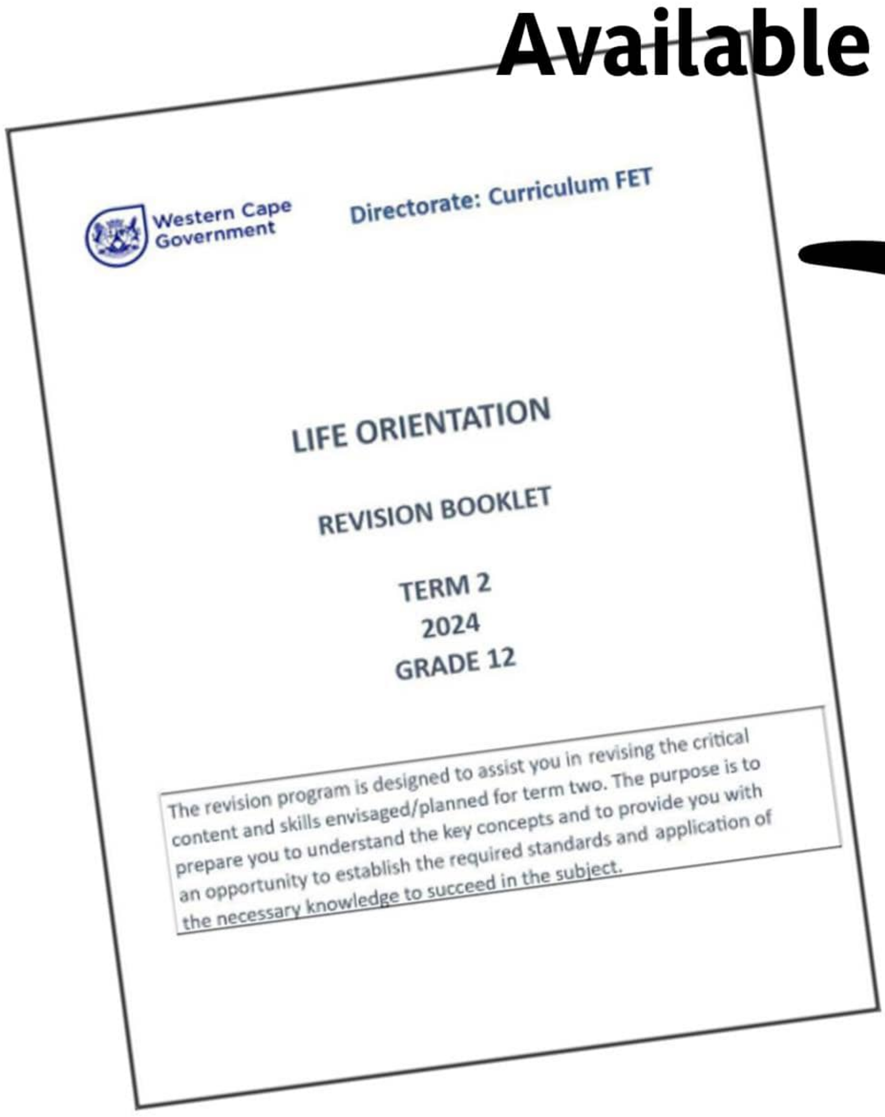

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2024
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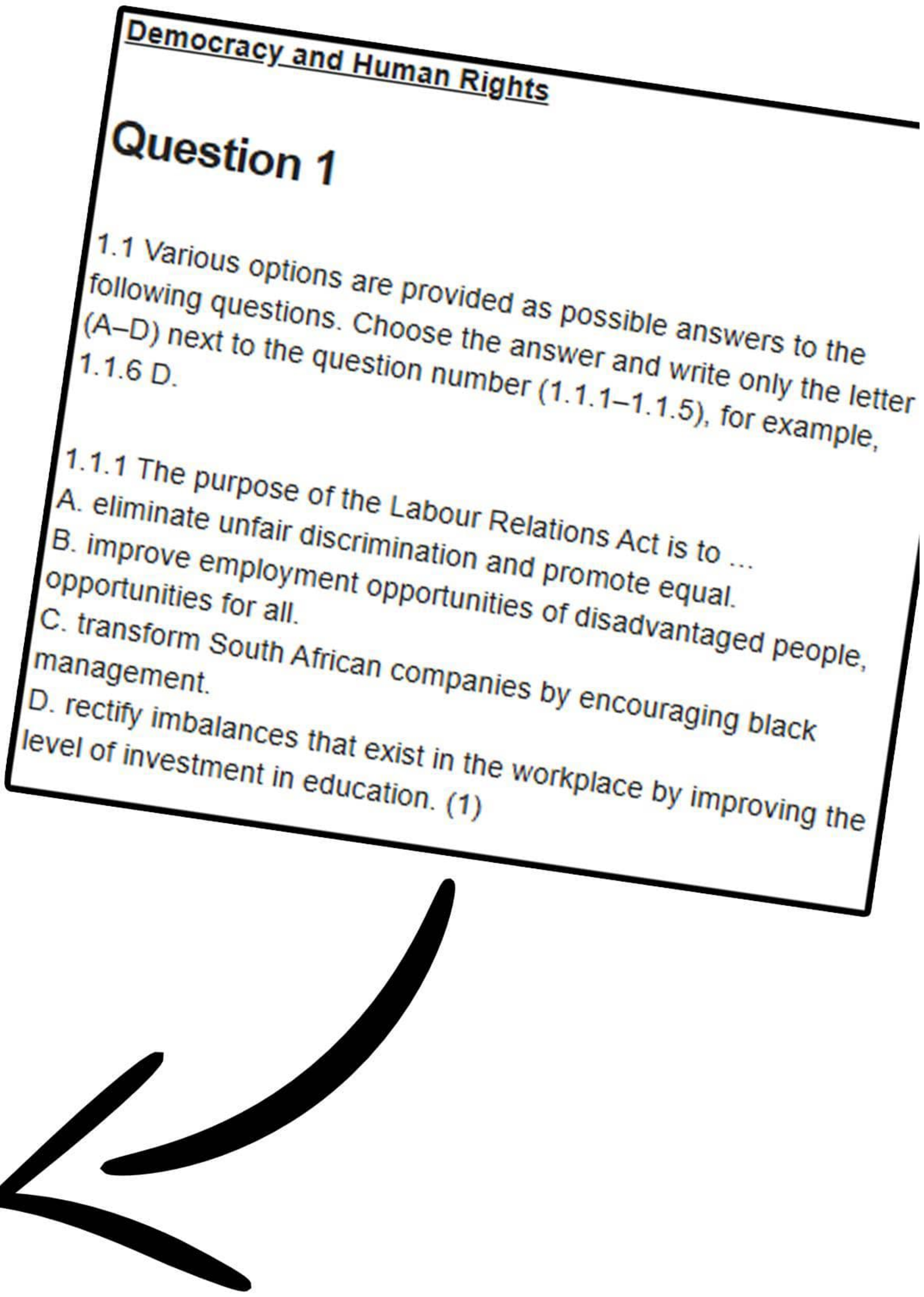
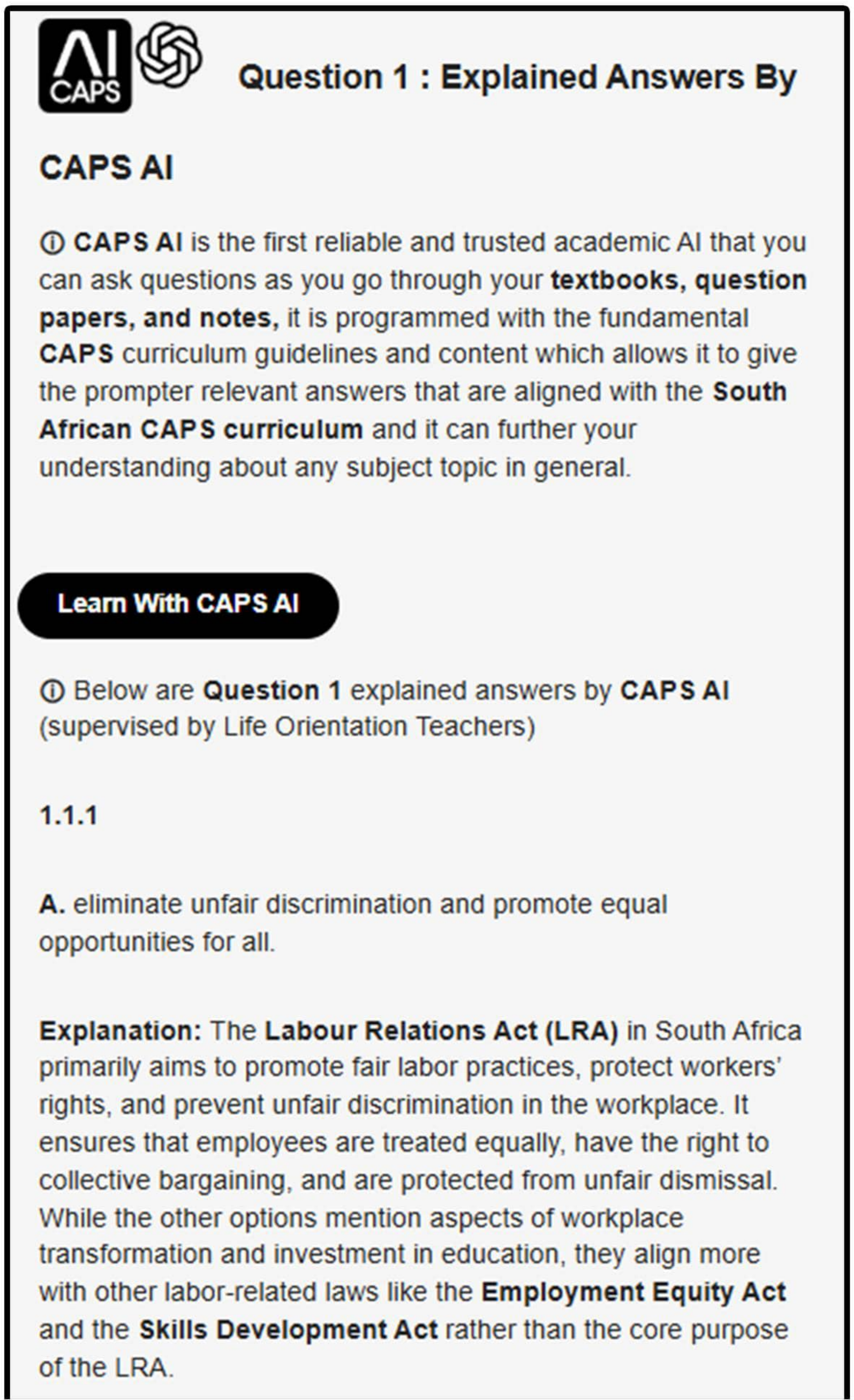


Democracy and Human Rights

Question 1

1.1 Various options are provided as possible answers to the following questions. Choose the answer and write only the letter (A–D) next to the question number (1.1.1–1.1.5), for example, 1.1.6 D.

1.1.1 The purpose of the Labour Relations Act is to …
A. eliminate unfair discrimination and promote equal opportunities for all.
B. improve employment opportunities of disadvantaged people.
C. transform South African companies by encouraging black management.
D. rectify imbalances that exist in the workplace by improving the level of investment in education. (1)



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1.1.1

A. eliminate unfair discrimination and promote equal opportunities for all.

Explanation: The **Labour Relations Act (LRA)** in South Africa primarily aims to promote fair labor practices, protect workers' rights, and prevent unfair discrimination in the workplace. It ensures that employees are treated equally, have the right to collective bargaining, and are protected from unfair dismissal. While the other options mention aspects of workplace transformation and investment in education, they align more with other labor-related laws like the **Employment Equity Act** and the **Skills Development Act** rather than the core purpose of the LRA.

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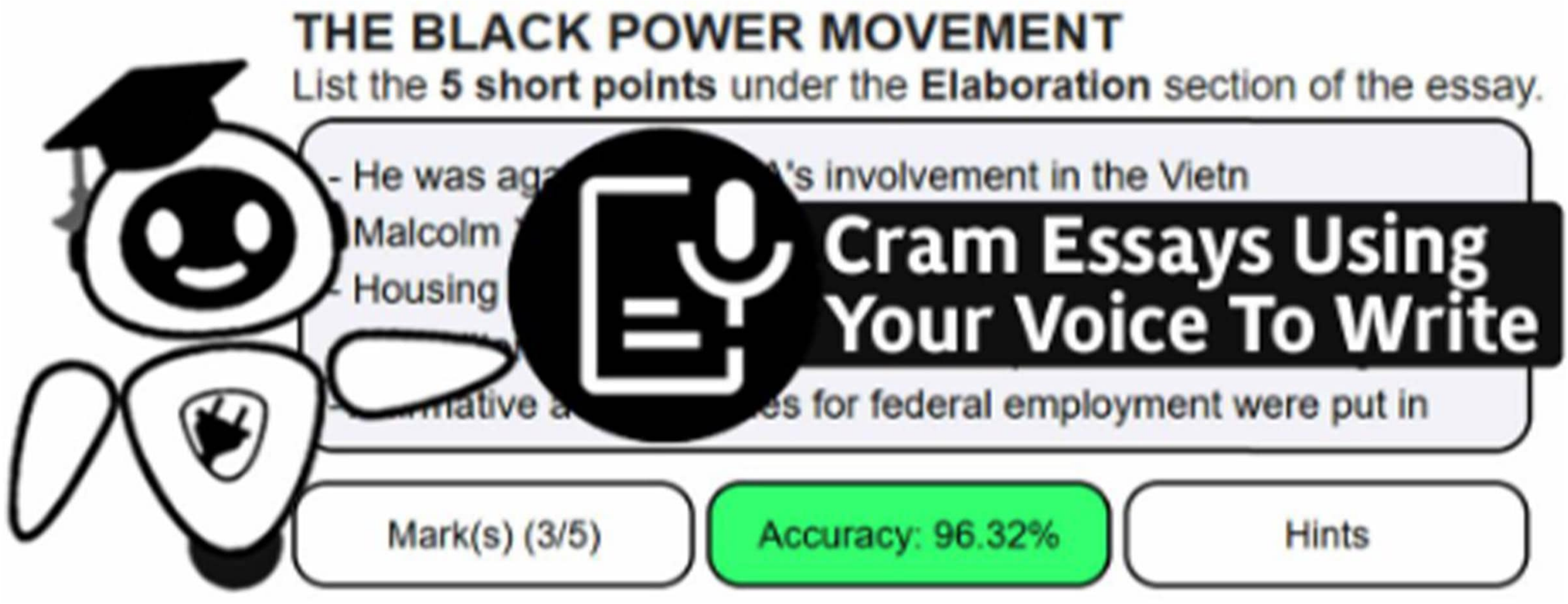

THE BLACK POWER MOVEMENT

List the **5 short points** under the **Elaboration** section of the essay.

- He was ag… 's involvement in the Vietn
- Malcolm …
- Housing …
- …
- …firmative a… es for federal employment were put in

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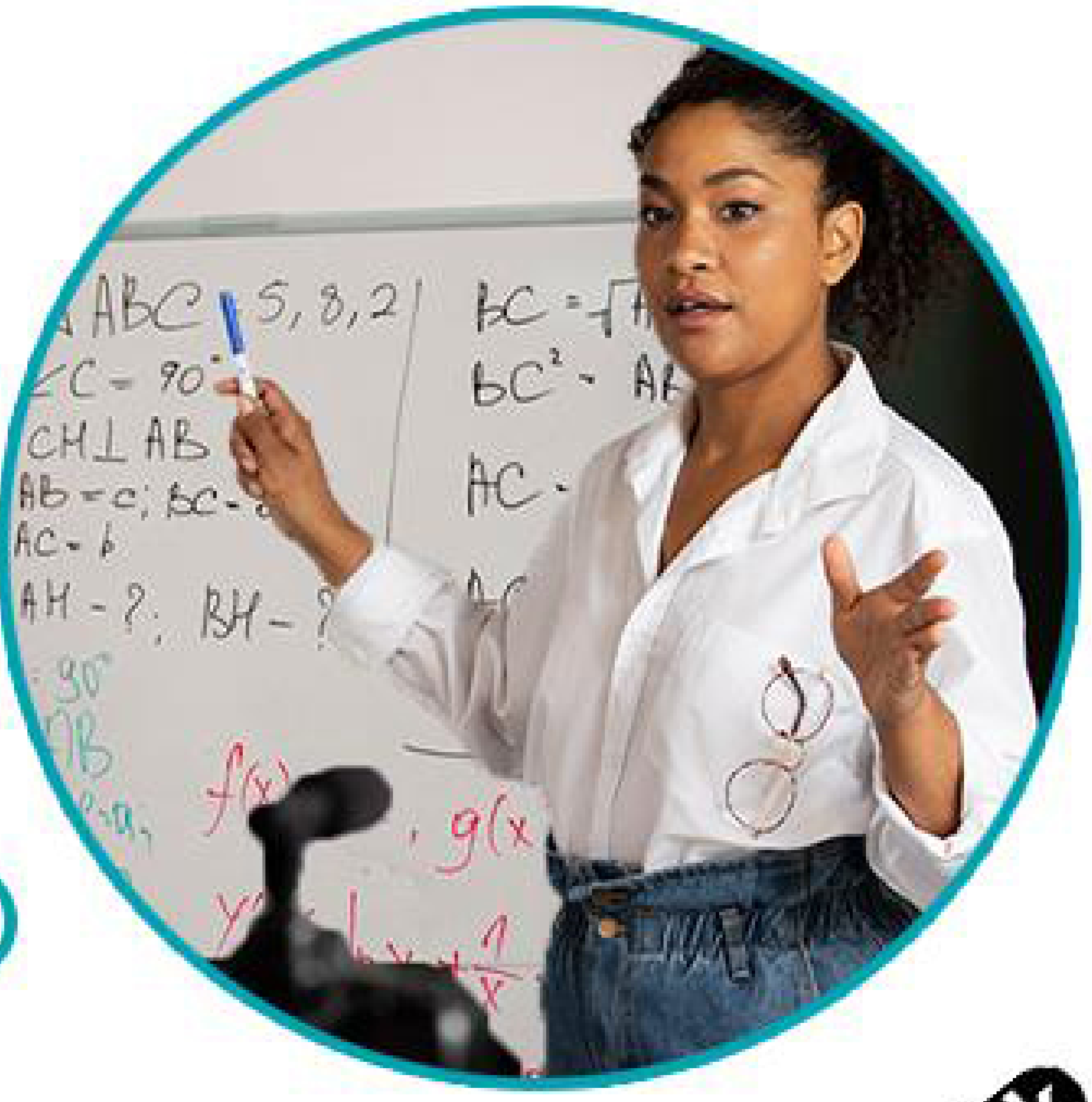

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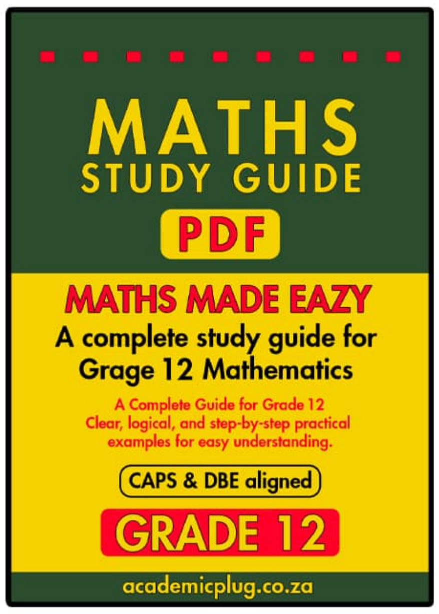
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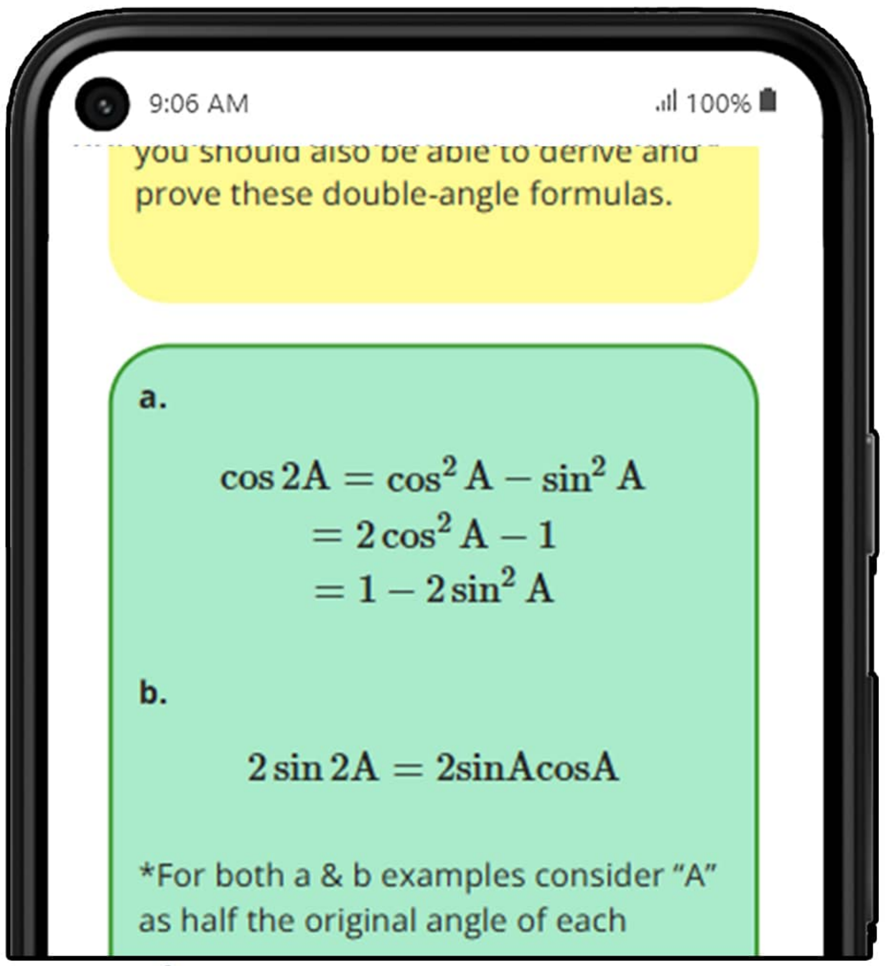
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you should also be able to derive and prove these double-angle formulas.

a.

$$\cos 2A = \cos^2 A - \sin^2 A$$
$$= 2\cos^2 A - 1$$
$$= 1 - 2\sin^2 A$$

b.

$$2\sin 2A = 2\sin A\cos A$$

*For both a & b examples consider "A" as half the original angle of each

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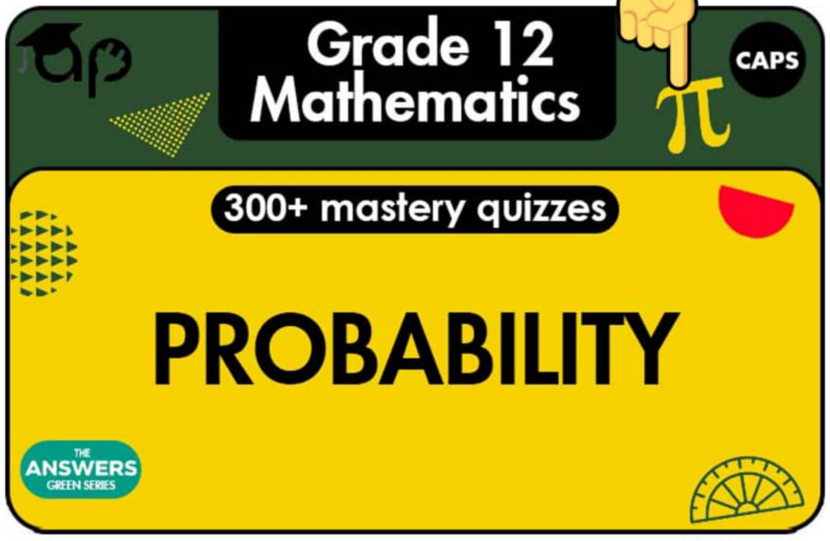
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CHAPTER 14
PROBABILITY

WHAT YOU WILL LEARN IN THIS CHAPTER

PAGE	CONTENT	CAPS	ASSESSMENTS YOU WILL BE ABLE TO:
294	**A. Grade 11 Revision**	6	• Use the fundamental counting principle to solve problems involving probability
294	• Notation		
294	• Mutually Exclusive Events		
294	• The Addition Rule		
294	• Complementary Events		
296	• Dependent and Independent Events		
296	• The Product Rule		
297	• Venn Diagrams		
300	• Tree Diagrams		
301	• Contingency Tables		
302	**B. The Fundamental Counting Principle**		
303	• Factorial Notation		
305	• Dealing with Repetition		
307	• Selections		

A. GRADE 11 REVISION

NOTATION

The probability of an event occurring will always lie between 0 and 1, where 0 is an impossible event and 1 is a certain event. It is essential that you are familiar with the notations below which are used in most probability questions:

- $P(A) \rightarrow$ The probability of event A occurring
- $P'(A) \rightarrow$ The probability of event A **not** occurring
  (this is also called the **complement** of A)
- $P(A \text{ or } B) = P(A \cup B) \rightarrow$ The probability of A **or** B occurring. $\cup$ is the symbol for **or**, it is also known as **union**.
- $P(A \text{ and } B) = P(A \cap B) \rightarrow$ The probability of A **and** B occurring. $\cap$ is the symbol for **and**, it is also known as **intersection**.
- $P(A|B) \rightarrow$ The probability of A occurring **given** that B has occurred.

MUTUALLY EXCLUSIVE EVENTS

If two events are mutually exclusive, it means that they cannot occur in the same trial. For example, if a trial is the tossing of a dice and event A is obtaining a 6 while event B is obtaining a 2, it is impossible to obtain events A and B together.

If a trial is picking a card and event A is selecting a queen and event B is selecting a heart, it is possible to obtain events A and B together. This is an example of events which are not mutually exclusive.

For mutually exclusive events : $P(A \text{ and } B) = 0$.

THE ADDITION RULE

The addition rule states: $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$.

If events A and B are mutually exclusive $P(A \text{ and } B) = 0$ and the addition rule simplifies to: $P(A \text{ or } B) = P(A) + P(B)$.

The addition rule can also be written as:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

COMPLEMENTARY EVENTS

If two events are complementary:

$$P(\text{not } A) = P(A') = 1 - P(A)$$

There are 2 conditions that have to be true for events to be complementary:

1) **The events have to be mutually exclusive**
2) $P(A) + P(B) = 1$

Example 1: If a fair dice is rolled what is the probability of getting a multiple of 3?

The only multiples of 3 in the sample space are 3 and 6.

$$\mathrm{P}(3 \text{ or } 6)=\mathrm{P}(3)+\mathrm{P}(6)-\mathrm{P}(3 \text{ and } 6)$$

$$=\frac{1}{6}+\frac{1}{6}-0 \rightarrow \text{mutually exclusive}, \therefore \mathrm{P}(3 \text{ and } 6)=0$$

$$=\frac{2}{6}=\frac{1}{3}=0,3333\ldots=33,33\%$$

Example 2: A card is selected at random from a fair pack of playing cards:

a) Determine the probability of selecting a jack.
b) Determine the probability of selecting a heart.
c) Determine the probability of selecting a jack or a heart.

a) $\mathrm{P}(\text{jack})=\frac{4}{52}$

b) $\mathrm{P}(\text{heart})=\frac{13}{52}$

c) $\mathrm{P}(\text{jack or heart})=\mathrm{P}(\text{jack})+\mathrm{P}(\text{heart})-\mathrm{P}(\text{jack and heart})$

$$=\frac{4}{52}+\frac{13}{52}-\frac{1}{52}$$

$$=\frac{4}{13}=0,3077=30,77\%$$

Example 3: Given sample space $s=\{6;7;8;9;10\}$ determine whether or not the following events are complementary:

a) Selecting an even number and a multiple of 5.
b) Selecting a multiple of 3 and a prime number.
c) Selecting an odd number and a multiple of 2.

a) The set of even numbers can be given by $\mathrm{A}=\{6;8;10\}$ and the set of multiples of 5 can be given by $\mathrm{B}=\{10\}$. As the events share a common element (10) they are not mutually exclusive. Therefore they are not complementary events.

b) The set of multiples of 3 can be given by $\mathrm{A}=\{6;9\}$ and the set of prime numbers can be given by $\mathrm{B}=\{7\}$. Therefore the events are mutually exclusive.

$\mathrm{P}(\mathrm{A})=\frac{2}{5}$ and $\mathrm{P}(\mathrm{B})=\frac{1}{5}$

$\therefore \mathrm{P}(\mathrm{A})+\mathrm{P}(\mathrm{B})=\frac{2}{5}+\frac{1}{5}=\frac{3}{5}\neq 1$

$\therefore$ the events are not complementary.

c) The set of odd numbers can be given by $\mathrm{A}=\{7;9\}$ and the set of multiples of 2 can be given by $\mathrm{B}=\{6;8;10\}$. Therefore the events are mutually exclusive.

$\mathrm{P}(\mathrm{A})=\frac{2}{5}$ and $\mathrm{P}(\mathrm{B})=\frac{3}{5}$

$\therefore \mathrm{P}(\mathrm{A})+\mathrm{P}(\mathrm{B})=\frac{2}{5}+\frac{3}{5}=\frac{5}{5}=1$

$\therefore$ the events are complementary.

DEPENDENT AND INDEPENDENT EVENTS

Two events are said to be independent if the outcome of one event does not affect the probability of another event occurring. If the outcome of one event changes the probability that another event occurs, the events are said to be dependent.

For example, if a card is selected from a deck of cards and then placed back in the deck and another card is drawn, the drawing of the first card will not affect the outcome of drawing the second card. These would be independent events.

If the first card was not placed back in the deck, drawing the first card would affect the outcome of drawing the second card. These would be dependent events.

In most cases, events that take place with replacement are independent events and those without replacement are dependent events.

THE PRODUCT RULE

The product rule for independent events:

$$\mathbf{P(A\ and\ B) = P(A) \times P(B)}$$

Example: **Given the following probabilities:**

$\mathbf{P(A) = 0{,}45\ ;\ P(B) = 0{,}3}$ **and** $\mathbf{P(A\ or\ B) = 0{,}615}$.

a) Determine whether or not events A and B are mutually exclusive.

b) Hence, determine $\mathbf{P(A\ and\ B)}$.

c) Are events A and B independent?

a) $P(A) + P(B) = 0{,}45 + 0{,}3 = 0{,}75$

$P(A\text{ or }B) = 0{,}615$

$\therefore P(A\text{ or }B) \neq P(A) + P(B)$

$\therefore$ The events are not mutually exclusive.

b) $P(A\text{ or }B) = P(A) + P(B) - P(A\text{ and }B)$

$\therefore 0{,}615 = 0{,}45 + 0{,}3 - P(A\text{ and }B)$

$\therefore P(A\text{ and }B) = 0{,}135$

c) $P(A) \times P(B) = 0{,}45 \times 0{,}3$

$= 0{,}135$

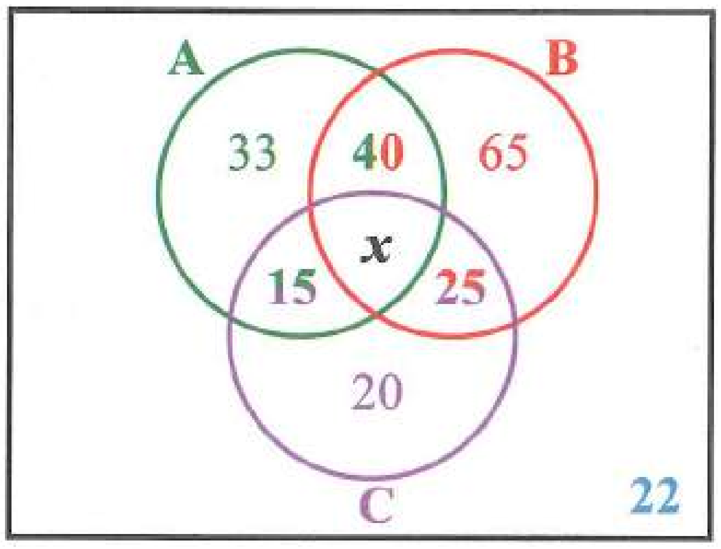
$\therefore P(A\text{ and }B) = P(A) \times P(B)$

$\therefore$ The events are independent.

VENN DIAGRAMS

Venn diagrams are a graphical way of representing a sample space and its events.

Example 1: Thabo polled 250 people to determine if they liked shampoo A, B or C. His results are shown in the Venn diagram below:



a) **How many people like all of the shampoos?**

b) **Determine the probability of randomly picking a person that likes shampoos A or B.**

c) **Determine the probability of randomly picking a person that likes shampoo C and doesn't like shampoo B.**

d) **Determine the probability of randomly picking a person that only likes shampoos A and B.**

e) **Determine the probability of randomly picking a person that doesn't like shampoos C or B.**

f) **What is the probability of randomly picking a person who doesn't like any of the shampoos?**

a) $x = 250 - (65 + 40 + 33 + 15 + 25 + 20 + 22)$ → don't forget the 22 who don't like any

$\therefore x = 30$

b) $P(A \overset{or}{\cup} B) = \dfrac{33 + 65 + 40 + 15 + 25 + 30}{250} = \dfrac{208}{250} = 0{,}832 = 83{,}2\%$

c) $P(C \overset{and}{\cap} B') = \dfrac{20}{250} + \dfrac{15}{250} = \dfrac{35}{250} = 0{,}14 = 14\%$

d) $P(A \overset{and}{\cap} B \text{ only}) = \dfrac{40}{250} = \dfrac{4}{25} = 0{,}16 = 16\%$

e) $P(C' \overset{or}{\cup} B') = \dfrac{33}{250} + \dfrac{22}{250} = \dfrac{55}{250} = \dfrac{11}{50} = 0{,}22 = 22\%$

f) $P(A' \overset{or}{\cup} B' \overset{or}{\cup} C') = \dfrac{22}{250} = \dfrac{11}{125} = 0{,}088 = 8.8\%$

Example 2: In a study 160 Grade 11 learners were asked if they take Biology, Accounting or Science. The results of the study are shown below:

- 80 people said they take Biology
- 70 people said they take Accounting
- 80 people said they take Science
- 42 people said they take Biology and Science
- 36 people said they take Accounting and Science
- 136 people said they take at least one of the subjects
- 14 said they take all 3 subjects.

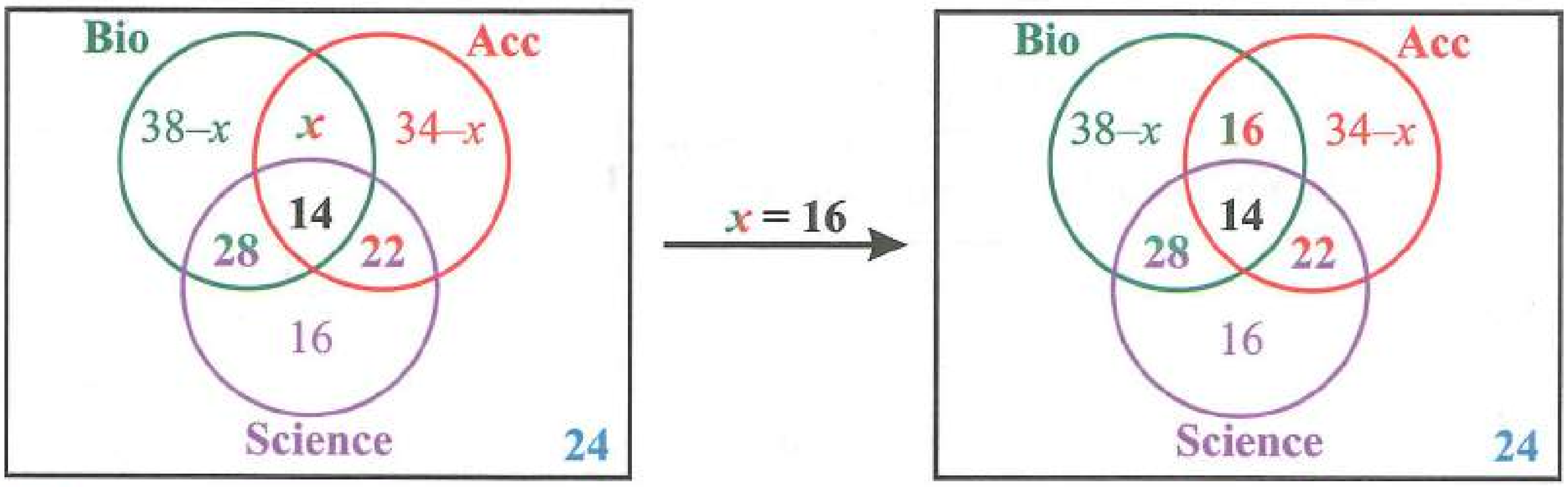
a) Record the above information in a Venn diagram.

b) How many people take none of the subjects?

c) How many people take Biology and Accounting but not Science?

d) What is the probability that a person chosen at random takes at least two of the subjects?

a)



Method:

Step 1: Determine the number of people that take none of the subjects. There are 160 people in the sample space and 136 people who take at least one of the subjects.

$\therefore$ the number of people who take none of the subjects $=160-136=24$

Step 2: It is given that **14** people take all of the subjects. Write this down in the intersection of all the subjects.

Step 3: It is given that 42 people take Biology and Science. However, this includes the **14** people that take all of the subjects. The 14 has to be deducted from the 42 people who take Biology and Science. The same must be applied to Accounting and Science.

$\therefore$ Biology and Science $=42-14=\mathbf{28}$ Accounting and Science $=36-14=\mathbf{22}$

Step 4: You can now work out the number of people who take only Science.

Number of people who take only Science $=80-28-14-22=\mathbf{16}$

Step 5: As you don't know the number of people who take Biology and Accounting let that be x.

You can now work out the number of people that take only Biology and only Accounting.

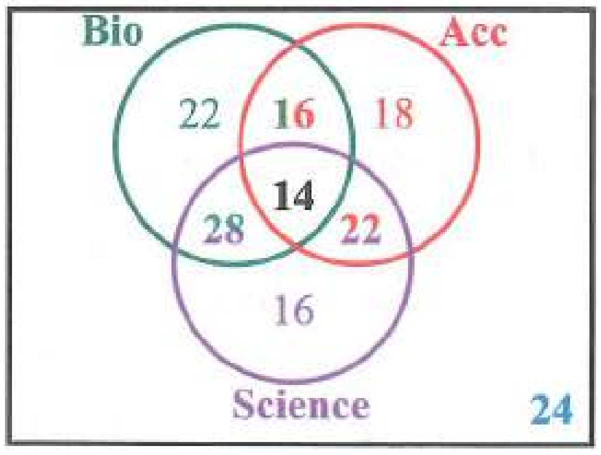
$\therefore$ Biology only $=80-28-14-x$
$=\mathbf{38-x}$

Accounting $=70-14-22-x$
$=\mathbf{34-x}$

Step 6: You can now calculate the value of x. There were a total of 160 people:

$\therefore 38-x+x+34-x+28+14+22+16+24=160$

$\therefore x=16$



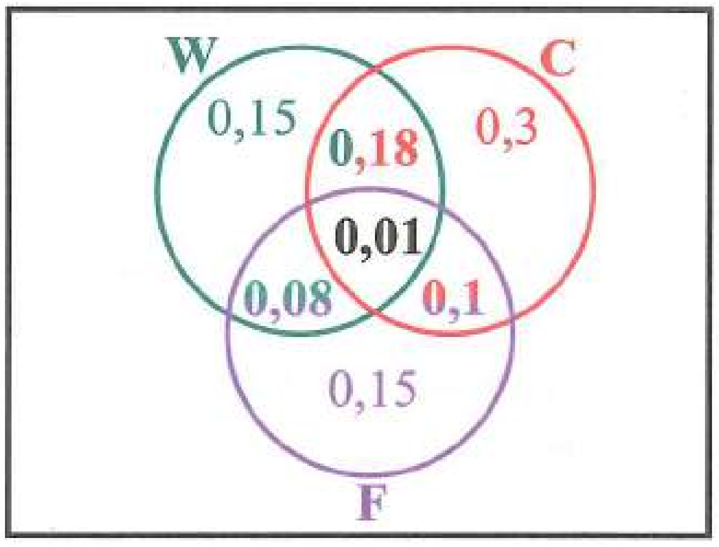
b) 24

c) 16

d) At least 2 subjects means 2 or more subjects (all intersections):
(Biology and Accounting) **or** (Biology and Science) **or** (Accounting and Science) **or** (Biology and Accounting and Science)

$$\therefore \text{P}(\text{at least two subjects}) = \frac{16}{160} + \frac{28}{160} + \frac{22}{160} + \frac{14}{160}$$
$$= \frac{80}{160} = \frac{1}{2} = 0{,}5 = 50\%$$

Example 3: **50 Grade 12 learners were asked if their school lunch consisted of a sweet (W), chocolate (C) or fruit (F). The probabilities of each are shown in the Venn diagram below.**



Determine the probability that a learner selected at random will have:

a) **A sweet and a chocolate and a fruit.**

b) **None of the above foods.**

c) **A sweet and a chocolate but not a fruit.**

d) **Only fruit.**

e) **A fruit or a sweet.**

a) $\text{P}(\text{W} \cap \text{C} \cap \text{F}) = 0{,}01 = 1\%$ → the intersection of all 3 sets

b) $\text{P}(\text{W}' \cup \text{C}' \cup \text{F}') = 1 - (0{,}15 + 0{,}18 + 0{,}3 + 0{,}01 + 0{,}08 + 0{,}1 + 0{,}15)$
$= 0{,}03 = 3\%$

c) $\text{P}(\text{W} \cap \text{C} \cap \text{F}') = 0{,}18 = 18\%$ → the intersection of sweets and chocolate

d) $\text{P}(\text{F only}) = 0{,}15 = 15\%$ → only fruit

e) $\text{P}(\text{F} \cup \text{W}) = 1 - \text{P}(\text{C only or none of the foods})$
$= 1 - (0{,}3 + 0{,}03)$
$= 0{,}67 = 67\%$

This question could also have been solved by adding all the elements in the sets excluding the choice of chocolate only.

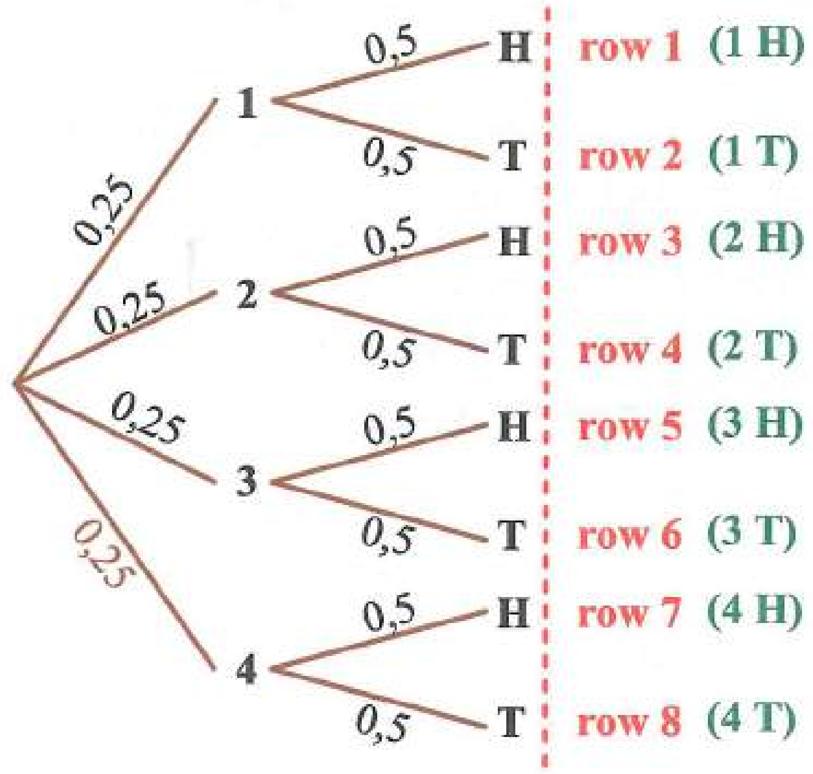
TREE DIAGRAMS

When there are 2 or more consecutive events taking place, it is often useful to represent the possible solutions on a tree diagram. Tree diagrams are constructed by showing all possible events. They can be used for dependent or independent events. When dealing with tree diagrams always multiply along branches and add probabilities moving down branches at the end. This is shown in the example below.

Example: **In a game, a four sided die with the numbers 1 to 4 is rolled and a fair coin is tossed.**

a) Draw a tree diagram to represent the above situation.
b) Determine the probability of rolling a 3 and getting a head.
c) Determine the probability of rolling an even number.
d) Determine the probability of rolling a 1 or a head.

a)



b) $P(3\text{ and a head}) = 0,25\times 0,5 = 0,125 = 12,5\% \rightarrow \text{row } 5$

c) $P(\text{even}) = 0,25\times 0,5 + 0,25\times 0,5 + 0,25\times 0,5 + 0,25\times 0,5 \rightarrow \text{rows } 3,4,7 \text{ and } 8$

$$= 0,5$$

$$= 50\%$$

d) $P(1\text{ or a head}) = 5(0,25\times 0,5) = 0,625 = 62,5\% \rightarrow \text{rows } 1,2,3,5 \text{ and } 7$

When dealing with tree diagrams, multiply across branches and add when moving down. Write the probability of an event occurring at the top of the branches and the event at the end of each branch.

CONTINGENCY TABLES

Contingency tables are statistical tables that show the relationships between two or more variables. They are often used to determine whether or not events are independent.

Example: **A group of 540 people with green or blue eyes were randomly selected in order to determine whether or not green or blue eyes are dependent on gender. The results are tabulated below:**

	Male	Female	Totals
Green eyes	183	147	**330**
Blue Eyes	117	93	**210**
Totals	**300**	**240**	**540**

a) If a person is selected at random, determine the probability that he or she is a female with green eyes.

b) If a person is selected at random and they have green eyes, determine the probability that he or she is female.

c) After analysing the results, a Grade 11 learner concludes that the probability of having green eyes is independent of gender. Is he correct? Substantiate your answer with relevant calculations. Give all answers correct to 2 decimal places.

a) $$\begin{aligned} P(\text{female} \cap \text{green eyes}) &= \frac{147}{540} \rightarrow \text{147 females with green eyes out of 540 people} \\ &= \frac{49}{180} \\ &= 27{,}22\% \end{aligned}$$

b) $$\begin{aligned} P(\text{female} \mid \text{green eyes}) &= \frac{147}{330} \rightarrow \text{there are 330 people with green eyes} \\ &= \frac{49}{110} \\ &= 44{,}55\% \end{aligned}$$

c) For events to be independent: $P(\text{female} \cap \text{green eyes}) = P(\text{green eyes}) \times P(\text{female})$

$$P(\text{green eyes}) \times P(\text{female}) = \frac{330}{540} \times \frac{240}{540} = \frac{22}{81} = 0{,}27$$

$$P(\text{female} \cap \text{green eyes}) = \frac{147}{540} = \frac{49}{180} = 0{,}27$$

$\therefore P(\text{female} \cap \text{green eyes}) = P(\text{green eyes}) \times P(\text{female})$

$\therefore$ the events are independent and the learner is correct.

Complete Exercise 1 on pages 308 and 309.

B. THE FUNDAMENTAL COUNTING PRINCIPLE

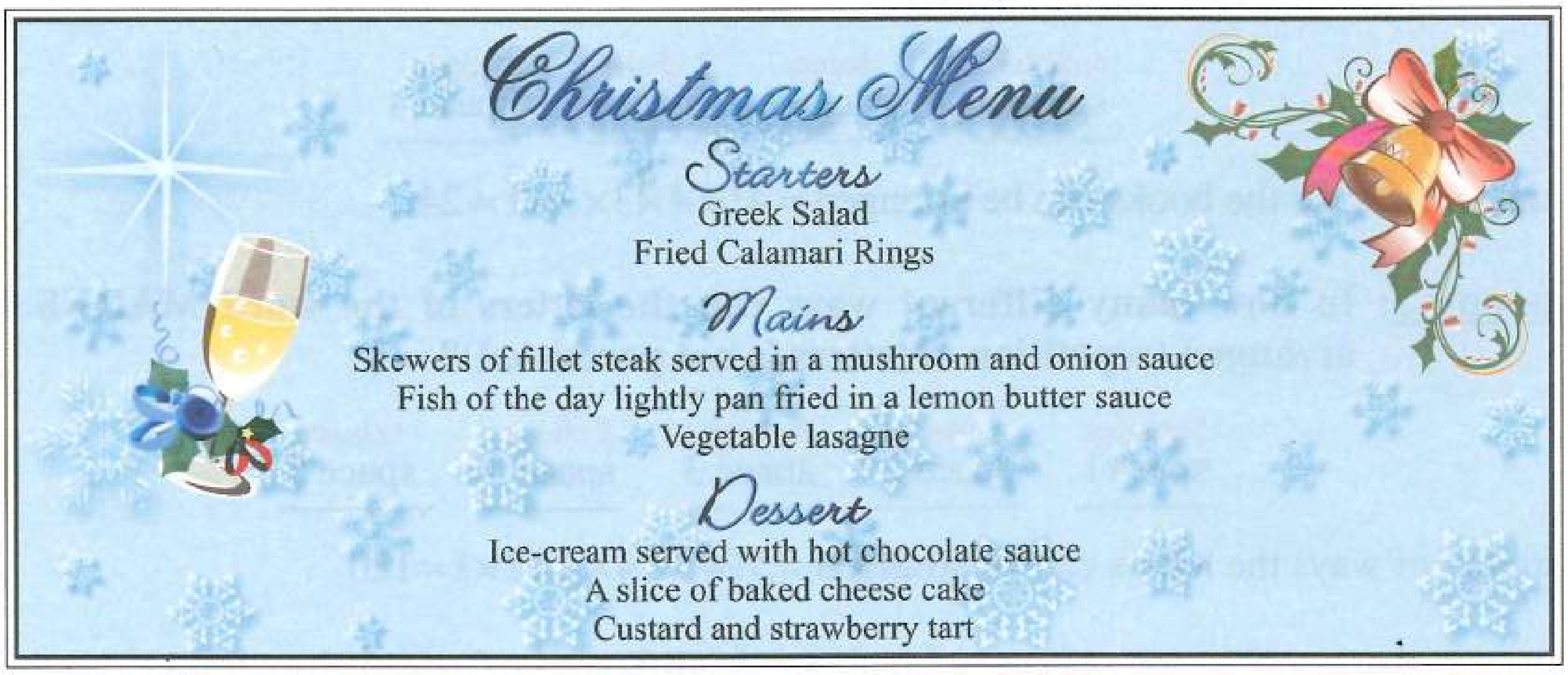
The fundamental counting principle is a principle used to determine the number of different ways to accomplish different tasks. It states that:
If there are m ways to perform a task and n ways to perform a second task, the total number of different ways in which both tasks can be performed is $m \times n$.

Example 1: A local café has 4 different flavours of ice-cream with a choice of 8 different toppings for dessert. If a customer chooses 1 topping and one flavour, how many different choices of dessert are there?

Using the Fundamental Counting Principle, there are 4 choices of ice-cream flavours and 8 choices for toppings.

$\therefore$ The total number of choices $= 4 \times 8 = 32$ choices

Example 2: At a Christmas dinner party, the set menu below shows the choices that patrons have for starters, main course and dessert. How many different meals can be made from the set menu?



Christmas Menu

Starters
Greek Salad
Fried Calamari Rings

Mains
Skewers of fillet steak served in a mushroom and onion sauce
Fish of the day lightly pan fried in a lemon butter sauce
Vegetable lasagne

Dessert
Ice-cream served with hot chocolate sauce
A slice of baked cheese cake
Custard and strawberry tart

There are 2 **choices for starters**, 3 **choices for mains** and 3 **choices for dessert**. Applying the Fundamental Counting Principle:

Total number of different meals $= 2 \times 3 \times 3 = 18$ meals

Example 3: A security system requires a user to choose a security code. The code consists of 4 digits. How many different codes can be chosen if:

a) The system allows repetition of digits.

b) The system does not allow repetition of digits.

a) There are 10 possible choices for each digit in the code, the numbers 0 through to 9. As repetition is allowed there are **10 choices for the 1st digit, 10 choices for the 2nd digit, 10 choices for the 3rd digit** and **10 choices for the 4th digit**.

Code: 10 choices 10 choices 10 choices 10 choices

$\therefore$ Number of different codes $= 10 \times 10 \times 10 \times 10 = 10^4$

b) If repetition is not permitted, there are **10 choices for the 1st digit**, **9 choices for the 2nd digit**, **8 choices for the 3rd digit** and **7 choices for the 4th digit**.

Code: 10 choices 9 choices 8 choices 7 choices

$\therefore$ Number of different codes $= 10 \times 9 \times 8 \times 7 = 5040$

Complete Exercise 2 on page 309.

FACTORIAL NOTATION

The factorial (!) of a natural number *n*, is the product of the positive integers less than or equal to *n*. This is denoted as *n*! and is pronounced *n* factorial.
The factorials below illustrate how factorial notation works:

$5! = 5\times4\times3\times2\times1$

$6! = 6\times5\times4\times3\times2\times1$

$7! = 7\times6\times5\times4\times3\times2\times1$

$n! = n\times(n-1)\times(n-2)\times(n-3)\times...\times1$

There is a ! button on your calculator. It is accessed by pressing shift x^{-1}.
It is important to note that $0! = 1$.

Example 1: **A Grade 12 learner has an Accounting, Science, Zulu and Mathematics textbook. How many different ways can they be arranged on a bookshelf?**

4 choices	3 choices	2 choices	1 choice
space 1	space 2	space 3	space 4

Number of ways the books can be arranged $= 4! = 4\times3\times2\times1 = 24$

Example 2: **In how many different ways can the letters of the word MARKS be arranged (repetition of letters is not permitted)?**

5 choices	4 choices	3 choices	2 choices	1 choice
space 1	space 2	space 3	space 4	space 5

Number of ways the letters can be arranged $= 5! = 5\times4\times3\times2\times1 = 120$

Example 3: **2 different History books, 3 different Geography books and 2 different Science books are placed on a book shelf.**

a) How many different ways can they be arranged?

b) How many ways can they be arranged if books of the same subject must be placed together?

a) Number of different arrangements $= 7! = 5\,040$

7 choices	6 choices	5 choices	4 choices	3 choices	2 choices	1 choice
space 1	space 2	space 3	space 4	space 5	space 6	space 7

b) In order to solve this problem we will treat each subject as one space:

History		Geography			Science	
2 choices	1 choice	3 choices	2 choices	1 choice	2 choices	1 choice
space 1	space 2	space 1	space 2	space 3	space 1	space 2

This means that there are 2! ways the history books can be arranged **and** 3! ways that the geography books can be arranged **and** 2! ways the science books can be arranged. You also have to take into account that the subjects can be arranged in any order. This means there are 3! ways the subjects can be arranged.

$\therefore$ Number of ways the books can be arranged $= 2!\times3!\times2!\times3! = 144$

Example 4: **In a school there are 8 prefects, 5 girls and 3 boys. They are to be seated on the stage for an assembly.**

- **a) How many different ways can they be seated on stage?**
- **b) How many ways can they be seated if all the boys and girls are to sit next to each other.**
- **c) If the prefects are randomly seated what is the probability of all the boys and girls being seated next to each other? (In other words all the boys together and all the girls together.)**

a) Number of different seating arrangments $= 8! = 40320$

8 choices	7 choices	6 choices	5 choices	4 choices	3 choices	2 choices	1 choice
seat 1	seat 2	seat 3	seat 4	seat 5	seat 6	seat 7	seat 8

b) In order to solve this problem we will treat the boys as one seat and the girls as another seat

Boys

3 choices	2 choices	1 choice
boy 1	boy 2	boy 3

Girls

5 choices	4 choices	3 choices	2 choices	1 choice
girl 1	girl 2	girl 3	girl 4	girl 5

This means that there are 3! Ways that the boys can be arranged **and** 5! ways that the girls can be arranged. You also have to take into account that the boys and girls can also swap places. In other words the girls could be seated on the left and the boys could be seated on the right. This means there are 2! ways the groups of boys and girls can be arranged

$\therefore$ Number of ways boys and girls can be seated together $= 3! \times 5! \times 2! = 1440$

c) It is important to remember that the probability of an event E occurring is:

$$\mathbf{P}(\mathrm{E}) = \frac{n(\mathrm{E})}{n(\mathrm{S})} \rightarrow \frac{\text{the number of ways the event can occur}}{\text{all of the possible events in the sample space}}$$

$$\begin{aligned}\therefore \mathrm{P}(\text{all the boys and girls are seated together}) &= \frac{1440}{40320} \\ &= \frac{1}{28} \\ &= 3{,}57\%\end{aligned}$$

DEALING WITH REPETITION

Consider the word BOB. In previous examples if there were 3 letters in a word there were 3! ways of arranging the letters. However in all of these cases there were no repeated letters. In the case of the word BOB, if the places of the B's were swopped the 'word' would still be BOB and it would not count as a different arrangement. In order to account for the repetition of letters, we divide 3! by 2! because the letter B is repeated twice. If the word was BOBBY then we would divide 5! by 3! as the letter is repeated 3 times.

Example 1: Given the word PROBABILITY:

a) Determine the number of different ways that the letters can be arranged.

b) If the word has to start with a B, how many different ways can the word be arranged?

c) What is the probability that the word starts with a B?

a) The letter B is repeated twice, and I is repeated twice.

$$\therefore \text{Number of arrangments}=\frac{11!}{2!\times 2!} \rightarrow \text{2 letters are repeated twice}$$
$$=9979200$$

b) If the 'word' is to start with the letter B, then the first letter is fixed and there are only 10 choices for the second letter, 9 for the third and so on. As the first letter is fixed, the letter B is not repeated in the remaining choices.

B _ _ _ _ _ _ _ _ _ _

$$\therefore \text{Number of arrangments beginning with B} =\frac{10!}{2!}$$
$$=1814400$$

c) $$P(\text{Word starting with a B})=\frac{1814400}{9979200}$$
$$=\frac{2}{11}$$
$$=0{,}1818$$
$$=18{,}18\%$$

It is important to note that $a!\times b!\neq(ab)!$

For example: $2!\times 3!\neq 6!$

Example 2: Given the word MISSISSIPPI:

a) Determine the number of different ways that the letters can be arranged.
b) If the word has to start with an M, how many different ways can the word be arranged?
c) What is the probability that the word starts with an M?

a) The letters I and S are repeated 4 times, and P is repeated twice.

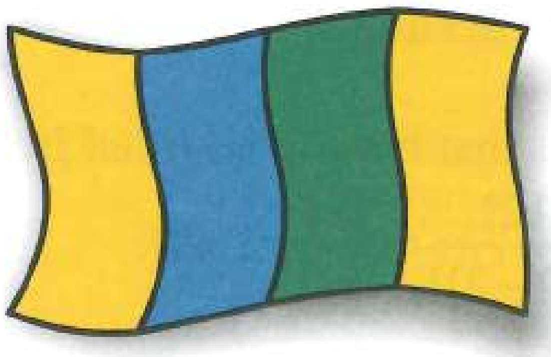
$$\therefore \text{Number of arrangments} = \frac{11!}{4! \times 4! \times 2!} = 34\,650$$

b) M _ _ _ _ _ _ _ _ _ _

$$\therefore \text{Number of arrangments beginning with M} = \frac{10!}{4! \times 4! \times 2!} = 3150$$

c) $$P(\text{Word starting with a M}) = \frac{3150}{34\,650} = \frac{1}{11} = 0{,}0909 = 9{,}09\%$$

Example 3: A flag is to be designed using the colours yellow, blue and green. There are four slots which the colours can be placed in as shown below and the colour yellow has to be used twice:



a) Determine the number of different flags that can be made.
b) How many different flags can be made that start on yellow and end on green?

a) The colour yellow is repeated:

$$\therefore \text{Number of arrangments} = \frac{4!}{2!} = 12$$

b) Yellow _ _ Green

$$\therefore \text{Number of arrangments beginning with yellow and ending in green} = 2!$$

Complete Exercise 3 on page 310.

SELECTIONS

We will now use factorial notation to deal with problems where there are a limited number of places which have to be filled from a group.

Illustrative Example:

On page 302 we dealt with the example where a security system required a user to choose a security code. The code consisted of 4 digits. It was seen that if repetition of digits was not allowed there are 10 choices for the 1st digit, 9 choices for the 2nd digit, 8 choices for the 3rd digit and 7 choices for the 4th digit.

Code: 10 choices 9 choices 8 choices 7 choices

$\therefore$ Number of different codes $= 10 \times 9 \times 8 \times 7 = 5040$

Using factorial notation, $10 \times 9 \times 8 \times 7$ can be written as $\dfrac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{6 \times 5 \times 4 \times 3 \times 2 \times 1}$.

This is the same as writing: $\dfrac{10!}{6!}$

In this example, there were 4 spaces to fill from 10 choices. It can be seen from this example that the general rule for this type of situation is:

If r spaces have to be filled from n different choices, the number of possible arrangements are:

$$\frac{n!}{(n-r)!}$$

Example 1: There are 12 contestants in a race. How many different ways can 1st, 2nd and 3rd place be arranged?

In this example there are 3 places that have to be filled from 12 choices:

$\therefore$ Number of arrangements $= \dfrac{12!}{(12-3)!} = \dfrac{12!}{9!} = 1320$.

Example 2: A local running club is holding a fundraising lottery. Four numbers are to be drawn from a possible 10 numbers. In order for a ticket to win the numbers on the lottery ticket must be in the same order that they are drawn. How many different winning arrangements are possible?

In this example there are 4 places that have to be filled from 10 choices:

$\therefore$ Number of winning arrangements $= \dfrac{10!}{(10-4)!} = \dfrac{10!}{6!} = 5040$

Complete Exercise 4 on page 310 and the Mixed Exercise on page 311.

Chapter 14 Probability (Exercises)

Grade 11 Revision

Exercise 1: (answers on pages 373 and 374)

1) **State whether the following events are mutually exclusive, complementary or neither:**
   a) Rolling an even number on a die and rolling a prime number.
   b) Rolling a number less than 6 and rolling a 6 on a die.
   c) Rolling an odd number on a die and rolling a 4.
   d) Rolling an even number and rolling an odd number.

2) **Two cards are drawn from a fair pack of playing cards without replacement. Draw a Tree Diagram to show the probabilities of drawing a specific suit in the first draw and a red or black card in the second draw. Giving answers as a percentage, use your diagram to determine the probability of:**
   a) Drawing 2 black cards.
   b) Drawing a spade and a red card.
   c) Drawing a spade and a black card.
   d) Not drawing a spade or a red card.

3) **Given the following probabilities say whether the following events are mutually exclusive, independent or neither.**

   a) $P(A)=0,2$
   $P(B)=0,35$
   $P(A \text{ or } B)=0,55$

   b) $P(A)=0,235$
   $P(B)=0,565$
   $P(A \text{ or } B)=0,7$

   c) $P(A)=0,2$
   $P(B)=0,5$
   $P(A \text{ or } B)=0,6$

   d) $P(A)=0,35$
   $P(B)=0,55$
   $P(A \text{ or } B)=0,7075$

4) **In a recent survey, 125 people were asked if they went on holiday in the summer, winter or spring in the past year. The results of the survey are shown below. Draw a Venn diagram and use it to answer the questions that follow:**
   - **39 went on holiday in summer**
   - **65 went on holiday in winter**
   - **5 went on holiday in all 3 seasons**
   - **23 went on holiday in winter and spring**
   - **43 went on holiday in spring**
   - **105 went on holiday in one of the seasons**
   - **17 went on summer and winter holidays**
   - ***x* went on holiday in summer and spring but not in winter**

   a) Determine the probability that a person selected went on holiday in one of the seasons?
   b) Determine the probability that a person selected went on holiday in summer **or** winter but not spring.
   c) Determine the probability that a person selected went on holiday in winter **and** summer.
   d) Determine the probability that a person selected went on holiday in spring **and** winter but not summer.

5) A study was performed on 800 patients over a 2 year period to determine the number of patients that have side effects from 3 different drugs. All patients were given Drugs A, B and C throughout the study at separate times to ensure that there was no interaction between the drugs. Draw a Venn diagram to represent the information below and use it to answer the questions which follow.

- **400 reported side effects from drug A**
- **350 reported side effects from drug B**
- **400 reported side effects from drug C**
- **210 reported side effects from drugs A and C**
- **180 reported side effects from drugs B and C**
- **680 reported side effects from at least 1 of the drugs**
- **70 reported side effects from all 3 drugs**
- ***x* reported side effects from drugs A and B but not C**

a) Determine the probability that a person had no side effects.
b) Determine the probability that a person in the trial had side effects.
c) Determine the probability that a person had side effects from drug A.
d) Determine the probability that a person had side effects from drug B or C.

6) In order to determine if side effects from drug A are related to gender, another study was conducted. The results are tabulated in the contingency table below. Complete the table below and use it to answer the questions below:

	Male	Female	Total
Had side effects	1500		3000
No side effects		1000	
Total			5000

a) Determine the probability that a person chosen at random will not have side effects and be male.
b) Determine the probability that a person that had side effects will be female.
c) Show that being female and not having side effects are independent events.

The Fundamental Counting Principle

Exercise 2: (answers on page 374)

1) A gift hamper consists of a chocolate slab, a bottle of wine, a tin of biscuits and a roll of dried fruit. There are 4 different types of chocolates, 3 different bottles of wine, 8 tins of biscuits and 3 different types of fruit to choose from. How many different hampers can be made up?

2) A binary number consists of only 1's and 0's and cannot start with a 0. How many different 6 digit binary numbers can be made?

3) Jimmy has to create a 6 digit password for his email account. The password must start with 3 letters and end in 3 numbers. How many different passwords can be formed in each of the following cases:

a) Repetition is allowed.
b) Repetition is not allowed.
c) Vowels may not be used and repetition is not allowed.
d) The number formed may not start with 0 or an even number and repetition is not allowed.
e) The first letter has to be a vowel and repetition is not allowed.
f) The number may not contain 5 and repetition is not allowed.

4) Every morning a father has to drop his children off at school on his way to work and pick them up again on his way home from work. There are 3 different roads from the father's home to the school and 5 different roads from the school to his office. In how many ways can the father:

a) Travel from his home to his office?
b) Travel from his home to his office and back to his home?
c) Travel from his home to his office, and back to his home, if he does not use the same road twice?

Factorial Notation

Exercise 3: (answers on page 374)

1) **Consider the word MATHEMATICS.**
   - a) How many word arrangements can be made with this word?
   - b) How many word arrangements can be made with this word if the word has to start with an S?
   - c) What is the probability that the word starts with an S?

2) **Consider the word PRACTICE.**
   - a) How many word arrangements can be made with this word?
   - b) How many word arrangements can be made with this word if the word has to start and end on the same letter?
   - c) What is the probability that the word starts with a C?

3) **4 different English books, 5 different Afrikaans books, and 3 different Zulu books are randomly arranged on a shelf.**
   - a) How many different ways can the books be arranged?
   - b) How many ways can the books be arranged if books of the same language have to be placed next to each other?
   - c) What is the probability that books of the same language are arranged together?

4) **There are 5 girls and 6 boys seated in a row:**
   - a) How many different ways can they be seated?
   - b) How many different ways can they be seated if all the boys are to be seated next to each other?
   - c) How many different ways can they be seated if a particular boy and girl have to be seated next to each other?
   - d) What is the probability that all the girls are seated next to each other?

Selections

Exercise 4: (answers on page 375)

1) **A hotel has 3 vacant rooms. In how many ways can 9 people occupy the rooms if only one person is allowed per room?**

2) **If a race has 15 contestants, how many ways can 1st, 2nd and 3rd place be awarded?**

3) **A company has 4 different job openings available. If there are 16 applicants for the jobs, in how many different ways can the openings be filled?**

4) **There are 3 participants in the finals of a swimming competition. If there are 8 lanes in the swimming pool, how many ways can the finalists be arranged in the different lanes?**

Mixed Exercise: (answers on page 375)

1) **A learner writes a Mathematics and History exam. He has a 40% chance of passing History and a 60% chance of passing Mathematics, He has a 30% chance of passing both subjects. What is the probability that he passes Mathematics or History?**

2) **A school installs a motion detection light system for their classrooms in order to save electricity. The system uses two motion sensors in the room. If there is motion, the probability that it will be detected by sensor A is 0,95. The probability that it will be detected by sensor B is 0,98 and the probability that it will be detected by both sensors simultaneously is 0,94.**

   a) If motion is present what is the probability that it will be detected by sensor A or B?
   b) In order to ensure that motion is detected 100% of the time, an additional 2 sensors are needed per classroom. Would you advise the school to invest in the additional sensors?

3) **A photographer has to take a picture of the school squash team. There are 3 boys and 3 girls in the team. In how many different ways can they be seated if:**

   a) Any member can be seated next to each other?
   b) The captain and vice-captain must be together?

4) **A man has 7 different shirts and 4 different pairs of pants which have to be hung in his cupboard.**
   a) In how many different ways can the clothes be hung?
   b) If all the shirts and all the pants have to be hung next to each other, in how many different ways can they be hung?
   c) What is the probability that a pair of pants will be hung at the beginning of the rail and a shirt will be hung at the end of the rail?
   d) If a single shirt and a single pair of pants is chosen, how many different outfits can be made?

5) **In a survey at a testing station 1 530 people were polled after taking their driver's license test. The results of the survey are as follows:**

	Test Failed	Test Passed	Total
Male	463	b	782
Female	a	c	d
Total	913	617	1 530

   a) Calculate the values of a, b, c and d.

   b) Calculate the probability of choosing, at random, a female who passed the test.

   c) Is being a female and having failed the test independent? Use calculations correct to two decimal places to motivate your answer.

6) **A password for a safe consists of 2 letters followed by 3 single digit numbers.**
   a) How many different outcomes are there for the three digits, if repetition is allowed?
   b) How many possible outcomes are there for the two letters if repetition is not allowed?
   c) What is the probability of the combined outcome PR 454 occurring if repetition is not allowed for the letters?
   d) What is the probability that the first letter is not an A?

7) **In a cricket match the winner of a coin toss between the teams' captains is allowed to choose who bats first. In 3 consecutive matches, what is the probability that a captain will win the toss:**

   a) Every time.
   b) At least once.

CHAPTER 15
EXEMPLAR PRELIMINARY EXAMINATIONS

A. PAPER 1

Marks: 150
Time: 3 hours

INSTRUCTIONS AND INFORMATION

Read the following instructions carefully before answering the questions:

1. This question paper consists of 10 questions.
2. Answer ALL the questions.
3. Clearly show ALL calculations, diagrams, graphs, et cetera that you have used in determining the answers.
4. Answers only will not necessarily be awarded full marks.
5. An approved scientific calculator (non-programmable and non-graphical) may be used, unless stated otherwise.
6. Round your answers off to TWO decimal places if necessary, unless stated otherwise.
7. Diagrams are NOT necessarily drawn to scale.
8. Number the answers correctly according to the numbering system used in this question paper.
9. Write legibly and present your work neatly.

Answers to this paper can be found on pages 376 and 377.

Question 1

1.1 Solve for x, rounded off to TWO decimal places where necessary:

1.1.1 $x(x-1)=2$ (3)

1.1.2 $2x^2-3x=8$ (4)

1.1.3 $3x^2+x-2\geq 0$ (4)

1.2 Given the equation $x^2+2xy-8y^2=0$:

1.2.1 Determine the values of the ratio $\frac{x}{y}$. (3)

1.2.2 Hence, determine the values of x and y if $x+y=6$. (5)

1.3 Simplify the following without the use of a calculator. Show all workings:

$$\left(\frac{\sqrt{3^{2011}}-\sqrt{3^{2009}}}{\sqrt{3^{2008}}}+\sqrt{3}\right)^2$$ (4)

[23]

Question 2

2.1 Given the sequence: $8\,;x\,;64$

Determine the value of x if the sequence is:

2.1.1 Arithmetic (2)

2.1.2 Geometric (3)

2.2 Determine the value of R if $\text{R}=\sum_{i=8}^{16}2^{i-3}$ (5)

2.3 Prove that for any arithmetic sequence of which the first term is a and the constant difference is d, the sum to n terms can be expressed as: $S_n=\frac{n}{2}\left[2a+(n-1)d\right]$. (4)

[14]

Question 3

3.1 Given the geometric series: $3.2^4+3.2^3+3.2^2+\ldots$

3.1.1 Determine the value of r, the constant ratio. (1)

3.1.2 Explain whether or not the sequence converges. (1)

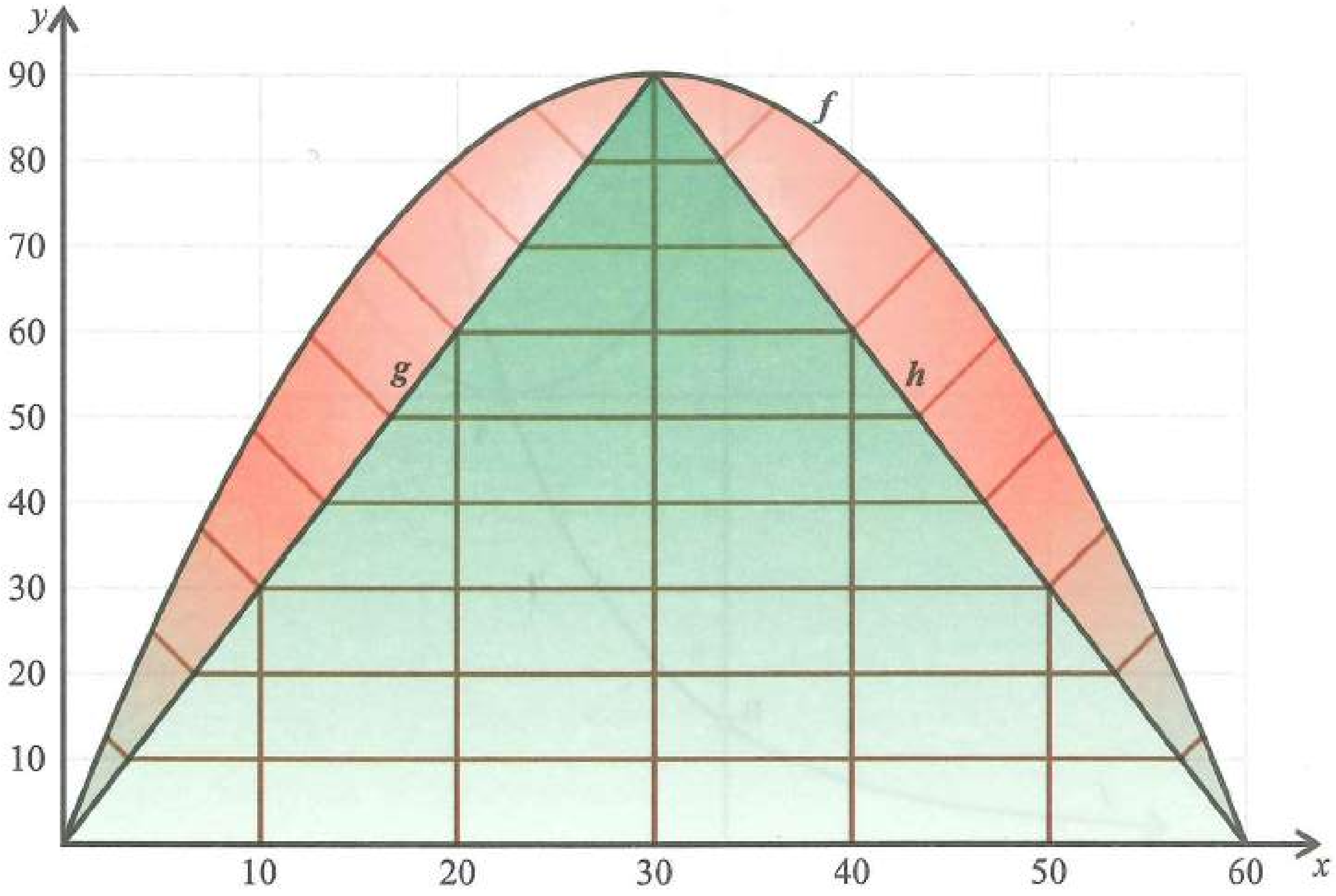
3.1.3 Determine the sum to infinity of this sequence. (2)

3.2 The first three terms of the sequence $27\,;x\,;y\,;-3$ form a geometric sequence, while the last three terms form an arithmetic sequence. Determine the values of x and y. (8)

[12]

Question 4

4.1 A new mega-roller coaster is being designed for a theme park. At its highest point, patrons will be 90 metres above the ground. The main arch of the roller coaster is modelled by parabola f and will be built onto a wooden frame. The wooden frame is modelled by two straight lines g and h. Support beams are attached to the wooden frame every 10 metres horizontally and vertically as shown in the diagram below. Determine the equations of f, g and h and write down the domain of h. (7)



4.2 Given: $h(x)=4^x$ and $f(x)=3(x+1)^2-12$.

4.2.1 Sketch the graphs of h and f on the same set of axes. Indicate ALL intercepts with the axes and any turning points. (8)

4.2.2 Without any further calculations, sketch the graph of $y=\log_4 x=g(x)$ on the same set of axes. (2)

4.2.3 The graph of p is the graph of f shifted 2 units to the RIGHT. Write down the new equation of p. (2)

4.2.4 Show algebraically that $h\left(x-\frac{1}{2}\right)=\frac{1}{2}h(x)$. (3)

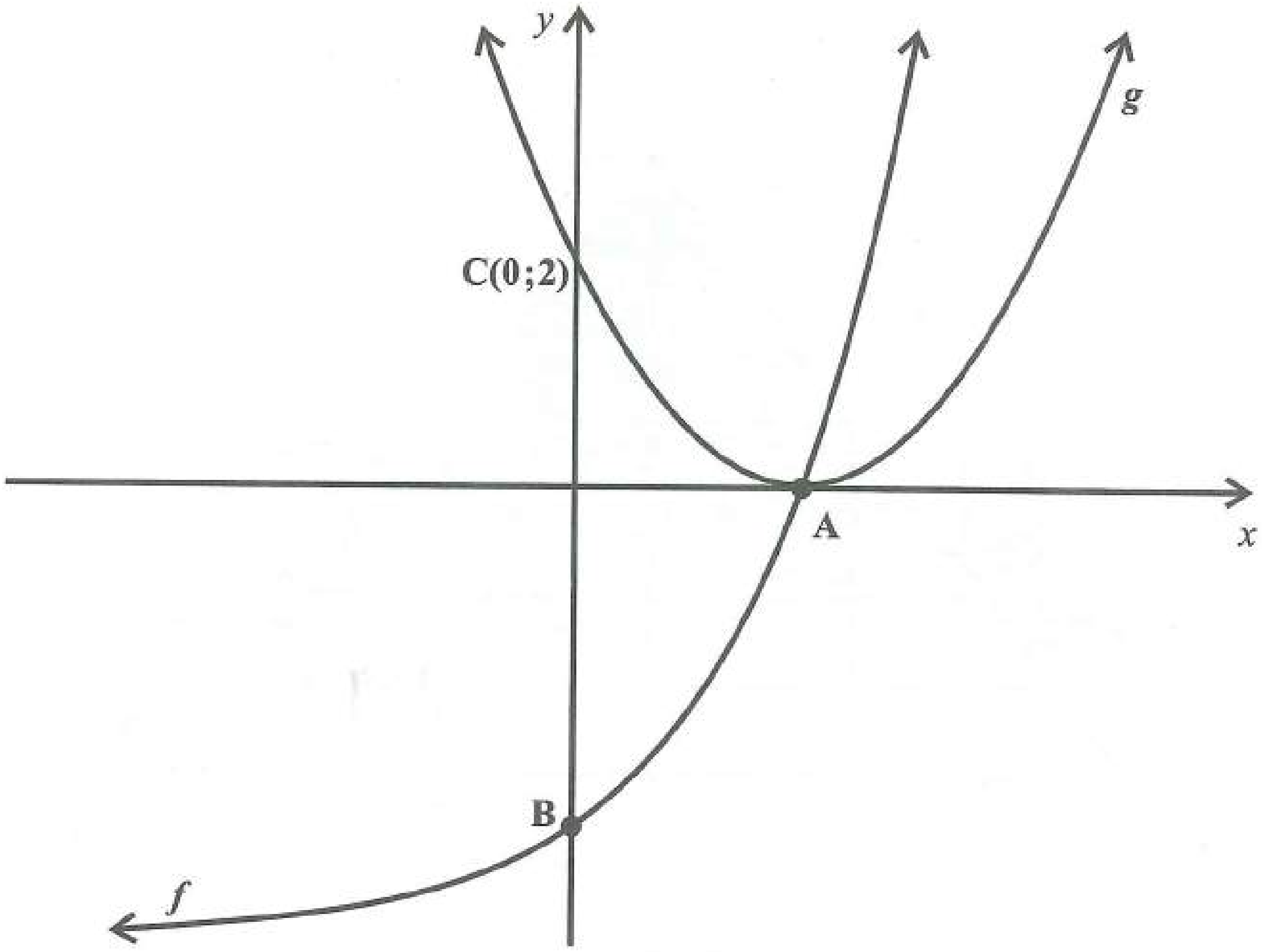
[22]

Question 5

The graphs of $f(x)=2^x-4$ and $g(x)=ax^2+bx+c$ are sketched below.

B and $C(0;2)$ are the y-intercepts of the graphs of f and g respectively.

The two graphs intersect at A, which is the turning point of the graph of g and the x-intercept of the graphs of f and g.



5.1 Determine the coordinates of A and B. (4)

5.2. Write down the equation of the asymptote of the graph of f. (1)

5.3 Determine the equation of p if $p(x)=f(2x)+8$. (2)

5.4 Determine the equation of p^{-1} in the form $y=\ldots$. (2)

5.5 Determine the equation of h, if h is the reflection of p^{-1} about the x-axis. (1)

5.6 Calculate $\sum_{k=0}^{2} g(k)-g(3)+f(0)$. Show ALL workings. (4)

[14]

Question 6

6.1 Tebogo plans to retire in 40 years' time. He has calculated that he needs an amount equivalent to the buying power of R1 300 000 in today's Rands. He expects inflation to be an average of 7,8% per annum during the next 40 years.

6.1.1 Show that he will need R26 223 689,04 in his account in 40 years' time. (2)

6.1.2 He plans to make equal monthly payments for the next 40 years, at the end of each month, into an account which offers an interest rate of 10,3% per annum compounded monthly. Determine Tebogo's monthly deposits. (5)

6.2 Rachael also wishes to save for retirement. She can comfortably afford to invest R8 500 at the end of each month into an account which offers an interest rate of 10,7% per annum, compounded monthly, for a period of 20 years. After 20 years she wants to stop depositing R8 500 each month and intends to let the investment sum grow at 11,2% compounded quarterly until she has an amount of at least R26 000 000.
How long, correct to the nearest month, will it take to realise her goal? (8)

[15]

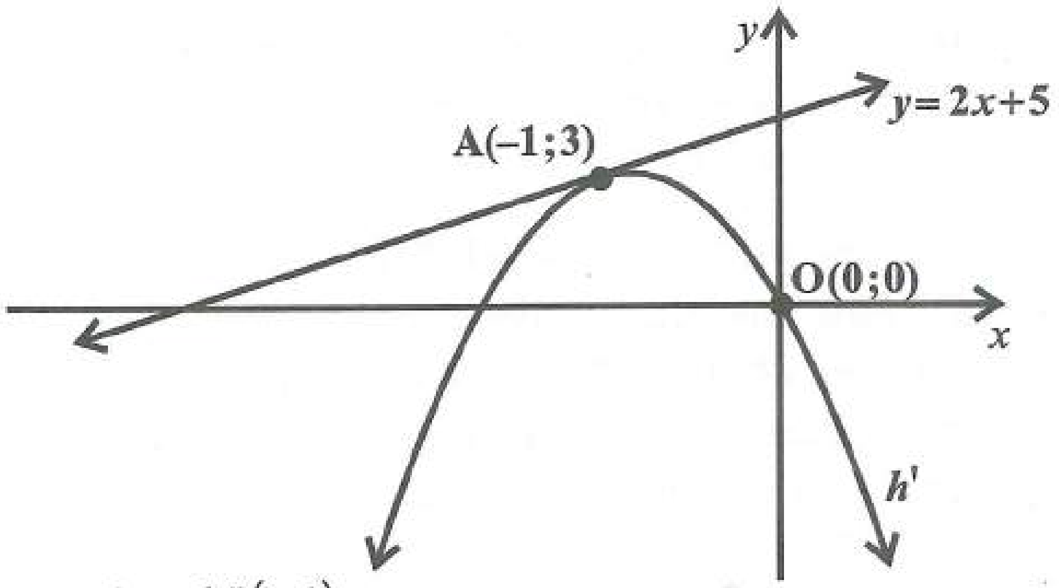
Question 7

7.1 The graph of *f* with equation $f(x)=ax^3+bx^2+cx+d$, where *a*, *b* and *c* are real constants, has the following properties:

- $a>0$
- $d=10$
- $f(-2)=0$, $f(2)=0$ and $f(4)=0$
- $f'(0)=0$, $f'(3)=0$ and $f(3)=-3$

Using the above information, draw a sketch graph of *f*. Clearly indicate all intercepts with the axes, as well as the turning points. (6)

7.2 In the figure below, the line of $y=2x+5$ is a tangent to the curve of *h*' with equation $h'(x)=ax^2+bx$ at the point $\text{A}(-1;3)$, with *a* and *b* being real constants, Note that *h*' is the derivative function of the cubic function *h*.



7.2.1 Determine $h''(-1)$. (1)

7.2.2 Hence, determine the values of *a* and *b* in $h'(x)=ax^2+bx$. (6)

7.2.3 Determine the *x*-coordinates of the turning points of *h*. (2)

[15]

Question 8

8.1 Determine $f'(x)$ from first principles if $f(x)=3-x^2$. (5)

8.2 Evaluate:

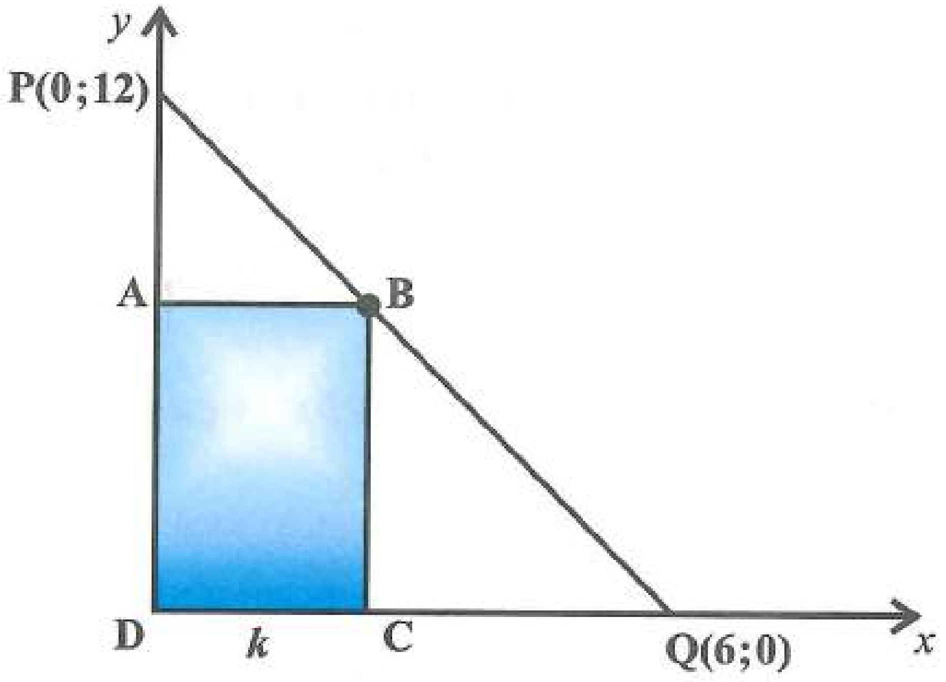
8.2.1 $D_x\left[-5+8\sqrt{x}\right]$. (2)

8.2.2 $\frac{dy}{dx}$ if $y=\frac{4-5x^4}{8x^3}$. (4)

[11]

Question 9

In the figure below, point B lies on the line joining points $P(0;12)$ and $Q(6;0)$. ABCD is a rectangle, where A, C and D lie on the axes. If $DC=k$:



9.1 Determine the length of BC in terms of *k*. (2)

9.2 Determine the coordinates of B for which ABCD has a maximum area. (5)

9.3 Hence, determine the maximum area of ABCD. (2)

[9]

Question 10

10.1 In a bag, there are *x* blue balls and 2 red balls. A ball is selected at random, the colour is recorded and then replaced. Another ball is then selected at random, the colour is recorded again and then replaced. The probability that the two balls are different colours is 0,375. Draw a tree diagram of the above scenario and use it to determine the value of *x*. (6)

10.2 Musical chairs is a game often played at birthday parties. The game is played by having one less chair than the number of players. Music is played and the players walk around the chairs which are arranged in a straight line. When the music stops the players have to sit down on the chairs. The player who does not have a chair to sit on is then eliminated and a chair is removed. Phindile, Alexi, Daniel, Nawaaz, Marijke, Savannah, Rose, Romy and Samir are playing a round of musical chairs.

10.2.1 How many different seating arrangements are possible? (2)

10.2.2 If Nawaaz is eliminated, what is the probability that Romy and Rose are seated next to each other? (4)

10.3 Using the letters in the word GUESS, determine:

10.3.1 The number of different 5 letter 'words' that can be formed. (1)

10.3.2 The probability that a word will start with the letter 'S'. (2)

[15]

Total: [150]

B. PAPER 2

Marks: 150
Time: 3 hours

INSTRUCTIONS AND INFORMATION

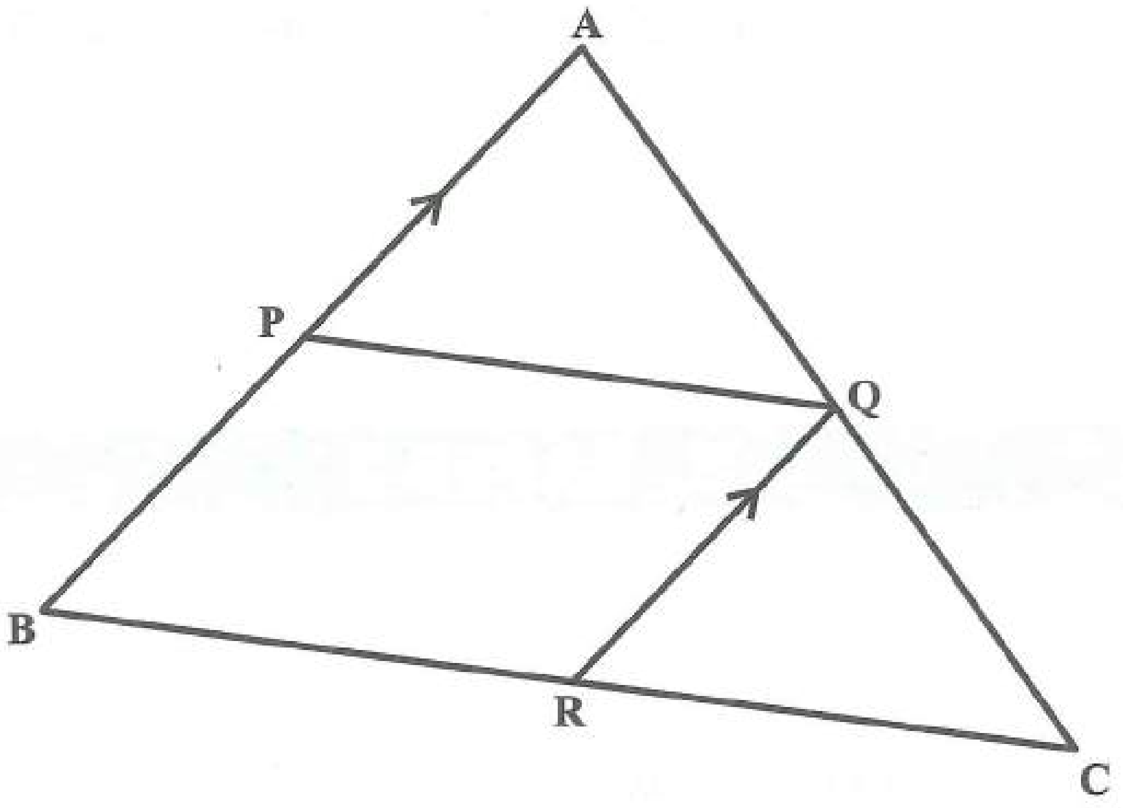
Read the following instructions carefully before answering the questions:

1. This question paper consists of 13 questions.
2. Answer ALL the questions
3. Clearly show ALL calculations, diagrams, graphs, et cetera that you have used in determining the answers.
4. Answers only will not necessarily be awarded full marks.
5. An approved scientific calculator (non-programmable and non-graphical) may be used, unless stated otherwise.
6. Round your answers off to TWO decimal places if necessary, unless stated otherwise.
7. Diagrams are NOT necessarily drawn to scale.
8. Number the answers correctly according to the numbering system used in this question paper.
9. Write legibly and present your work neatly.

Answers to this paper can be found on pages 378 and 379.

Question 1

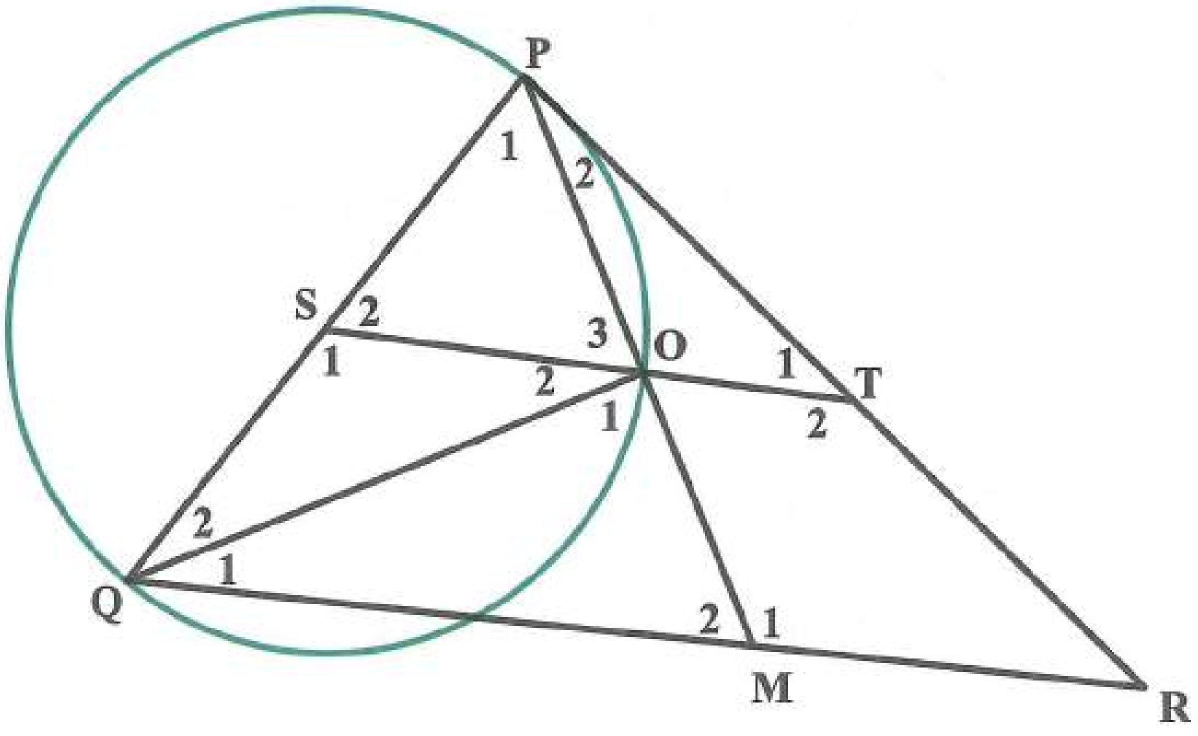
In ΔABC, AB//QR and $AP : PB = BR : RC$. With reasons, prove that PQ//BC. (6)



[6]

Question 2

In ΔPQR, $Q\hat{P}R : P\hat{Q}R : \hat{R} = 15:16:5$ S and T are the midpoints of PQ and PR. PSQ is a diameter of circle centre S and O is a point on the circumference of the circle.



2.1 Determine the size of:

2.1.1 $Q\hat{P}R$ (3)

2.1.2 $\hat{R}$ (1)

2.1.3 $\hat{T}_1$ (3)

2.2 Show that $OP = OM$. (2)

2.3 Prove that $\Delta PQO \equiv \Delta MQO$. (4)

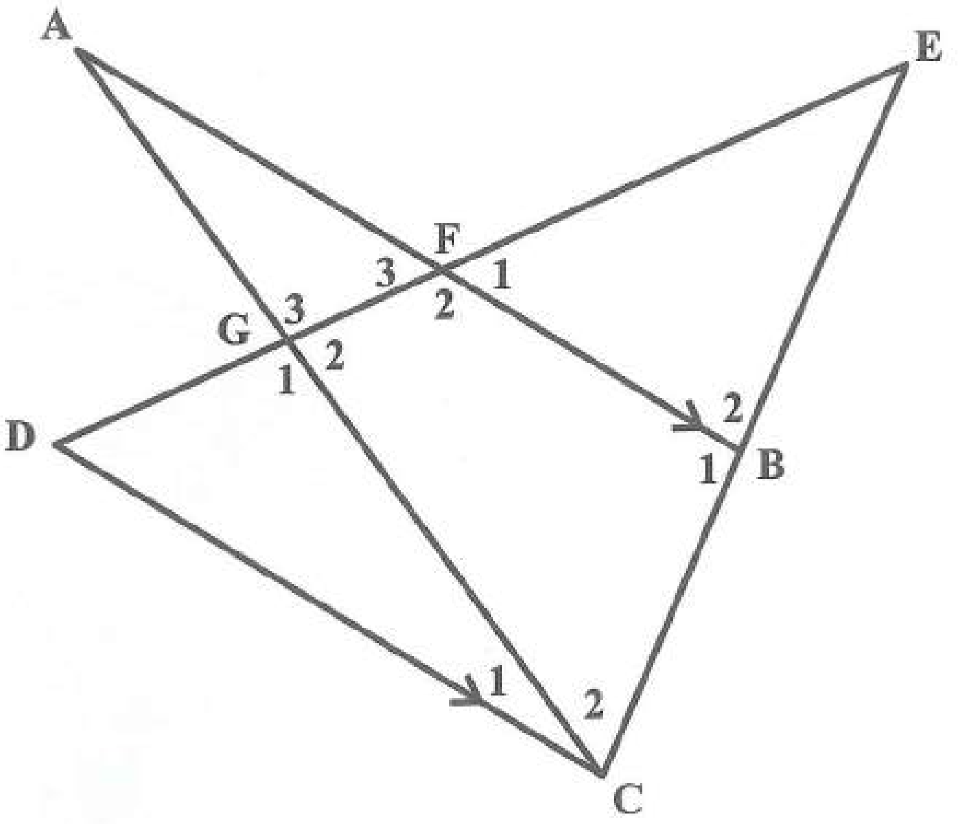
2.4 Hence, show that $QM = 2PS$. (2)

[15]

Question 3

In the diagram below, F lies on lines ED and AB and G lies on lines ED and AC. DGFE is a straight line and:

BFA // DC
AB = 40 cm
BC = 20 cm
EF = 16 cm
EB = 10 cm
FB = 12 cm



3.1 With reasons, determine the value of $\frac{EF}{ED}$. (3)

3.2 Determine the length of ED. (2)

3.3 Without any reasons complete: $\Delta EFB ||| \Delta ...$ (1)

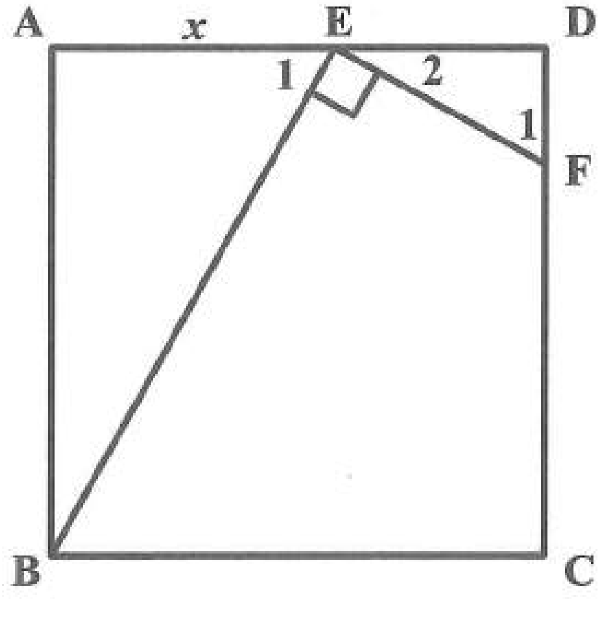
3.4 Hence, with reasons, determine the length of DC. (3)

3.5 Prove that $\frac{AF}{CD} = \frac{FG}{DG}$. (4)

[13]

Question 4

In the diagram below, ABCD is a square with sides 9cm in length and BE⊥EF.



4.1 Show that $\hat{F}_1 = a$, if $\hat{E}_1 = a$ and $\hat{E}_2 = y$. (2)

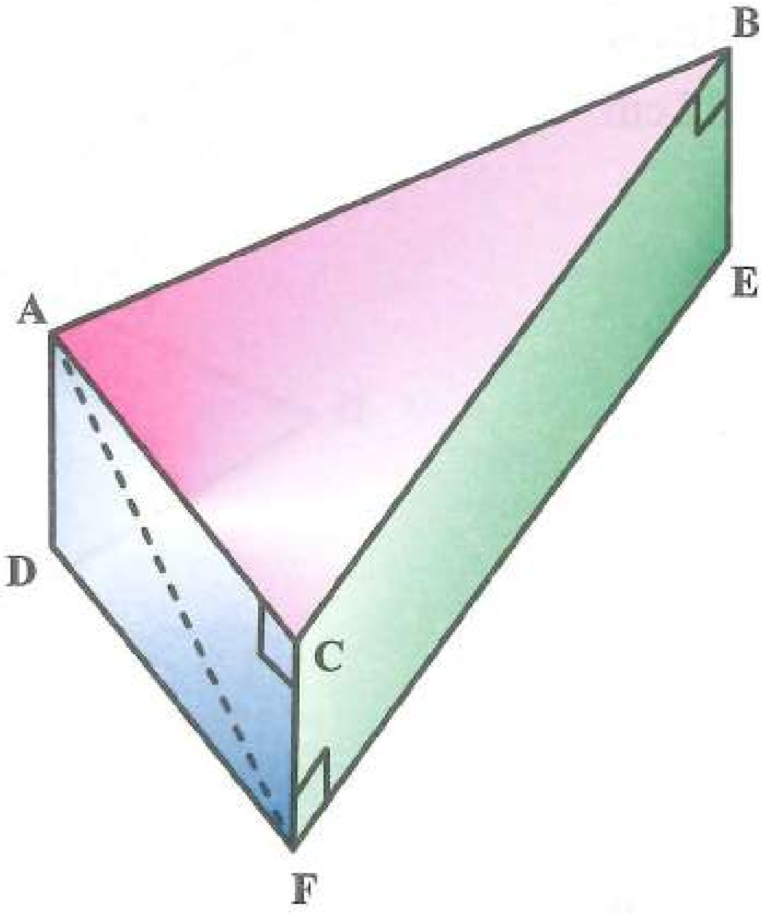
4.2 Prove that $\Delta ABE ||| \Delta DEF$. (2)

4.3 Given that AE = x cm, show that the area of $\Delta EDF = \frac{x(9-x)^2}{18}$. (5)

[9]

Question 5

The figure below represents a doorstop in the shape of a triangular right prism. $BA = BC = 5$ units, $A\hat{B}C = 50^\circ$ and $F\hat{A}C = 25^\circ$.



5.1 Determine the area of ΔABC. (2)

5.2 Determine the length of AC. (3)

5.3 Hence, determine FC, the height of the doorstop. (3)

[8]

Question 6

Points $A(1;-1)$, $B(7;1)$ and $C(x;3)$ are points on the Cartesian plane. Calculate the possible value(s) of x in each of the following cases:

6.1 $BC = 2\sqrt{5}$. (3)

6.2 A, B and C are collinear. (2)

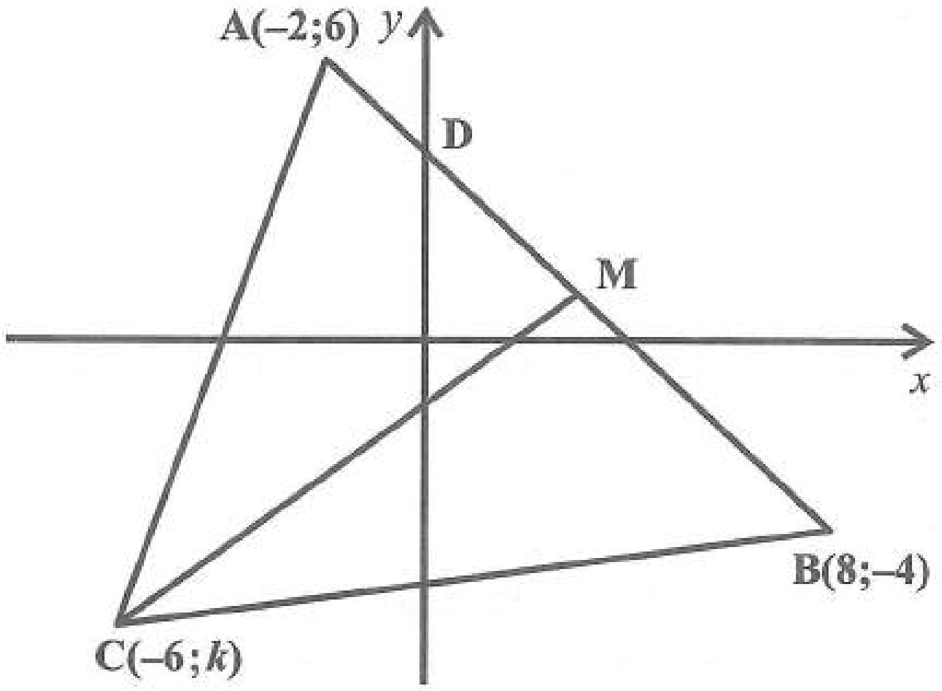
6.3 B is the midpoint of AC. (3)

6.4 ΔABC is right-angled at A. (4)

[12]

Question 7

In the diagram below, $A(-2;6)$, $B(8;-4)$ and $C(-6;k)$ are three points in a Cartesian plane with point M lying on line AB. $AM = BM$ and $AB = BC$.
D is the y-intercept of line AB.

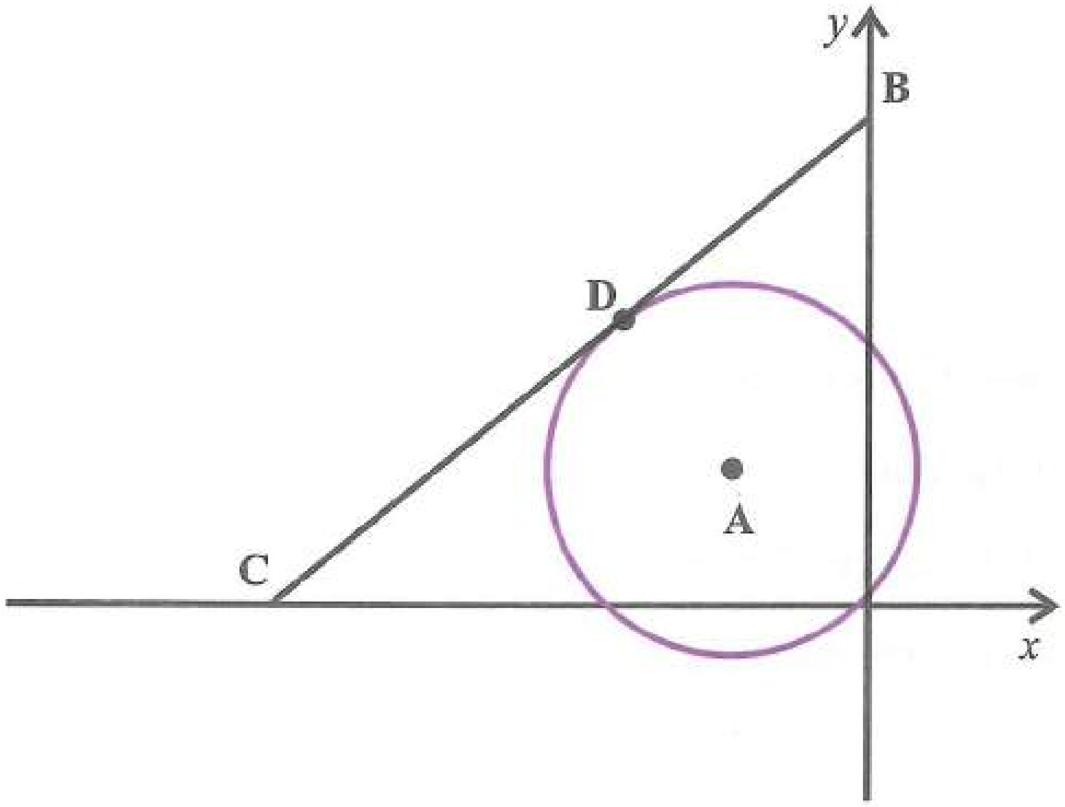


7.1	Determine the coordinates of M.	(2)
7.2	Determine the value(s) of k.	(6)
7.3	Hence, determine the equation of line CM if $k = -6$.	(5)
7.4	If AEBC is a parallelogram and $k = -6$, determine the coordinates of E.	(2)
7.5	Given that a straight line is defined by $2y - 1 = mx$, determine the value of m if the line has an angle of inclination of 116,56°.	(4)

[19]

Question 8

In the figure below, the circle with centre A has equation $x^2 + y^2 + 6x = -1 + 6y$. Line BC is a tangent to the circle at $D\left(-\frac{28}{5};\frac{31}{5}\right)$.



8.1	Determine the coordinates of A, the centre of the circle.	(4)
8.2	Write down the radius of the circle.	(1)
8.3	Determine the equation of tangent BC.	(4)

[9]

Question 9

A men's shoe store conducts a survey from a representative sample of regular customers to see the ages of customers against the amount they spend monthly on shoes. The results are tabulated below:

x **Age of customer**	18	21	23	25	30	32	36	38	39	45
y **Amount spent in Rands**	330	300	300	240	250	190	180	310	150	120

9.1 Use your calculator to determine the regression line which best fits the given data. Round off your answer to two decimal digits where necessary. (4)

9.2 Determine the correlation coefficient r for the data set. (2)

9.3 Predict how much money a 16 year old will spend on shoes at the store. (2)

9.4 Explain whether or not your prediction in the question above is reliable. (2)

[10]

Question 10

A flower shop records the distances that their delivery vans travel each day. The table below shows the results:

Distance (d) (km)	**Frequency**	**Midpoint**
$0 < d \leq 5$	5	
$5 < d \leq 10$	41	
$10 < d \leq 15$	77	
$15 < d \leq 20$	58	
$20 < d \leq 25$	39	
$25 < d \leq 30$	17	
$30 < d \leq 35$	3	

10.1 Copy and complete the above table. (1)

10.2 Estimate the mean of the data. (4)

10.3 Estimate the standard deviation for the data. (4)

10.4 In which interval does the median of the data lie? (2)

[11]

Question 11

11.1 Without the use of a calculator and showing ALL calculations, prove that:

11.1.1 $\dfrac{\tan 225^\circ \cos 210^\circ \sin 23^\circ \cos\left(-135^\circ\right)}{\cos 113^\circ \sin 315^\circ} = \dfrac{\sqrt{3}}{2}$ (6)

11.1.2 $\sin 75^\circ = \dfrac{\sqrt{2}\left(\sqrt{3}+1\right)}{4}$ (4)

11.2 Prove that $\cos\left(90^\circ + 2x\right)\tan\left(-x-180^\circ\right)+\sin^2\left(360^\circ - x\right) = 3\sin^2 x$. (6)

[16]

Question 12

12.1 If $\sin 32^\circ = p$, express the following in terms of p, without the use of a calculator:

12.1.1 $\cos 32^\circ$ (2)

12.1.2 $\sin 16^\circ \cos 16^\circ - \sin\left(-58^\circ\right)\tan 32^\circ$. (5)

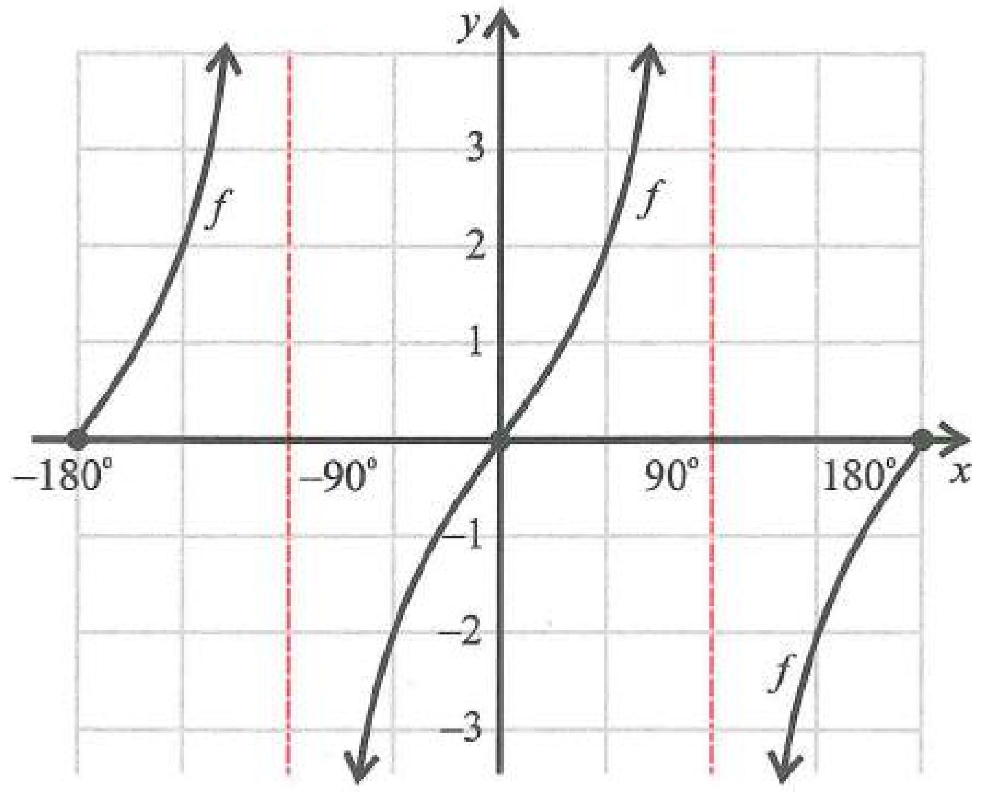
12.2 Determine the general solution of $\dfrac{3}{2}\sin x = 0{,}378$. (5)

12.3 Determine a value for x if $\sin x; \cos x; 2\cos x$ form a geometric sequence and $x \in \left[0^\circ ; 90^\circ\right]$. (3)

[15]

Question 13

The graph of $f\left(x\right) = a\tan x$ for $x \in \left[-180^\circ ; 180^\circ\right], x \neq \pm 90^\circ$ is sketched below:



13.1 Determine the value of a. (1)

13.2 If $h\left(x\right) = \cos\left(x + 45^\circ\right)$, sketch the graph of h for $x \in \left[-180^\circ ; 180^\circ\right]$. (3)

13.2 The graph of $q\left(x\right) = \sin bx + c$ has the same period as f and a y-intercept at the point $\left(0;1\right)$. Determine the values of b and c. (3)

[7]

Total: [150]

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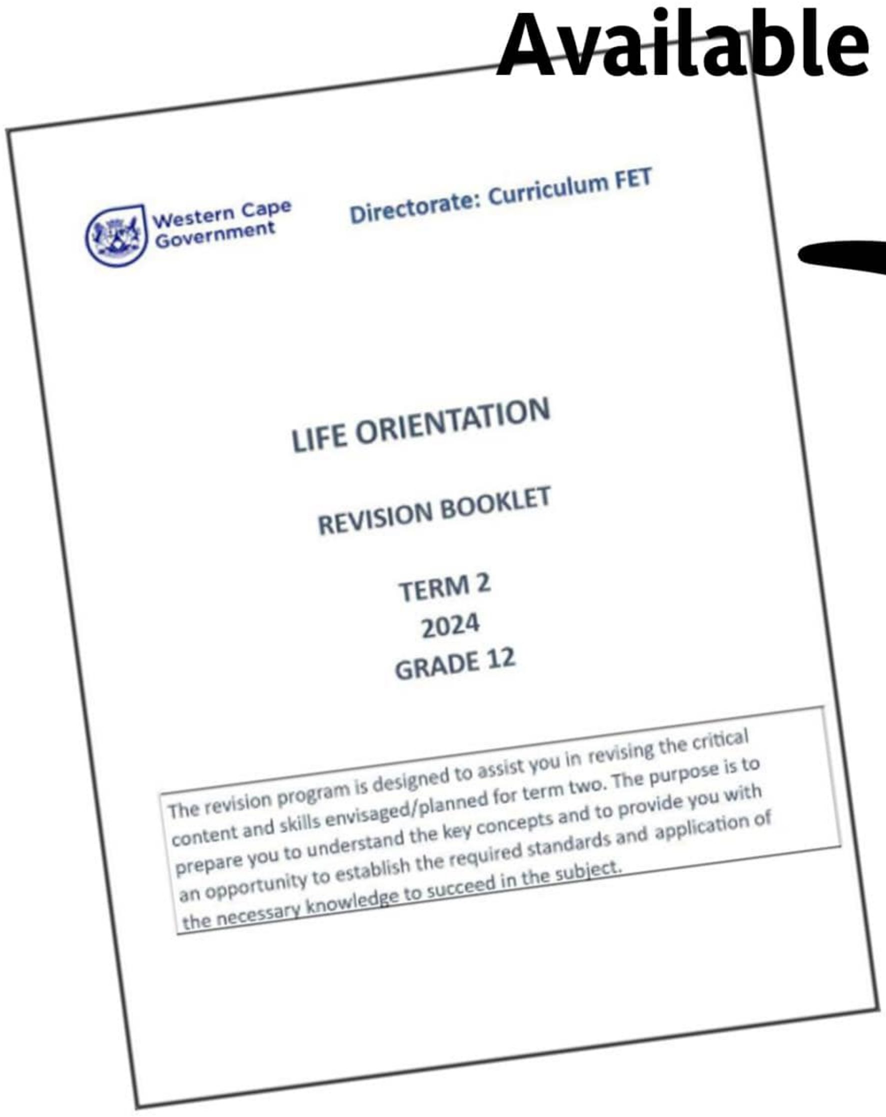

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TERM 2
2024
GRADE 12

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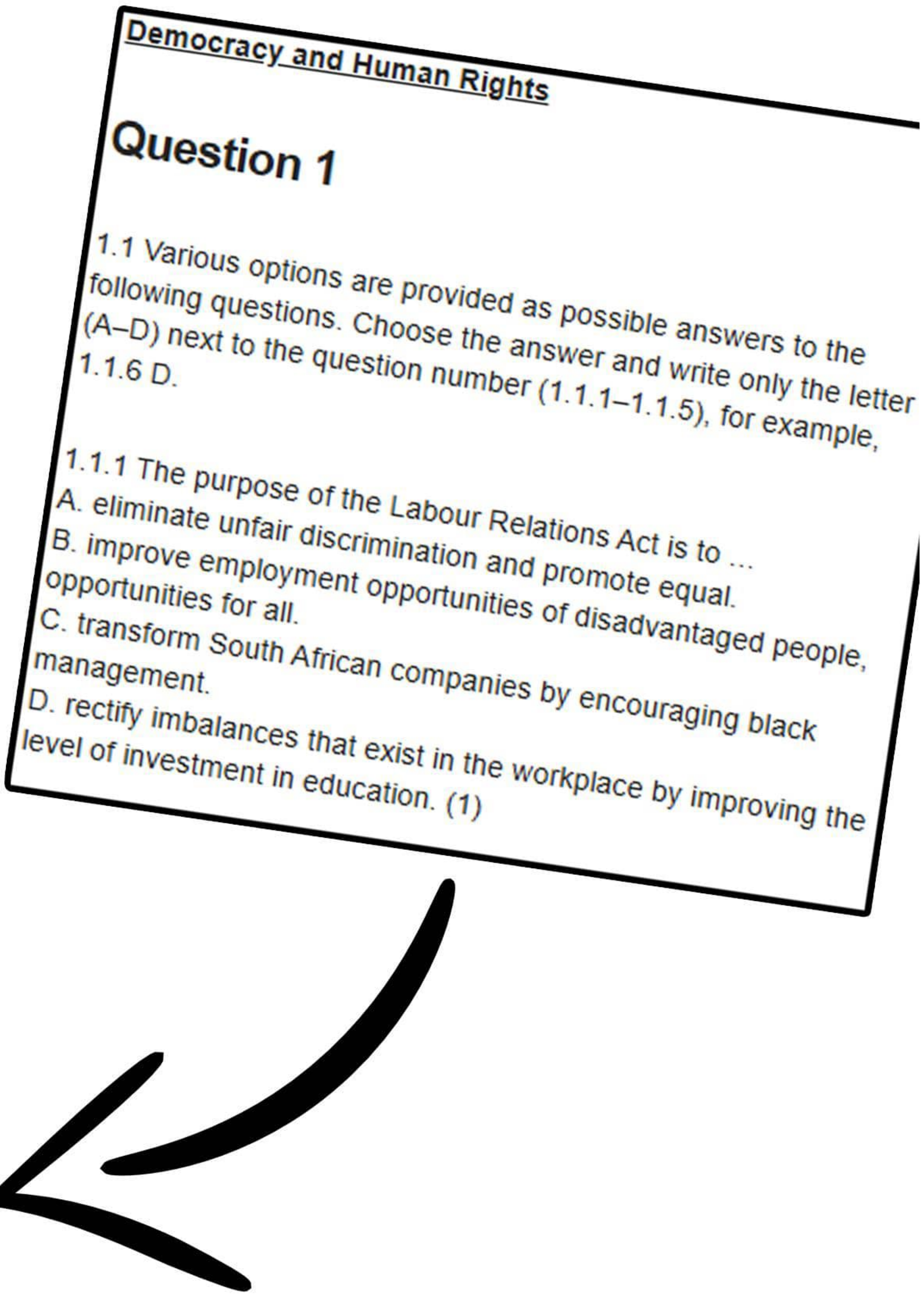
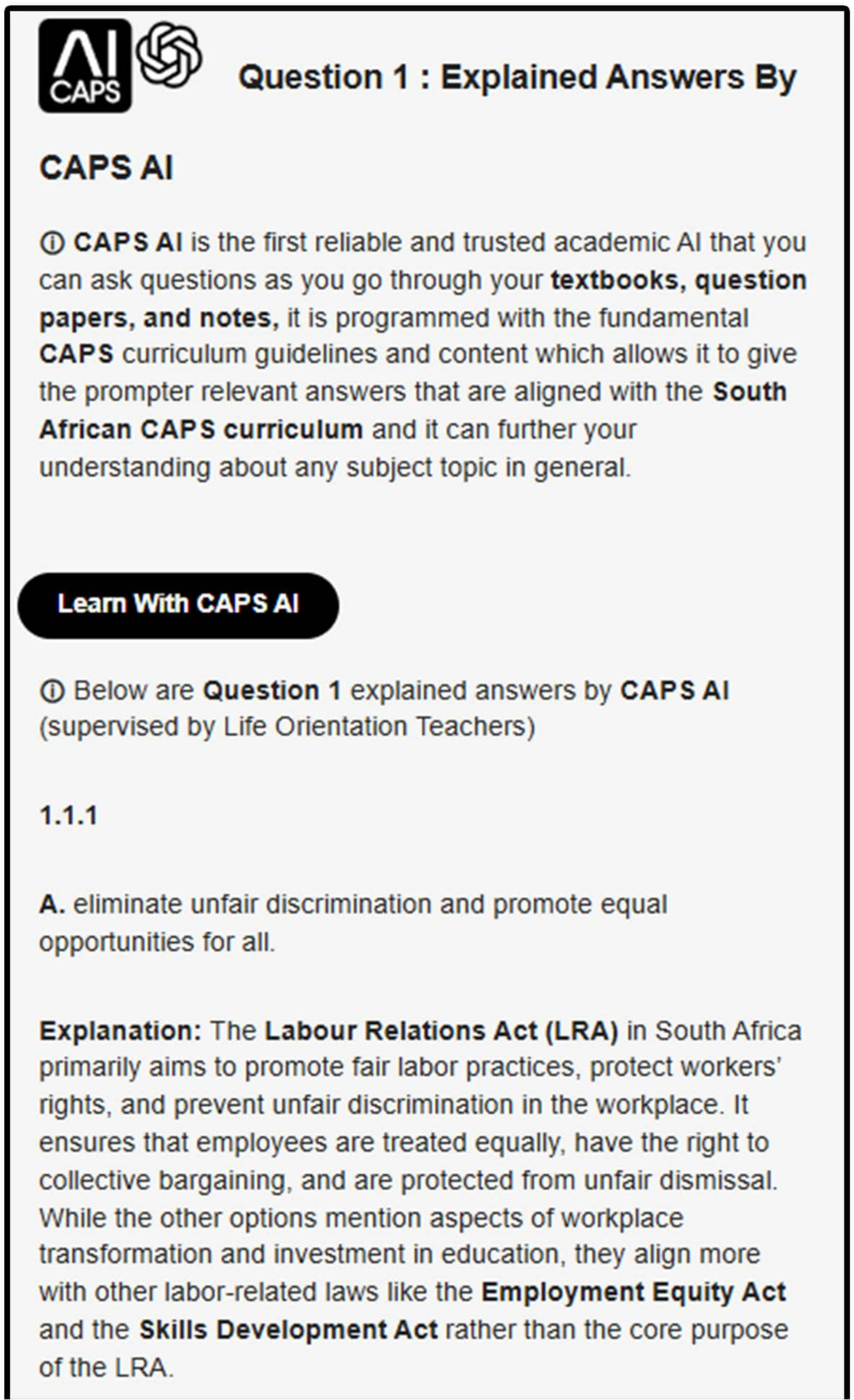


Democracy and Human Rights

Question 1

1.1 Various options are provided as possible answers to the following questions. Choose the answer and write only the letter (A–D) next to the question number (1.1.1–1.1.5), for example, 1.1.6 D.

1.1.1 The purpose of the Labour Relations Act is to …
A. eliminate unfair discrimination and promote equal opportunities for all.
B. improve employment opportunities of disadvantaged people, opportunities for all.
C. transform South African companies by encouraging black management.
D. rectify imbalances that exist in the workplace by improving the level of investment in education. (1)



Question 1 : Explained Answers By CAPS AI

ⓘ **CAPS AI** is the first reliable and trusted academic AI that you can ask questions as you go through your **textbooks, question papers, and notes,** it is programmed with the fundamental **CAPS** curriculum guidelines and content which allows it to give the prompter relevant answers that are aligned with the **South African CAPS curriculum** and it can further your understanding about any subject topic in general.

ⓘ Below are **Question 1** explained answers by **CAPS AI** (supervised by Life Orientation Teachers)

1.1.1

A. eliminate unfair discrimination and promote equal opportunities for all.

Explanation: The **Labour Relations Act (LRA)** in South Africa primarily aims to promote fair labor practices, protect workers' rights, and prevent unfair discrimination in the workplace. It ensures that employees are treated equally, have the right to collective bargaining, and are protected from unfair dismissal. While the other options mention aspects of workplace transformation and investment in education, they align more with other labor-related laws like the **Employment Equity Act** and the **Skills Development Act** rather than the core purpose of the LRA.

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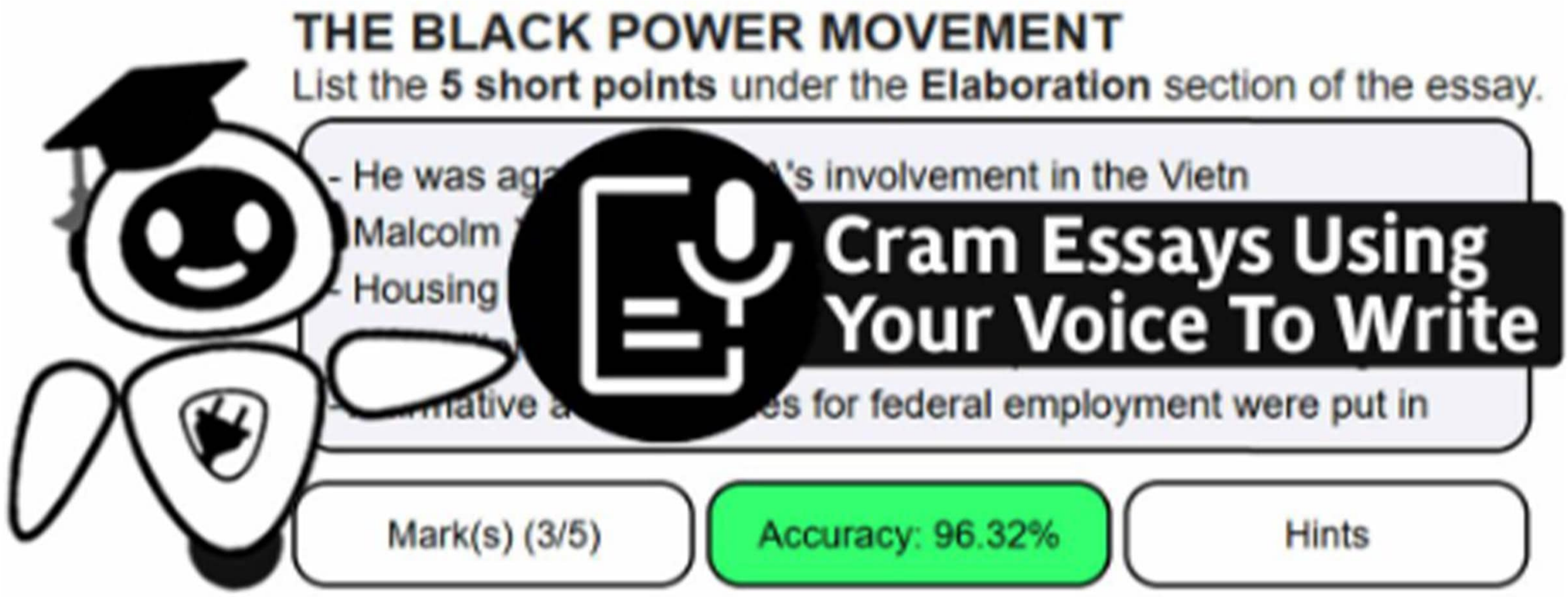

THE BLACK POWER MOVEMENT
List the **5 short points** under the **Elaboration** section of the essay.

- He was ag A's involvement in the Vietn
Malcolm
- Housing
- ative a es for federal employment were put in

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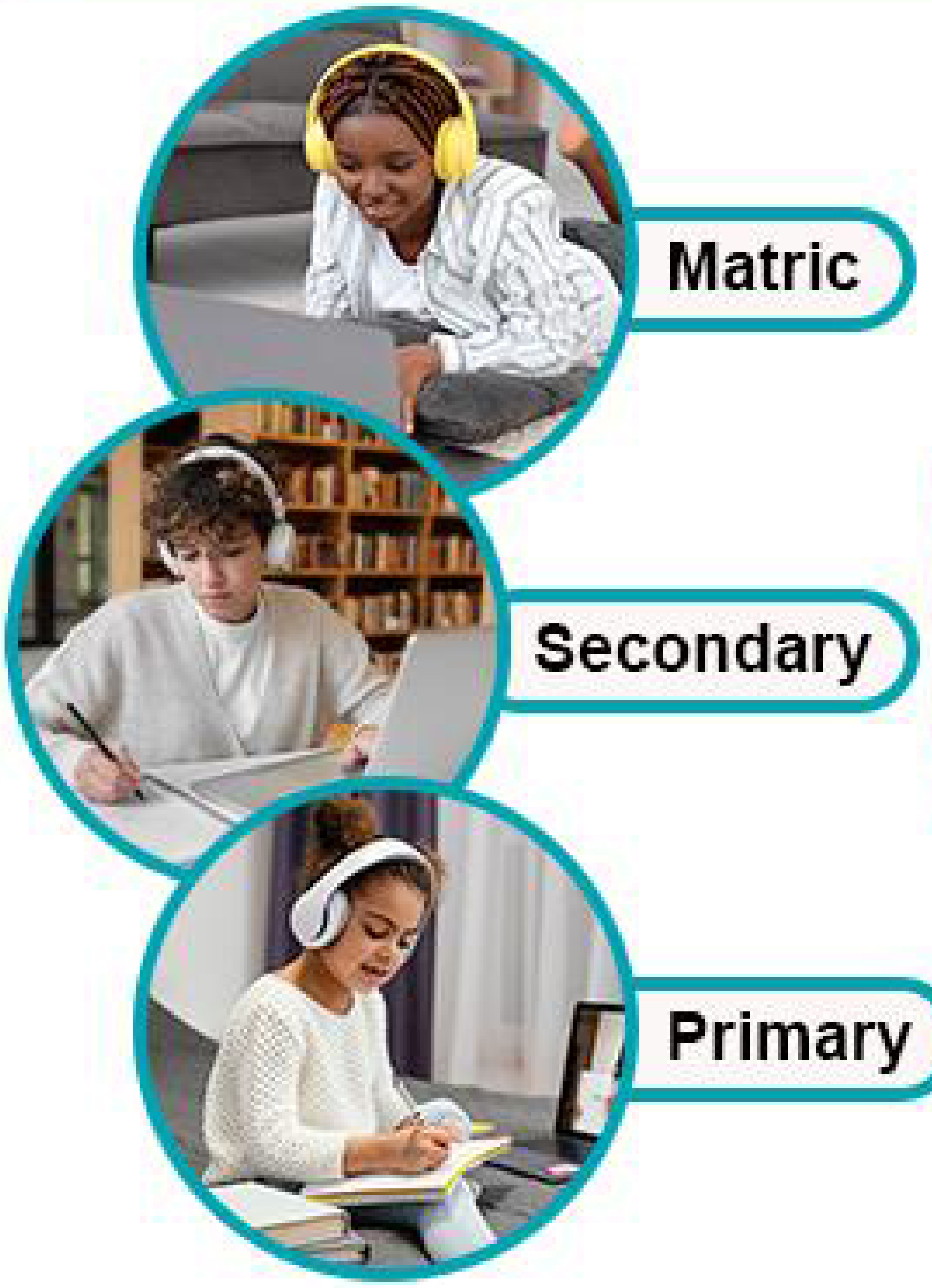

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CHAPTER 16
MAKING THE MOST OF MATHS

WHAT YOU WILL LEARN IN THIS CHAPTER

PAGE	CONTENT	CAPS	ASSESSMENTS YOU WILL BE ABLE TO:
326 326 326 327	**A. Maths in the Classroom** **B. Exam Study Techniques** **C. The Morning of the Exam** **D. Exam Writing Techniques**		• Get the most out of mathematics in your classroom. • Study with confidence and use your time effectively. • Relax and walk into your exam with confidence. • Write your exam neatly and use your time efficiently.

A. MATHS IN THE CLASSROOM

- Up to **80% of learning** should be achieved in the **classroom.** Try to sit as close as possible to the front of the classroom where you can see and hear properly.

- **Concentrate** and avoid the temptation to talk to your friends. Missing an explanation could result in a misunderstanding of an entire section.

- **Do not be shy!** If you don't understand an explanation that your teacher has given, put your hand up and say so. **Ask** your teacher to explain it again until you do understand. If you did not understand a concept, it is likely that the majority of your class is in the same position.

B. EXAM STUDY TECHNIQUES

- **Practising past exam papers** is the most helpful and practical form of revision. Past papers expose you to the required level of work, exam format and the necessary time limitations imposed on you.

- **Make sure you start studying early.** You should start with past papers at least two weeks before the exam. Make sure that as you do each question you check your answer. If you check your answers after you have completed the whole paper, you cannot remember what each question was about. This means that you do not learn from your mistakes. It is important that you revise each section before starting past papers.

- **In order to achieve the correct pace**, while timing yourself, do an entire past paper and only check your answers at the end of the paper. This should be done two to three days before your exam.

- Ensure the room that you are studying in is **brightly lit** and **properly ventilated**.

- Cellular phones can be your worst enemy whilst studying. **Turn your phone off** and check your messages only when you have finished studying.

C. THE MORNING OF THE EXAM

- Make sure that **everything you need for your exam is packed the night before** and ready to take in the morning. Ensure your **calculator is in the correct mode** with charged batteries. It is a good idea to take a spare calculator into your exam.

- **Sleep is very important.** Avoid going to bed too late or waking up very early to study. A maths test or exam requires a fresh brain.

- **Eat Breakfast!** Hunger affects concentration.

- **Avoid discussing work** before the exam. This is likely to cause confusion.

- You've worked hard and are **prepared** - now relax and enjoy the exam.

D. EXAM WRITING TECHNIQUES

- **Read through the entire question paper before you start**. You can answer the questions in any order you like. Answer the questions that you are most confident in first. However, you must answer an entire question in the same section, don't answer parts of a question in different places in your answer book. If you are unsure of a certain part of the question, leave it out and come back to it.

- **If you leave out part of a question, make sure you leave enough space to answer it**. Always leave at least half a page open if the question is worth more than two marks. If you are not sure how much space will be needed, be safe and leave a full page.

- **Mark questions** that you have **left out** on the question paper, so that you remember to come back to them.

- **Neatness is crucial**. Keeping your solutions **neat** and **legible** during an exam may result in up to a 5% increase in your results compared to an untidy and illegible answer book. Neat workings also assist you, as it is easier to check answers and find mistakes. In addition, neatness helps to avoid silly mistakes. Do not start a new question at the end of a page.

- **Do not** attempt to **save space** in your exams! Cramped work is messy and difficult to mark. Leave at least one line between answers. Similarly, do not write complicated fractions on a single line.

- **Never cross out an answer, unless** you have already written another answer to replace it. Once you have what you feel is the correct answer, only then cross out the incorrect workings. If you leave two solutions without crossing one out, only the first solution will be marked.

- **Do not use** any type of **correction fluid** (eg. tippex) in a test or exam.
  Rather **neatly cross out your answer** and carry on. Erasable pens should also not be used.

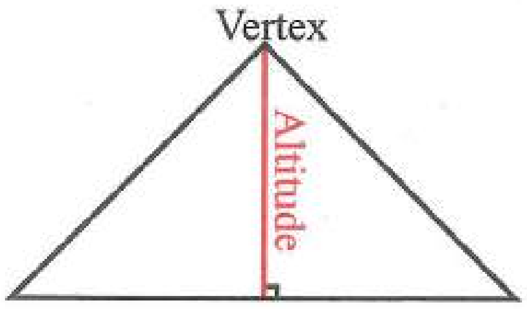
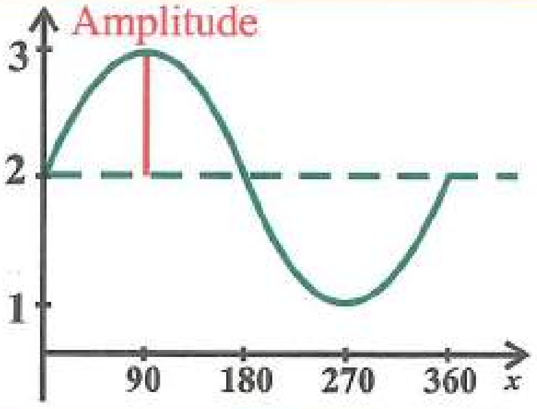
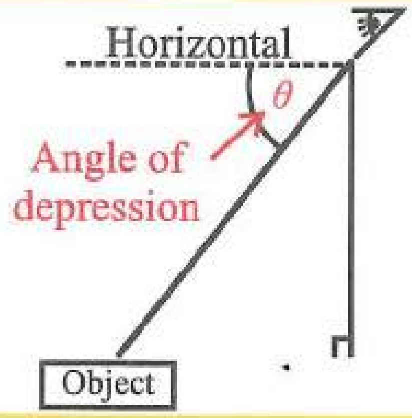
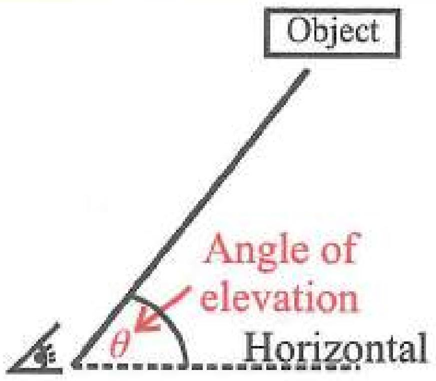
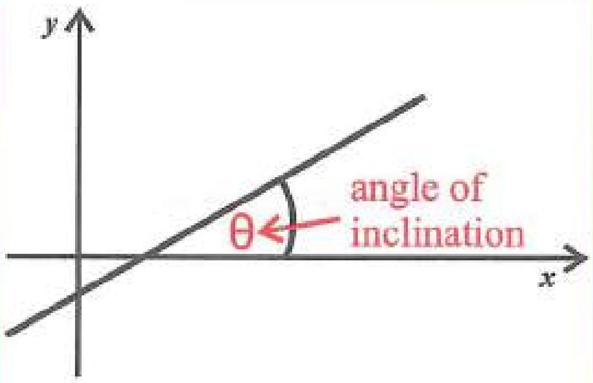
- **Follow the layout** shown in this book. Keep to the top to bottom layout and the exam will be a pleasure to write and mark.

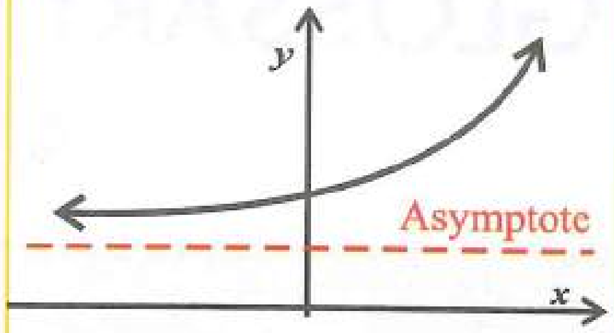
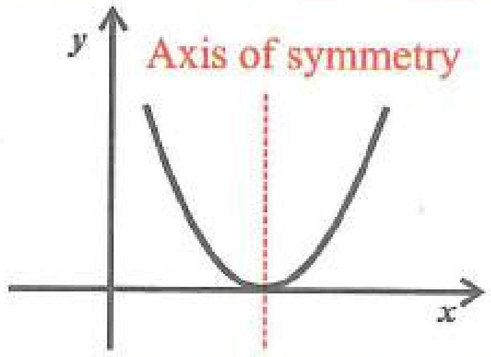
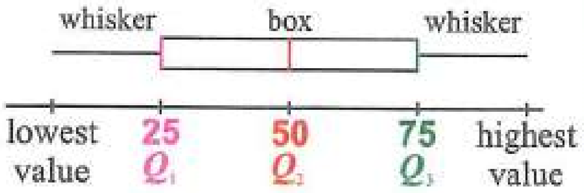
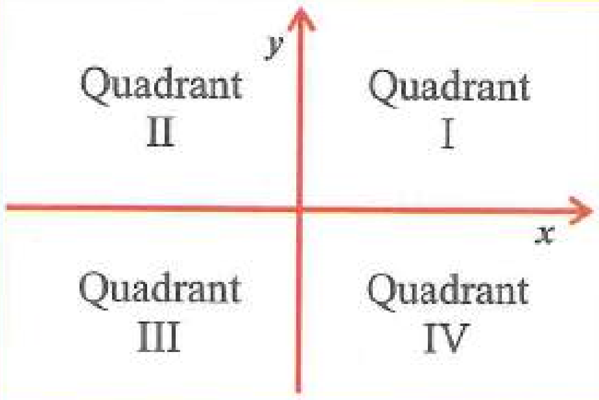
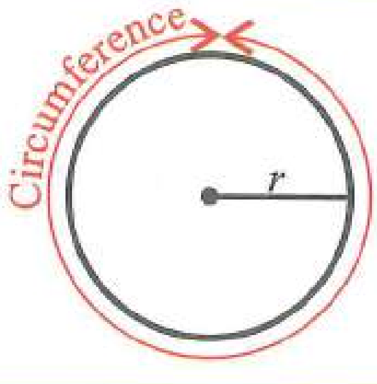
- An answer book that has the final answer of each question **underlined with an arrow** is very pleasant for a teacher to mark. However, this can be very costly in terms of time. **If you are a slow or untidy worker, use a ruler** and neatly underline your answers with arrows, **only once you are finished** your test or exam and have **checked** through **your solutions**. Make sure that you don't cross out your answer when doing this.

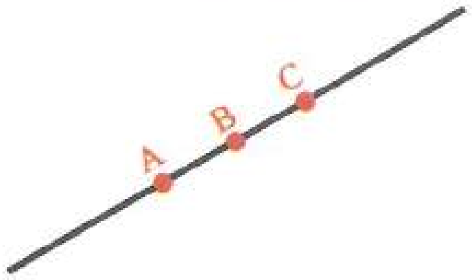
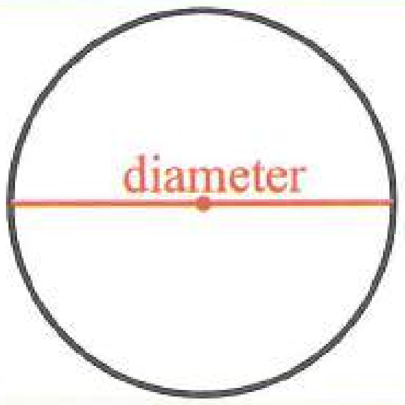
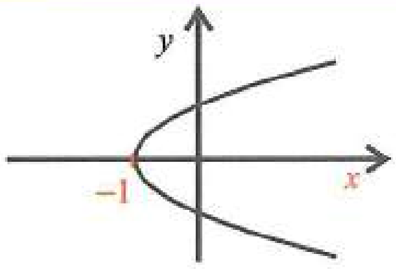
- Finally, **relax** and be **positive**. It is important that you **don't panic** at any stage during the exam.

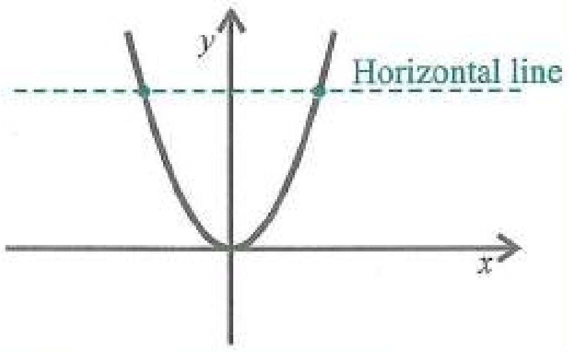
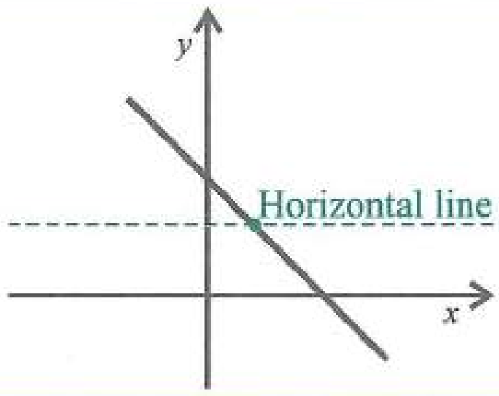
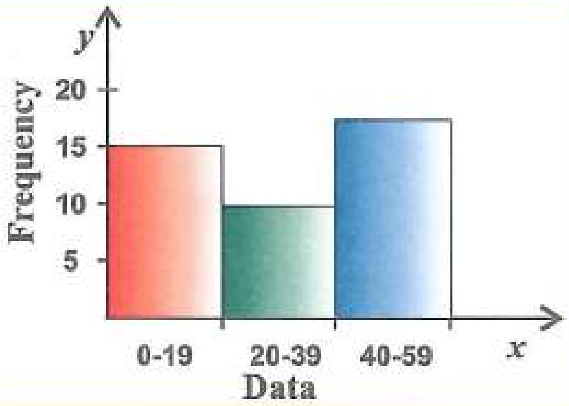
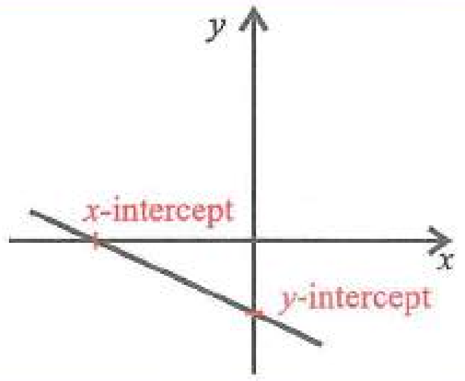
Good luck – you have earned it!

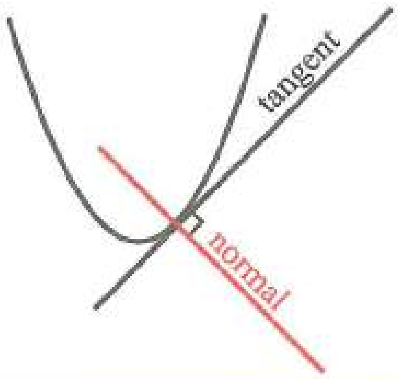
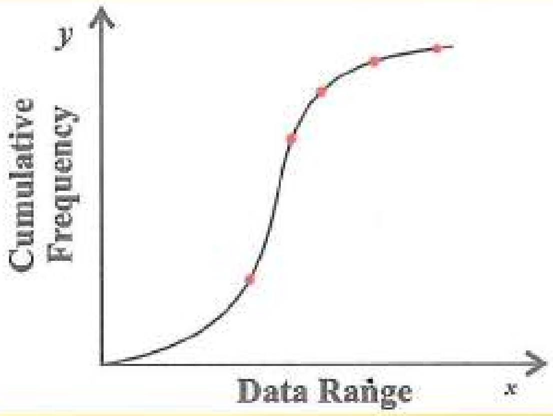
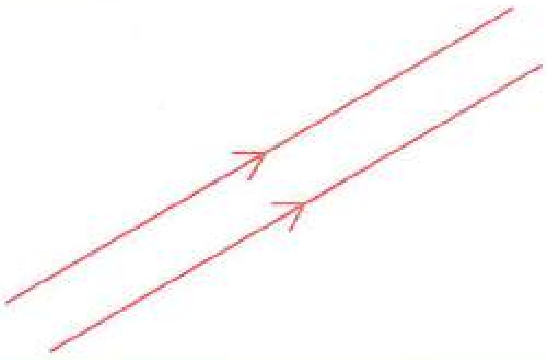
CHAPTER 17
GLOSSARY OF IMPORTANT TERMS AND SYMBOLS

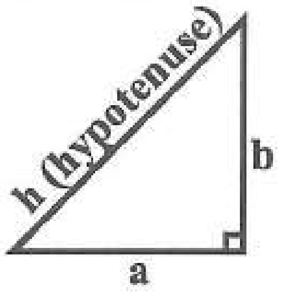
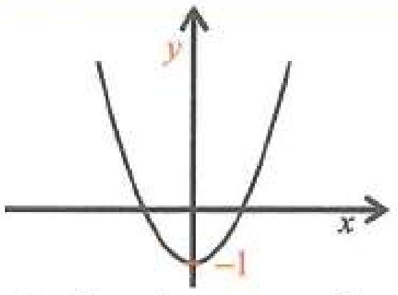
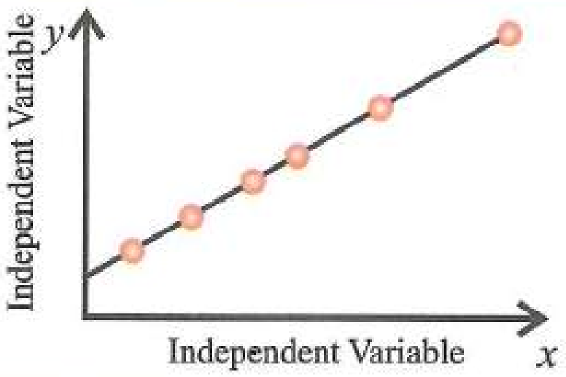
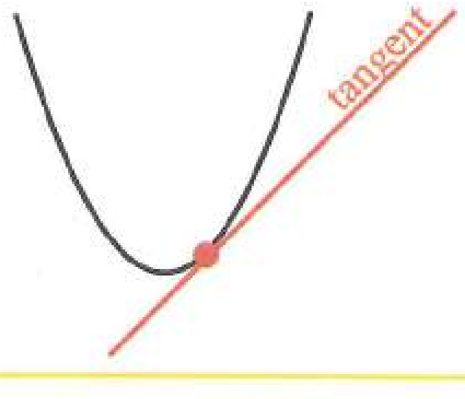
Term	Explanation	Example (where possible)
altitude	**A perpendicular line drawn from one side of a figure to the vertex of the other side.**	 Vertex Altitude
amplitude	**The length of maximum deviation from the zero line.** $\text{Amplitude} = \frac{y_{\max} - y_{\min}}{2}$	
angle of depression	**The angle between the horizontal and the line of sight when looking down at an object.**	 Horizontal θ Angle of depression Object
angle of elevation	**The angle between the horizontal and the line of sight when looking up at an object.**	 Object Angle of elevation θ Horizontal
angle of inclination	**The angle measured anticlockwise from the positive side of the x axis. It is always between 0° and 180°.**	 y angle of inclination θ x
arithmetic sequence	**A sequence where the difference between any sequential terms is constant.**	**In the sequence:** 2, 4, 6... **The constant difference** d **is** $T_n - T_{n-1} = 2$.

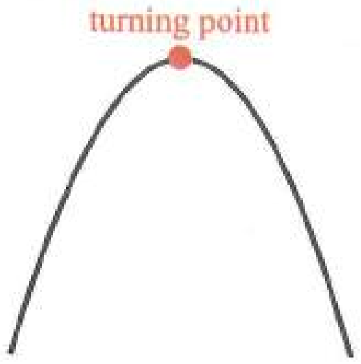
Term	Explanation	Example (where possible)
asymptote	An imaginary line that a graph can never touch.	 y Asymptote x
axis of symmetry	A line drawn so that the graphs or parts of the graph on either side of the line are mirror images of each other.	 y Axis of symmetry x
bivariate data	Data that is given in ordered pairs, an x and y coordinate. It is usually represented using a scatter plot.	The time a student spends studying and the mark that he or she obtains is an example of bivariate data.
box and whisker plots	A visual representation of the median (Q_2), lower quartile(Q_1), upper quartile (Q_3), as well as the highest and lowest values in a set of data, to show the spread around the median.	 whisker box whisker lowest value 25 Q_1 50 Q_2 75 Q_3 highest value
Cartesian Plane	Four quadrants that are separated by an x and y axis	 y Quadrant II Quadrant I x Quadrant III Quadrant IV
circumference	The physical distance around a circle. Formula: $C = 2\pi r$ or $C = \pi d$	 Circumference r
coefficient	The number in front of a variable.	In the expression $2x^2 + 4$ the coefficient of x^2 is 2.

Term	Explanation	Example (where possible)
collinear points	**When 3 points lie on the same line they are collinear.**	 $m_{AB} = m_{BC} = m_{AC}$
constant term	**The term in an expression that does not contain any variables.**	**In the expression** $2x^2 + bx + 4$ **the constant term is** 4**.**
denominator	**The number at the bottom of a fraction.**	**In the fraction** $\frac{3}{x}$, x **is the denominator.**
derivative (differentiation)	**The rate of change of one variable with respect to another. It can be found from first principles or by using differentiation rules.**	**The derivative of** $f(x) = x^2 + 2x + 3$ is: $f'(x) = 2x + 2$
diameter	**A line through the centre of the circle that touches the circumference of the circle at two points.**	
domain	**The domain of a function is the possible** x **values that the function can have.**	 **In the above graph the domain is** $x \geq -1$
effective interest rate	**The equivalent annual interest rate that would be received due to the effects of compounding interest.** $1 + i_{\text{eff}} = (1 + i_{\text{nom}})^n$	**If the interest rate received is 10% p.a. compounded monthly, the effective rate is:** $i_{\text{eff}} = \left(1 + \frac{0,1}{12}\right)^{12} - 1$

Term	Explanation	Example (where possible)
exponent (and base)	The power that a number or variable is raised to.	In the term x^a, a is the exponent and x is the base.
function (many-to-one)	An equation or graph where there are multiple x values for a single y value.	
function (one-to-one)	An equation or graph where there is a single y value for a single x value.	
geometric sequence	A sequence where the ratio of any two sequential terms is constant.	In the sequence: 4, 8, 16, ... the constant ratio is 2
gradient (m)	The rate of change of y with respect x. $m = \frac{y_2 - y_1}{x_2 - x_1}$	The average gradient between the points (5;2) and (10;5) is: $m = \frac{5-2}{10-5} = \frac{3}{5}$
histogram	A graph without gaps that uses rectangles to represent frequency.	
intercept (x or y)	The point where a line cuts the x-axis (x-intercept) and y-axis (y-intercept).	

Term	Explanation	Example (where possible)
logarithm	A way of expressing the exponent of an equation as the subject of the formula.	if: $a^y = x$ then: $y = \log_a x$
normal	A line that is perpendicular to a tangent at the point of contact.	
numerator	The number or expression at the top of a fraction.	$\frac{x+5}{2}$ $x+5$ is the numerator and 2 is the denominator
ogive	A graph where the data range (x) is plotted against the cumulative frequency (y).	
outlier (of a data set)	A value in a set of data that is much higher or lower than any of the other values in the data set.	2, 100, 125, 126, 130 In the above set of data points 2, is an outlier.
parallel lines (//)	A value in a set of data that is much higher or lower than any of the other values in the data set.	
period	The number of degrees it takes for a trigonometric function to complete one cycle.	Graphs of sine and cosine have a period of 360° whilst tan has a period of 180°

Term	Explanation	Example (where possible)
Pythagoras (Theorem of Pythagoras)	In a 90° triangle the $(\text{hypotenuse})^2$= sum of the squares of the opposite two sides.	 $h^2 = a^2 + b^2$
range	The possible y-values that a graph or function can have.	 In the above graph the range is $y \geq -1$
reciprocal	The reciprocal of a number is one divided by the number. A number multiplied by its reciprocal is 1.	The reciprocal of x is $\frac{1}{x}$
regression line (line of best fit) (trend line)	A line that shows the general shape of a scatter plot, thus showing the overall trend of the data.	
root (of an equation)	The value of a variable that satisfies an equation.	The root of $x-5=0$ is 5
surd form	An irrational number that is written with a root symbol (usually in its simplest form).	$\sqrt{50}$ written in simplest surd form is $5\sqrt{2}$
tangent	A straight line that touches a graph at a single point. The gradient of the tangent is equal to the gradient of the graph at the point of contact.	

Term	Explanation	Example (where possible)
turning point	A point on a curve where the gradient at the point is equal to zero. At the TP the derivative of the function equals zero.	
quadratic formula	Used to solve for the roots of any quadratic equation.	The roots of $ax^2+bx+c=0$ are: $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

GLOSSARY OF IMPORTANT SYMBOLS

Symbol	Meaning	Symbol	Meaning
$\sqrt{\ }$	square root	$\in$	element (of a set)
$\sqrt[n]{\ }$	nth root	$\neq$	not equal to
$\mathbb{N}$	natural numbers	$\approx$	approximately equal to
$\mathbb{N}_0$	whole numbers	$A > B$	A is greater than B
$\mathbb{Z}$	integers	$A < B$	A is less than B
$\mathbb{Q}$	rational numbers	$A \geq B$	A is greater than or equal to B
$\mathbb{Q}'$	irrational numbers	$A \leq B$	A is less than or equal to B
$\mathbb{R}$	real numbers	$\therefore$	therefore
$\cap$	intersection (and)	$\cup$	union (or)
!	factorial	σ	standard deviation

Notes

Use these pages to make your own important notes

CHAPTER 18
EXERCISE ANSWERS, INVESTIGATION, PROJECT AND ASSIGNMENTS

Chapter 1 Number Patterns, Sequences and Series (Answers)

Sequences (progressions)

Exercise 1:

1)

a) $T_{12}=82$

b) $T_{11}=-3072$

c) $T_8=\frac{3}{2048}$

d) $T_{21}=-58$

e) $T_{18}=4+34x$

f) $T_{25}=ab^{24}$

2)

a) $n=17$

b) $n=22$

c) $n=7$

d) $n=7$

3)

a) $T_n=3(2)^{n-1}$

b) $T_n=6+(n-1)(-5)=11-5n$

c) $T_n=\frac{1}{2}+(n-1)\left(\frac{5}{2}\right)=-2-\frac{5}{2}n$

d) $T_n=\frac{1}{9}(-3)^{n-1}$

4)

a) $x=-1$

b) $n=15$

5)

a) $q=\frac{1}{2}$

b) $T_{15}=\frac{71}{2}$

6)

a) $T_1=9;\ T_2=6;\ T_3=4$

b) $n=8$

7) $x=\frac{1}{2}$ and $y=-3$ or $x=8$ and $y=12$

Arithmetic and Geometric Series

Exercise 2:

1)

a) $S_{52}=1378$

b) $S_{17}=-255$

c) $S_{16}=-65535$

d) $S_9=\frac{8687}{8}$

e) $S_{81}=1660,5$

f) $S_{28}=3164$

2)

a) $d=2x$

b) $x=-10$ or $x=3$

3)

a) $S_n=n^2$

b) $S_n=n^2+n$

4) $a=-8$ and $d=4$

5) $r=\frac{2}{3}$ and $n=3$

6) $S_{12}=360\ (a=-3$ and $d=6)$

Sigma Notation

Exercise 3:

1)

a) $S_{12} = 144$

b) $S_8 = 765$

c) $S_9 = -270$

d) $S_9 = 59\,046$

2)

a) $n = 18 \quad \left(n \neq -\dfrac{33}{2}\right)$

b) $n = 24$ (there are 18 terms in the sequence)

3)

a) $n = 7 \quad (n \neq -9, 31)$

b) $n = 4$ (there are 3 terms in the sequence)

Infinite Geometric Sequences

Exercise 4:

1)

a) $S_\infty = 10$

b) $S_\infty = \dfrac{9}{2}x$

c) $S_\infty = \dfrac{200}{9}$

d) $S_\infty = 2$

2)

a) $\dfrac{325}{99}$

b) $\dfrac{1562}{999}$

3)

a) $-2 < x < 0$

b) $-5 < x < -1$
Note that if $x = 0$ the series will be trivial as it would only contain zeroes.

Applications of Sequences and Series

Exercise 5:

1)

a) $T_4 = 6\text{mm}$

b) $S_\infty = 60$ mm

2)

a) 120 m

b) 900 m

3)

a) $T_5 = 0{,}5\pi$ cm

b) $S_4 = 15\pi$ cm

c) $S_\infty = 16\pi$ cm

4) $S_6 = \text{R}1229114{,}70$

Mixed Exercise:

1) $S_{100} = 5050 + 5350 = 10\,400$

2)

a) $x = 18$

b) $x = \pm 8\sqrt{2}$

3)

a) $S_5 = 726$

b) $S_4 = 240$

c) c) $T_5 = 486$

4)

a) $T_{40} - T_{41} = 4718592 - 126 = 4718466$

b) Show that each sequence is a multiple of 3.

5) **a)**

n	1	2	3	4	5
Sequence A	4	6	8	10	**12**
Sequence B	3	8	15	24	**35**
Sequence C	5	10	17	$\mathbf{\sqrt{10^2+24^2}=26}$	37

b) $T_n = 2+2n$ **c)** $T_n = n^2+2n$ **d)** $T_n = \sqrt{(2+2n)^2+\left(n^2+2n\right)^2}$

6)

a) $d = \text{R}250$ **b)** In the 16th year.

7)

a) $-\frac{1}{3} < x < \frac{1}{3}$ **b)** $x = -\frac{1}{6}$

8) $r = -\frac{2}{3}$

$\therefore$ The sequence is: $9; -6; 4; \ldots$

9)

a) Second difference $= 4$ **b)** $x = 6$

10)

a) 82,5 km **b)** 15 km

11)

a) 1,5 m **b)** $T_n = 1{,}5(0{,}5)^{n-1}$ **c)** No, $S_\infty = 3\,\text{m}$

12) $T_{12} = \frac{54}{7}$

13) Proof

Chapter 2 Functions, Inverses and Logarithms (Answers)

Classifying Functions and Inverse Functions

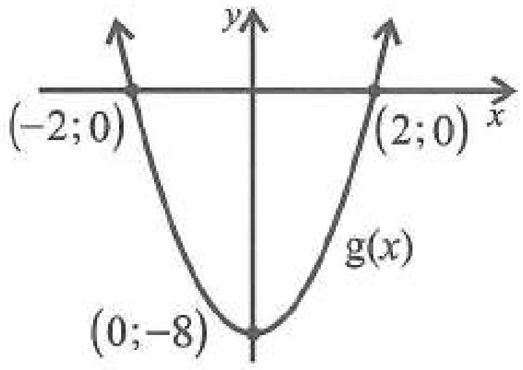
Exercise 1:

a) One-to-one function
b) Not a function
c) Many-to-one function
d) Not a function
e) Many-to-one function
f) Not a function
g) Many-to-one function
h) One-to-one function
i) One-to-one function

Exercise 2:

1)

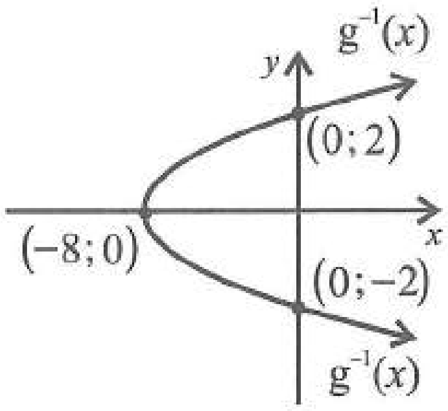
a)



b) Domain: $x \in \mathbb{R}$
Range: $y \geq -8$

c) $g^{-1}(x) = \pm\sqrt{\frac{x}{2}+4} \quad ; x \geq -8$

d)



e) Domain: $x \geq -8$
Range: $y \in \mathbb{R}$

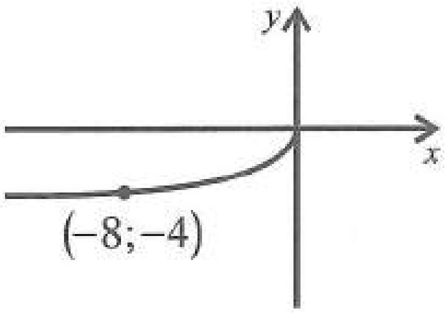
2)

a) $a = -\frac{1}{2}$

b) $h^{-1}(x) = -\sqrt{-2x}$ and $x \leq 0$

c) Range: $y \leq 0$

d)



e) $y = \sqrt{-2x}\ ;\ x \leq 0$

3)

a) $y = \pm\sqrt{2x} \quad ; x \geq 0$

b) $y = \pm 4$

c) Domain: $x \geq 0$
Range: $y \in \mathbb{R}$

4)

a) It is a many-to-one function as it fails the horizontal line test where $x > 2$.
b) $y \in (0;3]$
c) No, only the inverse of a one-to-one function will be a function.
d) $x \leq 2$ (you have to restrict the domain so that *f* will be a one-to-one function).

5) The turning point of the parabola is $(-5;3)$. Therefore the domain has to be restricted from -5.
$\therefore x \leq -5$ or $x \geq -5$, either answer would be correct.

Logarithms

Exercise 3:

1)

a) $5^2 = 25$
b) $28^3 = 21952$
c) $21^2 = 441$
d) $7^5 = 16807$
e) $5^x = 7$
f) $x^a = 9$

2)

a) $2^3 = 8$
b) $3^4 = 81$
c) $5^1 = 5$
d) $3^5 = x$
e) $10^{0,477} = 3$
f) $6^{-4} = 1296^{-1}$
g) $x^2 = 64$
h) $10^3 = 1000$
i) $10^5 = x$

3)

a) $x=3$

b) $x=1$

c) $x=\frac{1}{3}$

d) $x=0$

e) $x=5$

f) $x=256$

g) $x=1$

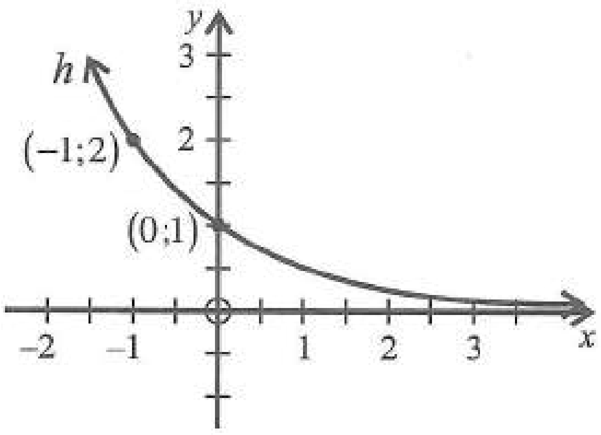
h) $x=8$

The logarithmic Function

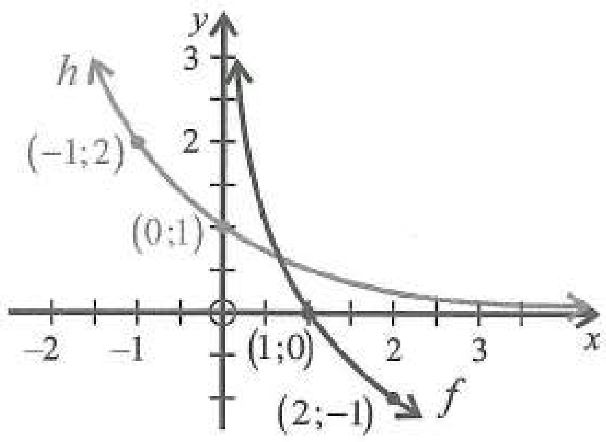
Exercise 4:

1)

a)



d)



b) $y>0$

c) $y=\log_{\frac{1}{2}} x$

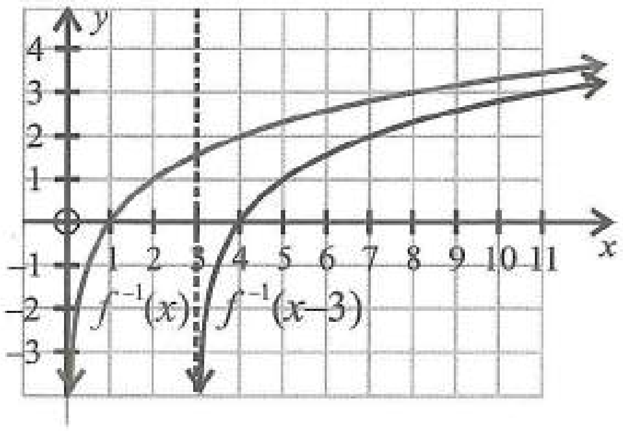
e) $g(x)=-\left(\frac{1}{2}\right)^x$

f) $p(x)=-\left(\frac{1}{2}\right)^{x+4}-2$

2)

a) $f^{-1}(x)=\log_2 x$

b)



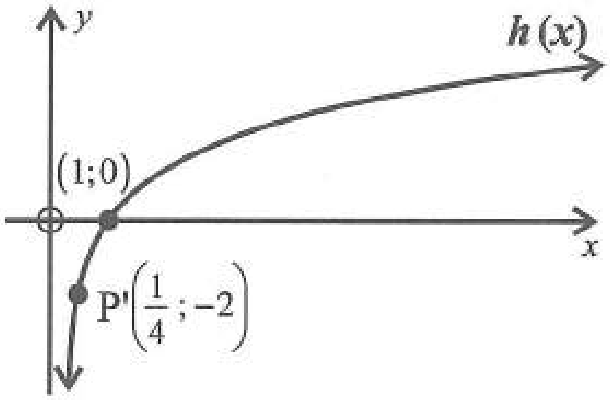
c) $x\in(3;11]$

3)

a) $a=2$

b) $y=\log_2 x$

c) $x>\frac{1}{4}$



4)

a) Q is the point $(0;1)$

b) $b=-2$

c) $y=\log_{\frac{1}{2}} x$

d) $g(x)=-\log_{0,5} x=\log_2 x$

e) $p(x)=\left(\frac{1}{2}\right)^x+4$

5)

a) $f(x)=\log_{\frac{1}{3}} x$

b) $f(x)=\left(\frac{3}{2}\right)^{-x}=\left(\frac{2}{3}\right)^x$

c) $f(x)=\log_4 x$

6)

a) $g(x)=\log_3 x$ and $h(x)=-\log_3 x$

b) $x\in\mathbb{R}$

c) $x\geq 1$

Laws of Logarithms

Exercise 5:

1)

a) $\log 4 - \log 10 - \log 4 = -\log 10$

b) $5\log 2 + 2\log 5 = \log 800$

c) $\log_2 6 + 3\log_2 2 = \log_2 48$

d) $\log(x+4)^2 - \log(x^2-16) = \log\left(\frac{x+4}{x-4}\right)$

2)

a) $\log(abc) = \log a + \log b + \log c$

b) $\log(10ab) = \log 10 + \log a + \log b$

c) $\log\left(\frac{xy}{z}\right) = \log x + \log y - \log z$

d) $\log\left(\frac{x+1}{x-1}\right) = \log(x+1) - \log(x-1)$

Mixed Exercise:

1)

a) $p = -2$

b) $p = 0$ or $p = \frac{1}{2}$

c) 15

2)

a) Either $x \in (-\infty; 0]$ or $x \in [0; \infty)$

b) $p^{-1}(x) = \log_{0,5} x$

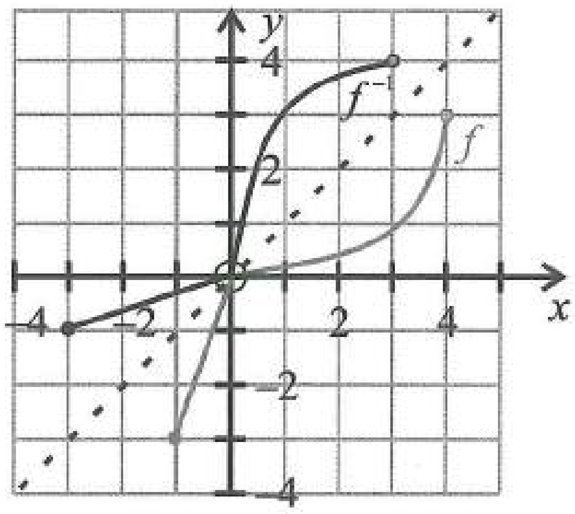
c) $g(x) = \frac{x}{2} + 2$

3)

a) Domain: $x \in [-1; 4)$

Range: $y \in [-3; 3)$

b)



c) The original function is a one-to-one function. Therefore, the inverse is also a function.

4) $y = \frac{2x}{x-1} = g(x)$

5)

a) $f^{-1}(x): y = 3^{-x}$

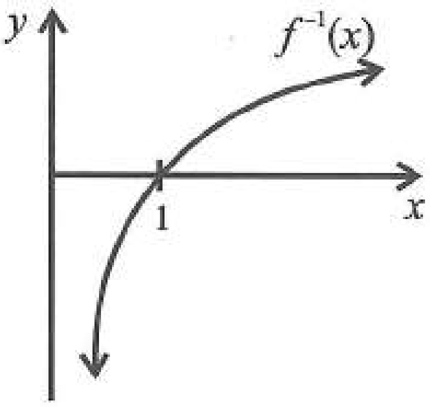
b) $y = 0$

c) Reflect the graph of f about the x-axis.

6)

a) $f^{-1}(x) = \log_3 x$

b)



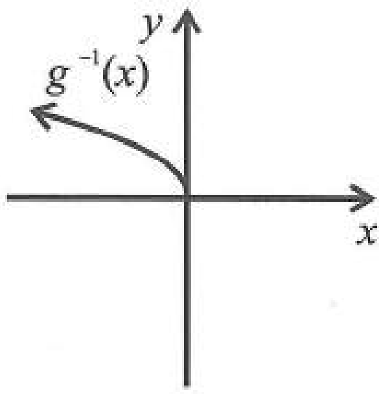
c) Domain: $x \in (0; \infty)$

Range: $y \in \mathbb{R}$

d) The inverse is not a function. Only the inverse of one-to-one functions are functions themselves. The graph of g is a many-to-one function.

e) Either $x \geq 0$ or $x \leq 0$.

f)



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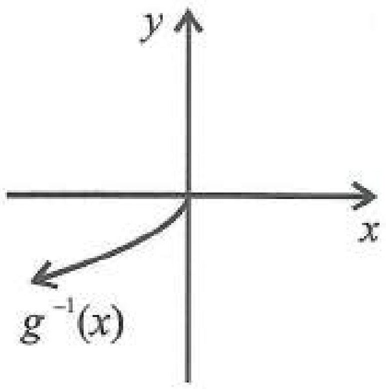


g) The graph of f needs to be reflected about the line $y = x$ and then reflected about the x-axis.

h) The graph of f needs to be reflected about y-axis and then translated 1 unit up.

Chapter 3 Financial Mathematics (Answers)

Future Value Calculations

Exercise 1:

1)
- a) R2333735,66
- b) R8025167,93
- c) R5691432,27
- d) R59905,38

2)
- a) R67959,45
- b) R333082,58
- c) R265123,13
- d) R11324,99

3)
- a) R33270,71
- b) R935,80

4) 264 months

5) 2 years

Future Value Calculations when payments are made at the beginning of the period

Exercise 2:

1)
- a) R280800
- b) R2478063,50
- c) R2458396,33
- d) Interest is earned for 1 extra month in b) as payments started at the beginning of the month.

2) R389290

3) R424274,04

Present Value Calculations

Exercise 3:

1)
- a) R4270,93
- b) R59216,67

2)
- a) R10467,74
- b) R412222,42

3)
- a) R100000
- b) R56180,96

Loans with Delayed Payments (Deferred Payments)

Exercise 4:

1)
- a) R154426,34
- b) R3057,83

2)
- a) R183163,78
- b) R4040,33

3)
- a) R360223,59
- b) R8094,19

Final Payments with Reduced Values

Exercise 5:

1)
- a) 131 payments of R6 000.
- b) R1 594,66
- c) R1 606,62

2)
- a) 32 payments.
- b) R28586,66
- c) R30373,32

Mixed Exercise:

1) 6 years.

2) 8% per annum, compounded monthly.

3)
- **a)** R975462,46
- **b)** R11944
- **c)** R12338,24

4)
- **a)** 9 full months.
- **b)** R2285,61
- **c)** R2301,23

5) R3 360 679,49

6) R4375,52

7)
- **a)** R2574,16
- **b)** R97647,74
- **c)** R2872,71

8)
- **a)** R3976,49
- **b)** R328088,26
- **c)** 111 payments of R5000
- **d)** R105,26

9) R7179,00

10)
- **a)** R578677,74
- **b)** No Solution. This means that Stephen will never have as much as Bongani.

Chapter 4 Trigonometry (Answers)

Grade 11 Revision

Exercise 1:

1)

- **a)** 1
- **b)** $-\tan\theta$
- **c)** -1
- **d)** $\frac{1}{4}$

2) Proofs

3) $\frac{12}{25}$

Compound Angles

Exercise 2:

1)

- **a)** $\sin\theta\cos 50^\circ + \sin 50^\circ \cos\theta$
- **b)** $\sin\theta\cos 25^\circ - \sin 25^\circ \cos\theta$
- **c)** $\cos 3x\cos 40^\circ + \sin 3x\sin 40^\circ$
- **d)** $\cos 2x\cos y - \sin 2x\sin y$

2)

- **a)** $\cos 2\mathrm{A}$
- **b)** $\sin 2\theta$
- **c)** $-\cos(x+y)$
- **d)** $\sin 10^\circ \rightarrow \cos 80^\circ$ is also correct

3)

- **a)** $\frac{\sqrt{3}}{2}$
- **b)** $\frac{\sqrt{3}}{2}$
- **c)** 1
- **d)** $-\frac{\sqrt{3}}{2}$

4) Proofs

5) Proofs

6)

- **a)** $\frac{\sqrt{6}-\sqrt{2}}{4}$
- **b)** $\frac{\sqrt{2}-\sqrt{6}}{4}$

7)

- **a)** $\mathrm{A} = 180^\circ - (\mathrm{B}+\mathrm{C}) = 180^\circ - \mathrm{B} - \mathrm{C}$
- **b)** Proof

Double angles

Exercise 3:

1)

- **a)** $\frac{1}{2}$
- **b)** $\frac{\sqrt{3}}{2}$
- **c)** 1
- **d)** $\frac{\sqrt{2}}{2}$
- **e)** $\frac{1}{4}$
- **f)** $-\frac{\sqrt{3}}{2}$
- **g)** $\sqrt{2}$
- **h)** $\frac{3}{2}$

2)

- **a)** $\sqrt{1-p^2}$
- **b)** $\sqrt{1-p^2}$
- **c)** $1-2p^2$
- **d)** $\frac{\sqrt{1-p^2}-p\sqrt{3}}{2}$

3)

- **a)** 2
- **b)** $\frac{\sqrt{2}}{4}$
- **c)** $-\frac{\sqrt{3}}{2}$
- **d)** 4

Trigonometric Identities

Exercise 4:

1) Proofs

2)

- **a)** Proof
- **b)** $\frac{\sqrt{2}}{4}$

3)

- **a)** Proof
- **b)** $\sqrt{3}$

4)

- **a)** Proof
- **b)** $\frac{\sqrt{2}}{2+\sqrt{2}}=\sqrt{2}-1$

Grade 11 Trigonometric Equations Revision

Exercise 5:

1)

- **a)** $\theta=0^\circ$ or $\theta=30^\circ$ or $\theta=150^\circ$ or $\theta=180^\circ$
- **b)** $\theta=\pm 60^\circ+360^\circ k$ or $\theta=\pm 120^\circ+360^\circ k\ ;\ k\in\mathbb{Z}$
- **c)** $\theta=-19{,}5^\circ+360^\circ k$ or $\theta=199{,}5^\circ+360^\circ k\ ;\ k\in\mathbb{Z}$
- **d)** $\theta=-45^\circ+180^\circ k$ or $\theta=26{,}6^\circ+180^\circ k\ ;\ k\in\mathbb{Z}$

Trigonometric Equations Using Double and Compound Angles

Exercise 6:

1)

- **a)** $\theta=\pm 109{,}47^\circ+360^\circ k;k\in\mathbb{Z}$
- **b)** $\theta=-45^\circ+180^\circ k$ or $\theta=-18{,}43^\circ+180^\circ k;k\in\mathbb{Z}$
- **c)** $\theta=\pm 90^\circ+360^\circ k$ or $\theta=\pm 0^\circ+360^\circ k;k\in\mathbb{Z}$
- **d)** $\theta=-71{,}57^\circ+180^\circ k$ or $\theta=-45^\circ+180^\circ k;k\in\mathbb{Z}$
- **e)** $\theta=15^\circ+180^\circ k$ or $\theta=-45^\circ+180^\circ k;k\in\mathbb{Z}$
- **f)** $\theta=-5^\circ-180^\circ k$ or $\theta=-75^\circ-180^\circ k;k\in\mathbb{Z}$

2)

- **a)** Proof
- **b)** $x=\pm 90^\circ+360^\circ k$ or $x=\pm 120^\circ+360^\circ k\ ;\ k\in\mathbb{Z}$

3)

- **a)** Proof
- **b)** $x=-135^\circ$ or $x=-45^\circ$ or $x=45^\circ$ or $x=135^\circ$

Mixed Exercise:

1)

- **a)** $-\tan x$
- **b)** $\frac{\sqrt{3}}{2}$
- **c)** $\frac{1}{2}$
- **d)** 2

2)

a) Proof

b) $\theta = \pm 180^\circ + 360^\circ k;\ k \in \mathbb{Z}$

3)

a) Proof

b) $\theta = \pm 90^\circ + 360^\circ k$ or $\theta = \pm 120^\circ + 360^\circ k;\ k \in \mathbb{Z}$

4)

a) Proof

b) 0,94

5) $-\dfrac{k}{8\sqrt{1-k^2}}$

6)

a) m

b) m

7) $x = 45^\circ + 180^\circ k$ or $x = \pm 60^\circ + 360^\circ k;\ k \in \mathbb{Z}$

8)

a) Proof

b) Proof

9) $\dfrac{\sqrt{2}}{2}$

Chapter 6 Two and Three Dimensional Trigonometry (Answers)

Grade 11 Revision

Exercise 1:

1)

a) $a = 43{,}5$ units

c) $c = 20{,}0$ units

b) $\theta = 36{,}8^\circ$ or $\theta = 143{,}2^\circ$

d) $\theta = 8{,}6^\circ$

2)

a) 37,1 cm^2

b) 86,4 cm^2

Three Dimensional Trigonometry

Exercise 2:

1)

a) $\text{RQ} = 14{,}28\,\text{m}$ and $\text{SQ} = 11{,}11\,\text{m}$

b) $\text{RS} = 13{,}90\,\text{m}$

2)

a) $\text{A}\hat{\text{D}}\text{B} = 46^\circ$

b) $\text{AD} = 61{,}08\,\text{m}$

c) $\text{CD} = 55{,}36\,\text{m}$

3)

a) $\theta = 62^\circ$

b) $\text{AB} = 42{,}57\,\text{m}$

c) $\text{AD} = 43{,}52\,\text{m}$

4)

a) $\text{PR} = 7{,}17\,\text{m}$

b) $\text{SP} = 6{,}43\,\text{m}$

5)

a) $\text{DB} = \dfrac{y}{\sin\theta}$

b) $\text{AB} = \dfrac{y\sin(\alpha+\beta)}{\sin\theta\sin\alpha}$

6)

a) $\text{AD} = \dfrac{y}{\tan\alpha}$

b) $\text{BD} = \dfrac{y}{\tan\theta}$

c) Area $\triangle\text{ABD} = \dfrac{y^2\sin\beta}{2\tan\alpha\tan\theta}$

7)

a) $\text{DE} = \dfrac{x\tan\theta - y}{\sin\alpha}$

b) The cliffs are 750m apart.

Mixed Exercise:

1)

a) $\dfrac{6\cos\alpha + 6\sqrt{3}\sin\alpha}{\sin\alpha}$

b) Proof

c) $\alpha = 58{,}56^\circ$

2)

a) 294 units^2

b) Proof

c) $\theta = 111{,}04^\circ$

3)

a) Proof

b) 63,6m

4)

a) $\text{PB} = 22{,}65\,\text{m}$

b) $\text{PA} = 19{,}55\,\text{m}$

c) $\text{AB} = 15{,}40\,\text{m}$

5)

a) Proof

b) Proof

c) $\text{AP} = 65{,}07\,\text{m}$

6) Proof

Chapter 7 Algebra and Differential Calculus (Answers)

The Remainder and Factor Theorem and Factorisation of Cubic Polynomials

Exercise 1:

1)

a) $q=1$

b) $n=-3$ and $m=5$

2)

a) $f(x)=(x-1)(x+2)(x+3)$

b) $f(x)=(x-1)(x-2)(x+1)$

c) $f(x)=(x-2)(x+2)(x+3)$

d) $f(x)=(x-3)^2(x+3)$

e) $f(x)=(x-2)(x-2)(x+4)$

f) $f(x)=x(x+3)(x-1)$

Limits

Exercise 2:

1)

a) 8

b) 3

c) 4

d) 0

e) 5

f) $\frac{5}{2}$

2) 2

3) $6x$

Average gradient, gradient at a point and differentiation from first principles

Exercise 3:

1)

a) $m=8$

b) $m=8$

2)

a) $m=-7$

b) $f'(5)=-9$

Exercise 4:

1)

a) $f'(x)=1$

b) $f'(x)=4x$

c) $g'(x)=3x^2$

d) $f'(x)=\frac{1}{2\sqrt{x}}$

2)

a) $1-2x$

b) $f'(5)=-9$. This means that the gradient is -9 at the point where $x=5$.

3)

a) Yes, $f(8)=0$ as it is the x-intercept. $f'(8)$ is the gradient of f at the point where $x=8$. At this point the gradient is positive and therefore greater than 0.

b) No, from the turning point the gradient of f increases as x increases. $\therefore f'(8)<f'(12)$

Differentiation Rules

Exercise 5:

1)

a) $\frac{dy}{dx} = 6$

b) $f'(x) = 0$

c) $g'(x) = -16x$

d) $g'(x) = \frac{1}{2x^{\frac{1}{2}}}$

e) $\frac{dy}{dx} = 4x^3 + 6x + 5$

f) $f'(x) = -\frac{3}{x^2}$

g) $\frac{dy}{dx} = 1$

h) $\frac{dy}{dx} = 6x^2 - 8 - 6x$

i) $f'(x) = 4x^3 + \frac{5}{2}$

j) $f'(y) = -18y - 6$

2)

a) $D_x\left[\left(1+\sqrt{x}\right)^2\right] = \frac{1}{\sqrt{x}} + 1$

b) $\frac{d}{dx}\left[\frac{1}{4}x^4 + 2x^{\frac{1}{2}}\right] = x^3 + \frac{1}{x^{\frac{1}{2}}} = x^3 + \frac{1}{\sqrt{x}}$

3)

a) $f'(x) = 6x^2 + 6x + 1$

b) $f'(3) = 73$

Tangents to Graphs of Functions

Exercise 6:

1)

a) $y = -x + 3$

b) $x = -4$

c) $y = 11x - 6$

2)

a) Point of contact is $(2;2)$

b) Point of contact is $\left(\frac{1}{2};\frac{17}{4}\right)$

3)

a) $y = -3x + 23$

b) $y = 3x - 2$ and $y = 3x + 2$

4)

a) $a = 4$ and $c = 0$

b) $y = -\frac{1}{48}x + \frac{25}{24}$

Stationary Points and Points of Inflection

Exercise 7:

1)

a) $(2;2)$

b) $x \geq 2$

2)

a) $(2;-15)$ and $(-1;12)$

b) $\left(\frac{1}{2};-\frac{3}{2}\right)$

c) $x \in (-\infty;-1]$ or $x \in [2;\infty)$

Mixed Exercise:

1)

a) $\lim_{x \to -1} \frac{x^2 - 1}{x+1} = -2$

b) $\lim_{x \to 0} \frac{x^3 + 5x^2 + 6x}{x^2 + 2x} = 3$

2)

a) $f'(x) = \frac{4}{\sqrt[3]{x}}$

c) $\frac{dy}{dx} = \frac{3}{4x^{\frac{1}{4}}} + 3$

b) $\frac{dy}{dx} = \frac{2}{x^{\frac{1}{2}}} - \frac{2}{x^2}$

d) $h'(x) = -\frac{3}{x^3} + \frac{2}{x^2} + 1$

3)

a) $q = -4$

b) $y = -\frac{1}{8}x + \frac{33}{4}$

4)

a) $(4;22)$

b) $g(x)$ is increasing at the point $(1;13)$

5)

a) $f'(x) = 1$

b) $f'(x) = \frac{-2}{x^2}$

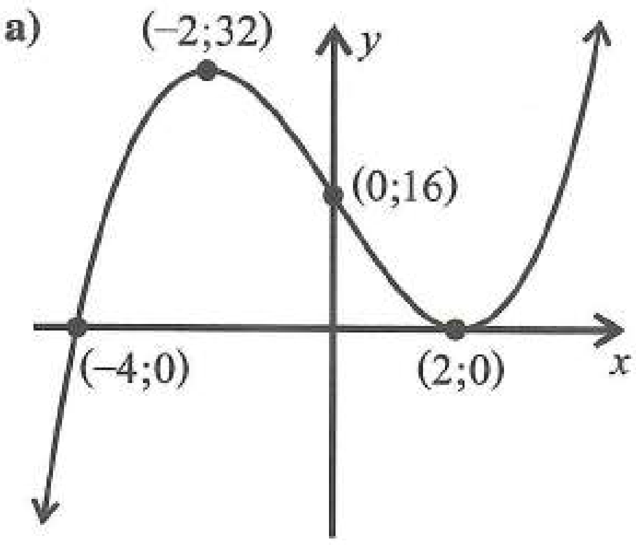
Chapter 8 Curve Sketching and Graph Interpretation (Answers)

Sketching Cubic Functions

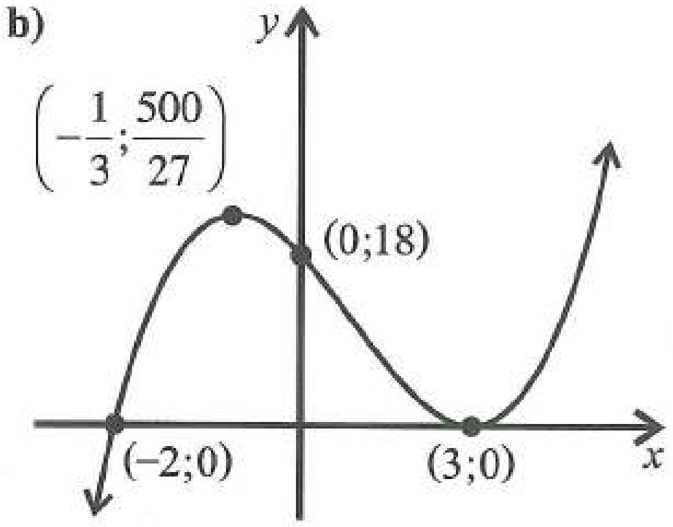
Exercise 1:

1)

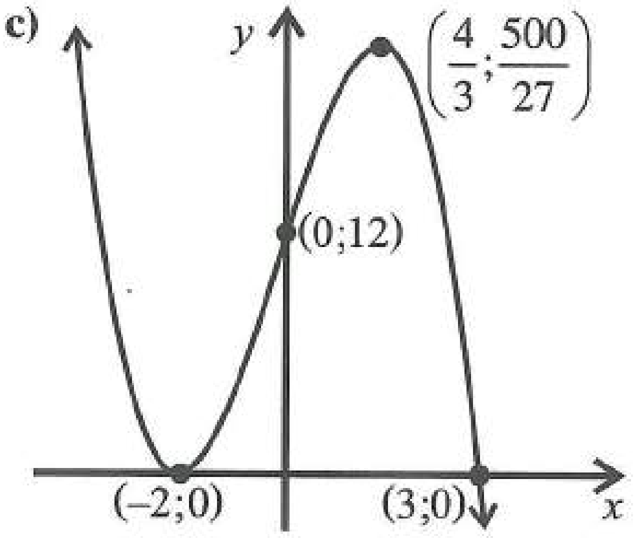
a)



b)



c)



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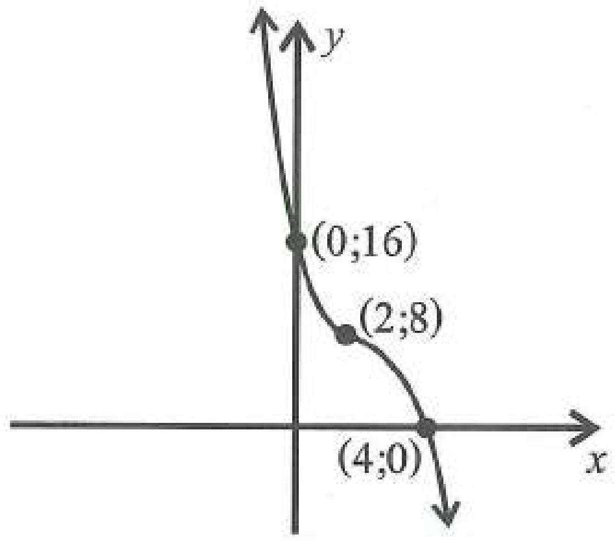

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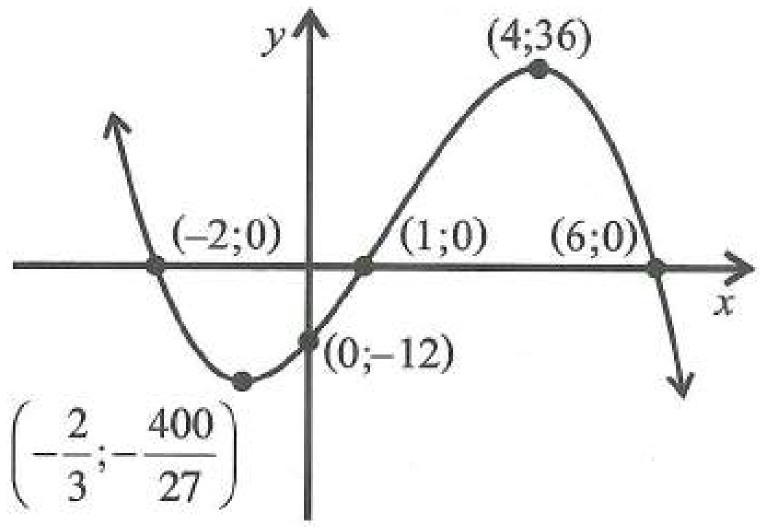
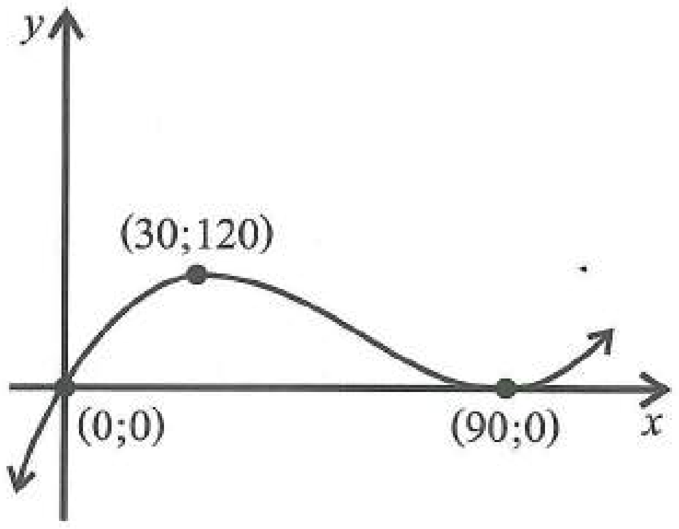


f)



Graph Interpretation

Exercise 2:

1)

a) $a=5$ and $b=-3$

b) $(-2;9)$ and $\left(\frac{1}{3};-\frac{100}{27}\right)$

2)

a) $a=-20$ and $b=-32$ and $c=192$

b) $\left(-\frac{2}{3};\frac{5488}{27}\right)$ and $(4;0)$

3)

a) $b=0$ and $c=3$ and $d=2$

b) $(-1;0)$ and $(2;0)$

4)

a) M is the point $\left(1;-\frac{8}{3}\right)$

b) N is the point $\left(-2;\frac{46}{3}\right)$

5)

a) $a=-3$ and $b=0$

b) C is the point $(2;0)$

c) $y=9x+9$

d) Point of inflection is $(1;2)$

6)

a) C is $(-3;0)$ and A is $(3;0)$

b) Points of contact are: $(-2;25)$ and $(4;7)$

c) The new y-intercept will be $(0;5)$

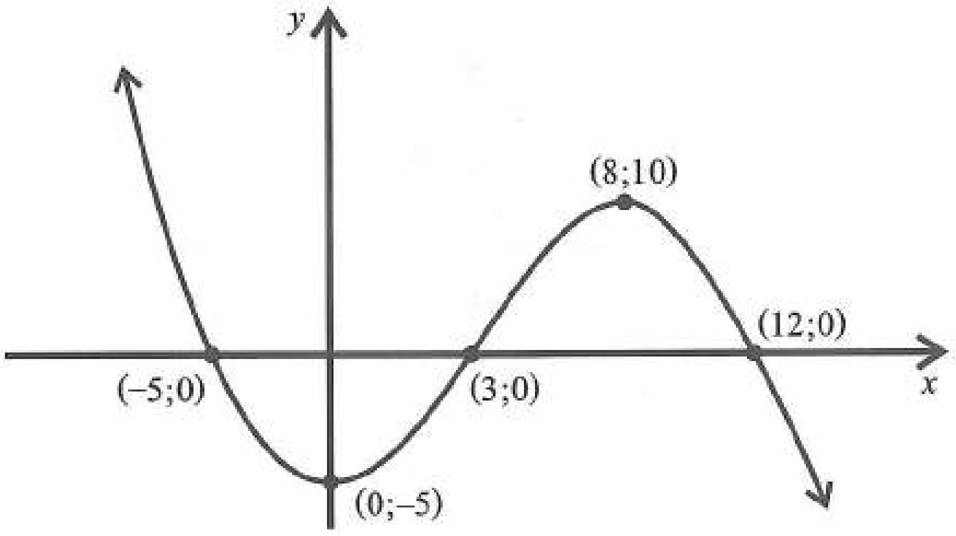
d) $-32 < k < 0$

7)

a) $b=-3$ and $a=2$

b) R is the point $(-1;11)$

8)



Mixed Exercise:

1)

a) AB = 3 units, OC = 6 units and OD = 12 units

b) E is the point $(4;36)$

c) $y = 8x - 12$

2) **It is important to note that the derivative of a cubic function can be represented by a parabola.**

a) Stationary points occur when the derivative is equal to 0. These are the x-intercepts of the parabola. As the parabola is concave up (happy face) the value of a in $f(x)$ is greater than 0.
$\therefore$ local max at $x=-1$ and local min at $x=5$.

b) The point of inflection occurs where the 2nd derivative is equal to 0. This is the turning point of g. The x-value of the turning point is the midpoint of the x-intercepts.
Therefore, the point on g which represents the point of inflection is $(2;-10)$.

c) The graph of g is negative on the interval $2 < x < 5$. Therefore, f is decreasing.

3)

a) Bridge length $= 40\text{m}$

b) Cost = R3000000

c) The mountain is 328m high.

d) The bridge is 162m above the valley.

4)

a) $f'(x) = \frac{1}{3}x^2 - \frac{4}{3}x - 4$

b) $x=-2$ and $x=6$

c) $x=2$

d) A maximum as $f(x)$ is increasing for $x<-2$ and decreasing for $-2<x<6$.

5)

a) $c=43$ and $a=21$ and $b=-60$

b) $y=-24x+26$

6)

a) $f(x)$ is increasing where $f'(x)>0$. This is where $-5<x<1$.

b) $f'(x)$ is increasing up to the turning point of $f'(x)$. The x- value of the turning point of $f'(x)$ lies halfway between the x-intercepts. Therefore, $f'(x)$ is increasing where $x<-2$

c) The point of inflection of f is where the 2nd derivative is equal to 0. This occurs at the turning point of $f'(x)$. Therefore, the point of inflection of f is where $x=-2$.

Chapter 9 Optimisation of Functions and Rates of Change (Answers)

Maxima and Minima Problems

Exercise 1:

1)

a) Proof

b) $x = 10\,\text{cm}$

2)

a) Proof

b) $x = y = 50\,\text{m}$

3)

a) $\text{RS} = -x + 3 - \frac{1}{x}$

b) $\text{RS}_{\text{max}} = 1\,\text{unit}$

4)

a) $\text{SA}(r) = 2\pi r^2 + \frac{800}{r} = 2\pi r^2 + 800r^{-1}$

b) $r = 4\,\text{cm}$ and $h = 8\,\text{cm}$

5)

a) $\text{PQ}(x) = x^2 - 6x + 14$

b) $\text{PQ}_{\text{max}} = 5\,\text{units}$

6) $\text{T} = 19^\circ\text{C}$

Rates of Change

Exercise 2:

1)

a) Initial mass $= 800\mu g$

b) Mass after 5 minutes $= 1350\mu g$

c) Mass is increasing at a rate of $60\mu g/\text{min}$

d) Maximum mass is $1440\mu g$

2)

a) Initial sales $= 8000$ copies

b) 9560 copies

c) 380 copies per month

3)

a) Initial volume $= 100l$

b) Volume is decreasing at a rate of $16l/\text{min}$

c) The tank will be empty after 5 minutes

4)

a) Initial temp $= 28^\circ\text{C}$

b) Temp is decreasing at a rate of $0{,}056\,^\circ\text{C/min}$

c) Min temp $= 21{,}6\,^\circ\text{C}$

Calculus of Motion

Exercise 3:

1)

a) $v(0) = 10\,\text{m/s}$

b) $s_{\text{max}} = 50\,\text{m}$

c) $a(10) = -1\,\text{m/s}^2$

2)

a) $v(t) = s'(t) = 5{,}88t$

b) $a_{\text{on earth}} = 9{,}8\ \text{m/s}^2$

c) $t = 1{,}65s$ to hit the ground

d) $v(1{,}65) = 9{,}70\ \text{m/s}$

3)

a) $v(0) = 8\,\text{m/s}$

b) The brick will not reach the scaffolding

Mixed Exercise:

1)

a) He will make the jump

b) Maximum distance apart is 80 m

c) He travelled 11,72 m

d) He is ascending at 0,075 m/s

2)

a) $L(5) = 33{,}125\,\text{m}$

b) It is decreasing at $6{,}875\,\text{m/h}$

c) Capacity $= 640\,000\,000\,l$

3)

a) Proof

b) $x = 1{,}19\,\text{km}$ and $y = 0{,}56\,\text{km}$ and $\text{A}_{\text{max}} = 7{,}13\,\text{km}^2$

4)

a) It takes 0,5s

b) $v(0{,}5) = 3\,\text{m/s}$

c) He does not make the goal

5)

a) Proof

b) $\text{A}_{\text{max}} = 3(-2\times 3 + 12) = 18\,\text{units}^2$

6)

a) Proof

b) Maximum area is 8 units^2

Chapter 10 Analytical Geometry (Answers)

Grade 11 Revision

Exercise 1:

1)

a) $y=-\frac{1}{2}x-4$

b) $y=2x+1$

c) Point of intersection is $(-2;-3)$

2)

a) $AB=\sqrt{29}$ units

b) $\alpha=111,8^\circ$

c) D is the point $(2;-4)$

d) $\hat{A}=90^\circ$

e) ABDC is either a rectangle or square

f) ABDC is a square

3)

a) $a=-3$ and $p=-1$

b) $q=7$

4) The line AB was rotated through and of $60,26^\circ$ anticlockwise

5)

a) $m_{CD}=4$

b) $y=4x-16$

The Circle

Exercise 2:

1)

a) $x^2+y^2=121$

b) $(x-1)^2+y^2=8$

c) $\left(x+\frac{1}{2}\right)^2+(y-3)^2=7$

d) $(x-a)^2+(y-b)^2=4r^2$

2)

a) Centre: $(0;0)$ and $r=4,5$ units

b) Centre: $(1;0)$ and $r=10$ units

c) Centre: $(-5;-1)$ and $r=3\sqrt{5}$ units

d) Centre: $(a;b)$ and $r=\sqrt{t^2+16}$ units

e) Centre: $(0;0)$ and $r=5$ units

f) Centre: $(-1;0)$ and $r=\sqrt{13}$ units

g) Centre: $(-5;0)$ and $r=1$ unit

h) Centre: $(-4;3)$ and $r=2\sqrt{13}$ units

i) Centre: $\left(-\frac{3}{2};0\right)$ and $r=1$ unit

j) Centre: $\left(-1;-\frac{3}{2}\right)$ and $r=\frac{\sqrt{21}}{2}$ units

k) Centre: $(4;1)$ and $r=\sqrt{37}$ units

l) Centre: $(a;0)$ and $r=a$ units

m) Centre: $\left(-5;\frac{1}{2}\right)$ and $r=\frac{\sqrt{4m+101}}{2}$ units

3)

a) $p=\pm 4$

b) $p=\pm\sqrt{11}$

4)

a) $(x-5)^2+(y+5)^2=100$

b) $(x+4)^2+(y-4)^2=25$

5)

a) Equation of the circle: $x^2+(y-4)^2=5$

b) Equation of the circle: $(x+1)^2+(y-4)^2=8$

6)

a) $(x-5)^2+(y+4)^2=p+41$

b) $p=34$

Intersection of Circles

Exercise 3:

1)

a) Radii are: $r_1 = 2$ units and $r_2 = 3$ units

b) The circles intersect each other at two points

2) Proof

Straight Lines and Tangents to Circles

Exercise 4:

1)

a) Equation of tangent: $y = \frac{3}{5}x + \frac{34}{5}$

b) Equation of tangent: $y = -\frac{1}{4}x - \frac{15}{4}$

c) Equation of tangent: $y = -2x + 10$

d) Equation of tangent: $y = 2x - 8$

e) Equation of tangent: $y = x$

f) Equation of tangent: $y = 9$

2)

a) $r = \sqrt{10}$ units

b) Proof

c) The length of the tangent is $\sqrt{19}$ units

3) $\text{RP} = 6$ units

4)

a) The point of intersection is $(3;3)$

b) The line is a tangent

5)

a) $(x+2)^2 + (y-3)^2 = 20$

b) $p = 7$

c) $y = 2x - 3$

d) T is the point $(4;5)$

Mixed Exercise:

1)

a) $d = \sqrt{245} = 7\sqrt{5}$ units $= 78,3$cm

b) $2,65$ units $= 133$ mm

c) $y = \frac{4}{3}x + 8$

d) Intersection at $\left(-\frac{12}{5}; \frac{24}{5}\right)$

e) The front and rear wheels have a common tangent

2)

a) $\text{M}(-2;3)$ and $r = 5\sqrt{2}$ units

b) $y = -x + 7$

c) D is the point $(-3;10)$

d) $y = \frac{1}{7}x + \frac{73}{7}$

3)

a) $2y - x - 4 = 0$

b) D is the point $(-4;0)$

c) C is the point $\left(1;\frac{5}{2}\right)$

d) $(x-1)^2 + \left(y - \frac{5}{2}\right)^2 = \frac{125}{4}$

e) $y = -2x + 17$

4)

a) $a = 6$ or $a = -6$

b) Centre at $(3;5)$

5)

a) $A(-4;-1)$ and $B(2;5)$

b) $AB = \sqrt{72} = 6\sqrt{2}$ units

c) N is the point $(-1;2)$

d) $ON \perp AB$

e) $y = -2x - 9$

f) C is the point $(-10;11)$

6)

a) P is the point $(-6;-3)$

b) $y = -2x - 15$

7)

a) $(x-3)^3 + (y-2)^2 = 16$

b) $y = 6$

Chapter 11Exemplar June Exams (Answers)

Paper 1

Question 1:

1.1

1.1.1 $x=-2$ or $x=3$

1.1.2 $x=-6,77$ or $x=1,77$

1.1.3 $x<-\frac{3}{2}$ or $x>1$ this is also written as $x\in\left(-\infty;-\frac{3}{2}\right)$ or $(1;\infty)$

1.2 $x=\frac{15}{2}$ and $y=-5$ or $x=3$ and $y=4$

1.3 $p=-5$ and $q=-5+2=-3$

Question 2:

2.1 1 312 km

2.2 48 km

Question 3:

3.1 $6+8+10+12$

3.2 10 094

Question 4:

4.1 1 594 317

4.2 1 062 882

4.3 Proof

Question 5:

5.1 8,31 years

5.3 R560,48

5.2 R3230

5.4 31 years and 2 months

Question 6:

6.1 90%

6.2 30 days

Question 7:

7.1 $y=2(x-2)^2+1$

7.4 $p(x)=2(x-3)^2+4$

7.2 $y=\frac{2}{x-2}+1$

7.5 $2<x\leq 3$ this can also be written as $x\in(2;3]$

7.3 $y\geq 1$ this can also be written as $y\in[1;\infty)$

Question 8:

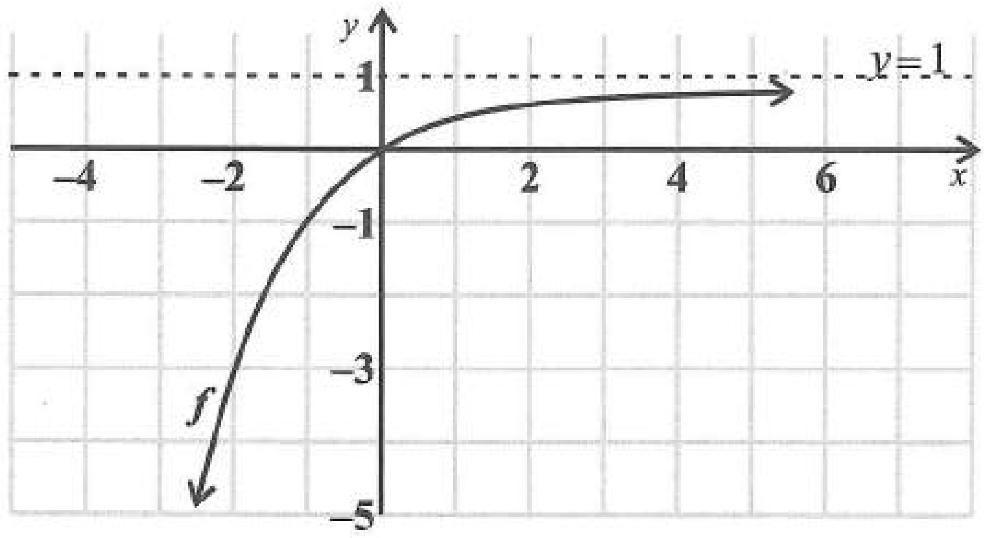
8.1 x-int : $(0;0)$ and y-int : $(0;0)$

8.4 $g(x)=-2^{-x-1}+1$

8.2 $y=1$

8.5 $x=-2,6$

8.3



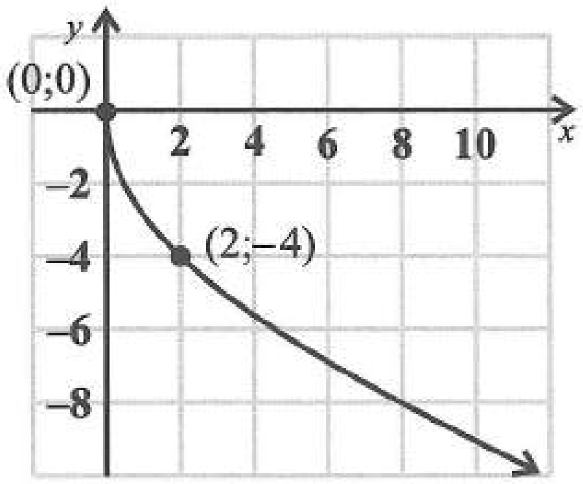
Question 9:

9.1 $a = \frac{1}{8}$

9.2 $y = -\sqrt{8x}\ ;\ x \geq 0$

9.3 $y \leq 0$ this can also be written as $y \in (-\infty;0]$

9.4



9.5 $y = -\sqrt{-8x}\ ;\ x \leq 0$

Question 10:

10.1 $-2x$

10.2 $\frac{dy}{dx} = -\frac{5}{2x^2} + \frac{1}{4\sqrt{x}}$

Question 11:

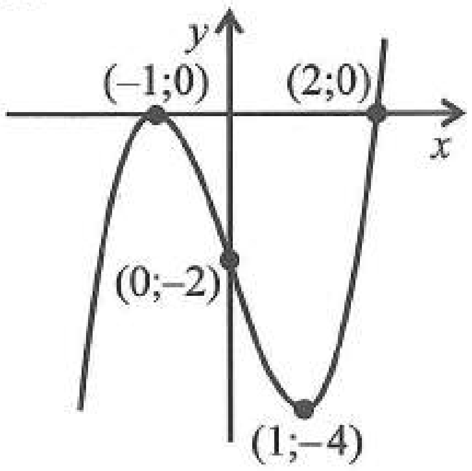
11.1 $(0;-2)$

11.2 $(-1;0)$ and $(2;0)$

11.3 $(-1;0)$ and $(1;-4)$

11.4 $(0;-2)$

11.5



11.6 $x < 1$ or $0 < x < 1$ this can also be written as $x \in (-\infty;-1)$ or $(0;1)$

Question 12:

12.1 Proof

12.2 Proof

12.3 $x = 14$ m

12.4 Cost = R22 407,08

Question 13:

13.1 $123456788 \times 123456790 - (123456789)^2 = -1$

13.2 1997

Paper 2

Question 1:

1.1 $C(-6;2)$ and $D(-12;14)$

1.2 $A\hat{B}K = 18,43^\circ$

1.3 $(x-6)^2(y+4)^2 = 180$

1.4 $y = \frac{1}{2}x + 8$

Question 2:

2.1 Proof

2.2 $A\left(3;\frac{-3}{2}\right)$

2.3 $y = -2x + \frac{9}{2}$

2.4 $BQ = 7,5$ units

2.5 Proof

2.6 $R\left(6;-\frac{15}{2}\right)$

Question 3:

3.1 Proof

3.2 $y = -2x + 10$

3.3 $y = 2x + 3$

3.4 $D\left(\frac{2}{5};\frac{19}{5}\right)$

3.5 $\theta_{BC} = 116,57^\circ$

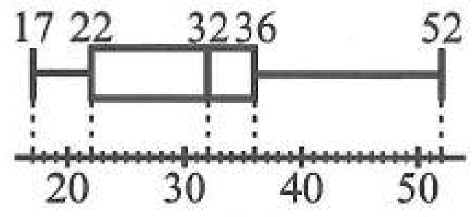
3.6 $B\hat{C}A = 36,86^\circ$

Question 4:

4.1 median $= 32$

4.2 $IQR = 36 - 22 = 14$

4.3



4.4 The data is skewed to the right. There is a greater spread of scores to the right of the median.

Question 5:

5.1 $-p$

5.2 $\frac{1}{\sqrt{1+p^2}}$

5.3 $\frac{p^2-1}{1+p^2}$

5.4 $\frac{1+p}{\sqrt{2+2p^2}}$ can also be written as $\frac{\sqrt{2}(1+p)}{2\sqrt{1+p^2}}$

Question 6:

6.1

6.1.1 $b = -1$

6.1.2 amplitude $= 1$

6.1.3 $k = 2$

6.1.4 $(-45^\circ;0)$ and $(45^\circ;0)$

6.1.5 maximum $= 0$

6.2

6.2.1 $x \in [-45^\circ;45^\circ]$

6.2.2 $x = 0^\circ$

6.2.3 $x \in (-60^\circ;60^\circ); x \neq 0^\circ$

Question 7:

7.1 $-\sin\theta$

7.2 Proof

7.3 $x = \pm 101,5^\circ + 360^\circ k; k \in \mathbb{Z}$

Question 8:

8.1 A$(-2;6)$ and $r=\sqrt{2}$

8.2 $y=-x+2$

8.3 C$(-6;2)$

Question 9:

9.1 $\text{AC}=185,53\,\text{m}$

9.2 $\text{AD}=259,46\,\text{m}$, $\therefore$ He is incorrect

9.3 Each side is equal to 230m

9.4 6,5 Empire State buildings would fit on the base

Question 10:

10.1 5,33

10.2 The mean would increase by 5 and the standard deviation would not be affected

Question 11:

11.1 Proof

11.2 $\text{BQ}=32\,\text{m}$

Chapter 12 Euclidean Geometry (Answers)

Ratios

Exercise 1:

1)

a) $\frac{AR}{RB}=\frac{5}{3}$

b) $\frac{AR}{AB}=\frac{5}{8}$

c) $\frac{AB}{RB}=\frac{8}{3}$

2)

a) $\frac{PA}{AQ}=\frac{1}{3}$

b) $\frac{PA}{PQ}=\frac{1}{4}$

c) $\frac{PQ}{QA}=\frac{4}{3}$

3)

a) $\frac{DC}{CE}=\frac{1}{1}$

b) $\frac{DC}{DE}=\frac{1}{2}$

c) $\frac{DE}{EC}=\frac{2}{1}$

4) $PR = 196\text{ units}$

5) $AB = 130\text{ mm}$

Theorem 1

Exercise 2:

1)

a) $x = 42\text{ units}$

b) $x = 40\text{ units}$

c) $x = 8\text{ units}$

d) $x = 8\text{ units}$

e) $x = 20\text{ units}$

f) $x = 10\text{ units}$

2)

a) Proof

b) Proof

c) $DG : GB = 2:1$

d) Proof

e) $QS : QT = 23:34$

f) $PO : OC = 5:2$

g) $BF = 26{,}67\text{ units}$

h) $BE : BG = 4:3$

3)

a) $\frac{EH}{HC}=\frac{4}{3}$

b) $\frac{BG}{CG}=\frac{2}{1}$

Theorem 1 (Converse)

Exercise 3:

1) Proof
2) Proof

Midpoint Theorem

Exercise 4:

1)

a) Proof

b) Proof

c) Proof

2)

a) Proof

b) Proof

3)

a) Proof

b) Proof

4) Area $\triangle AEG = 6\text{ cm}^2$

5) Perimeter $= 6x - 6\text{ units}$

6) $CG = 270\text{ cm}$

Similarity

Exercise 5:

1)
a) Proof b) $EC = 13,5\text{m}$

2)
a) Proof b) Proof

3)
a) Proof b) Proof

4)
a) Proof b) Proof

5) Proof

6) Proof

7)
a) Proof b) $\frac{EB}{ED} = \frac{1}{3}$

Pythagoras

Exercise 6:

1)
a) $\triangle ABD \mathbin{|||} \triangle CBA \mathbin{|||} \triangle CAD$ ⊥ From rt. ∠ Vertex to hyp.
b) $AD^2 = DB \times DC$
$AB^2 = DB \times BC$
$AC^2 = CB \times CD$
c) $AD = 3,75\text{m}$

2)
a) ⊥ From rt. ∠ Vertex to hyp.
b) $AD^2 = DB \times DC$
$AB^2 = DB \times BC$
$AC^2 = CB \times CD$
c) Proof
d) Proof

Mixed Exercise:

1)
a) Proof
b) Proof
c) Proof
d) $AB = 8\text{units}$
e) $AG = 5\text{units}$
f) $AF = 4\text{units}$

2)
a) $\frac{YP}{YR} = \frac{5}{7}$ b) $\frac{QR}{PT} = \frac{7}{5}$ c) $\frac{\text{Area } \triangle TPY}{\text{Area } \triangle QRY} = \frac{25}{49}$

3)
a) Proof b) $BD = \sqrt{6}$ units c) $AE = 6\sqrt{10}$ units

4)
a) $BF : FC = 1 : 1$ b) $AB : AC = 3 : 5$

5)
a) $\frac{AH}{ED} = \frac{4}{3}$ b) $\frac{BE}{CD} = 2$ c) $AD \times HE = 72\text{cm}$

6)
a) Proof
b) Proof
c) Proof
d) $KD = 2,67\text{cm}$

7)
a) Proof
b) $AC = 5\text{cm}$
c) Proof
d) Proof

8)

a) $\hat{T}_2 = \hat{S}_1 = x$ $(\angle\text{'s in same seg.})$

$\hat{T}_2 = \hat{R}_1 = x$ (tan chord thm.)

$\hat{S}_1 = \hat{M}_1 = x$ $(\text{alt. } \angle\text{'s, RS//QT})$

$\hat{M}_1 = \hat{M}_4 = x$ $(\text{vert. opp. } \angle\text{'s})$

$\hat{T}_2 = \hat{R}_3 = x$ $(\text{alt. } \angle\text{'s, RS//QT})$

b) Proof

c) PM is a tangent to the circle passing through M, Q and R. This is because the angles between the line and chord are equal in the opposite segment.

d) $PM = 2\text{cm}$

9)

a) Proof

b) Proof

c) $PR = 2\sqrt{41}\,\text{cm}$

d) Proof

10)

a) Proof

b) $AF = 6,4$ units

c) $\triangle ABC|||\triangle BDE|||\triangle AFB$

Chapter 13 Statistics (Answers)

Grade 11 Revision

Exercise 1:

1)

a) $d = 2$

$a = 3 \rightarrow (5-2)$

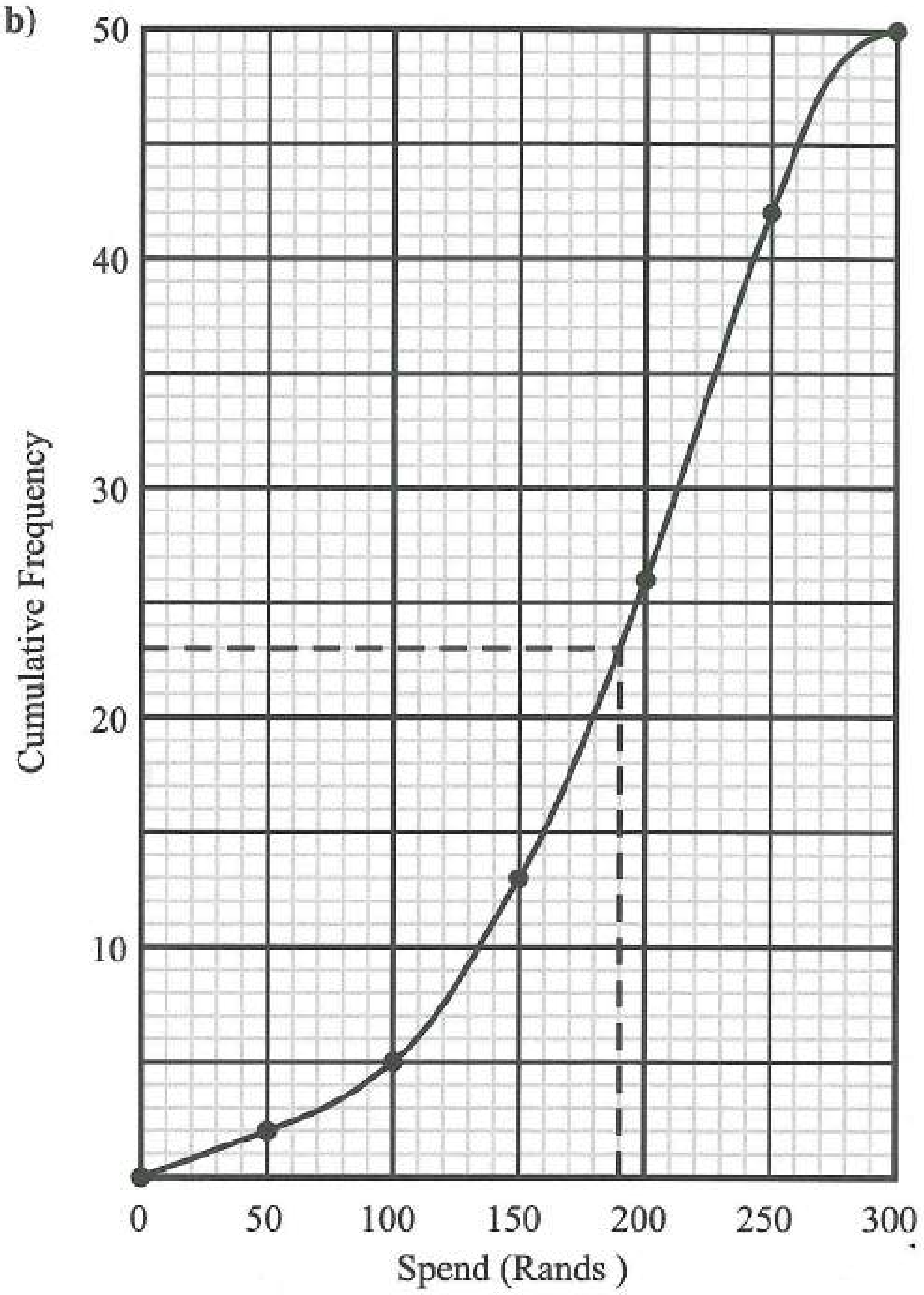
$b = 13 \rightarrow (26-13)$

$f = 50 \rightarrow (\text{total})$

$e = 42 \rightarrow (50-8)$

$c = 16 \rightarrow (42-26)$

b)



c) 27

2) $w = 2$

$x = 8$

$y = 15$

$z = 13$

3)

a) 3 700 cars **b)** 700 cars **c)** 250 cars **d)** 45 minutes

4)

a) 48 points **b)** 16,04 points **c)** 70%

5)

a) $40 \leq d \leq 54$ **b)** $d = 91$

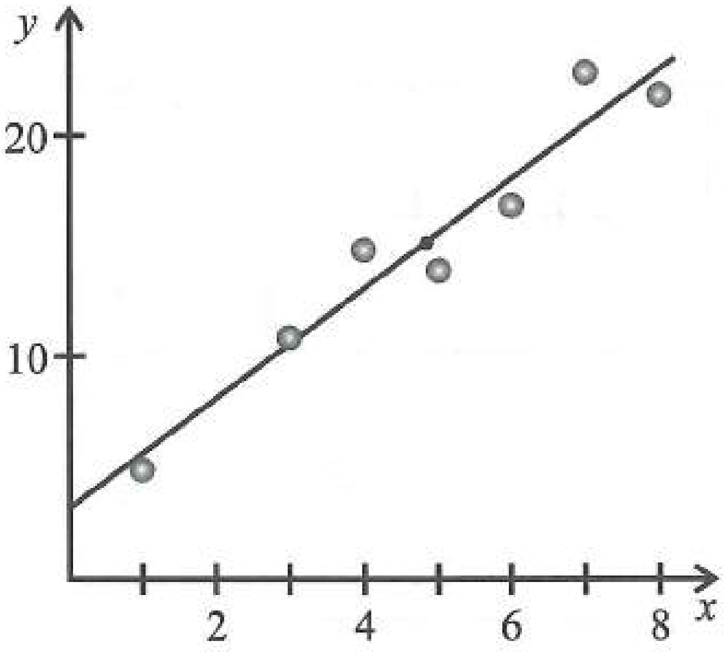
6)

a) 62,24 km/h **b)** 65 km/h **c)** 17,23 **d)** Yes

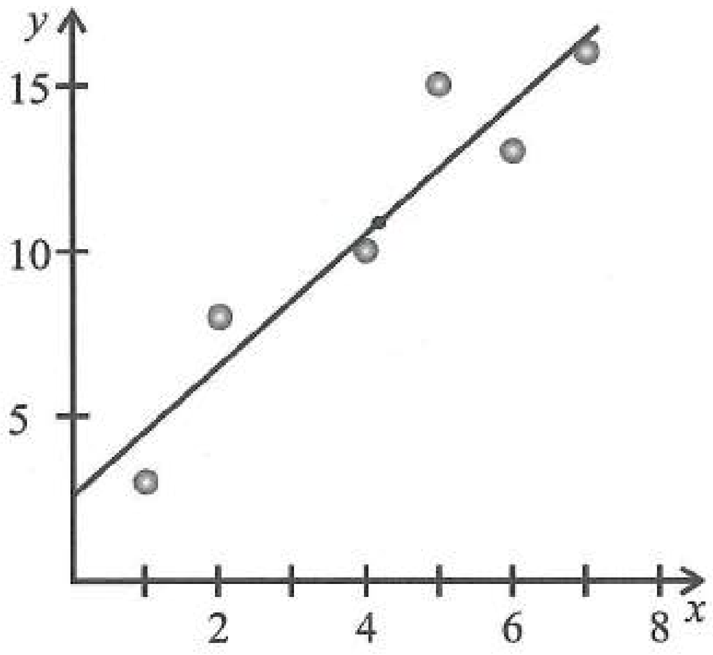
Exercise 2:

1)

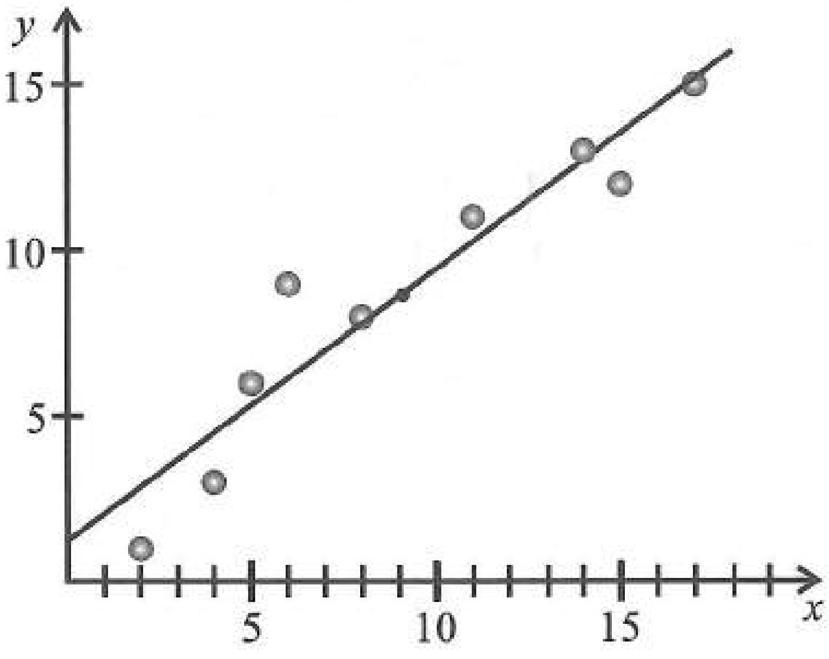
a) $y = 2{,}50x + 3{,}12$; $(4{,}86 ; 15{,}29)$



b) $y = 1{,}98x + 2{,}58$; $(4{,}17 ; 10{,}83)$



c) $y = 0{,}83x + 1{,}13$; $(9{,}11 ; 8{,}67)$



2)

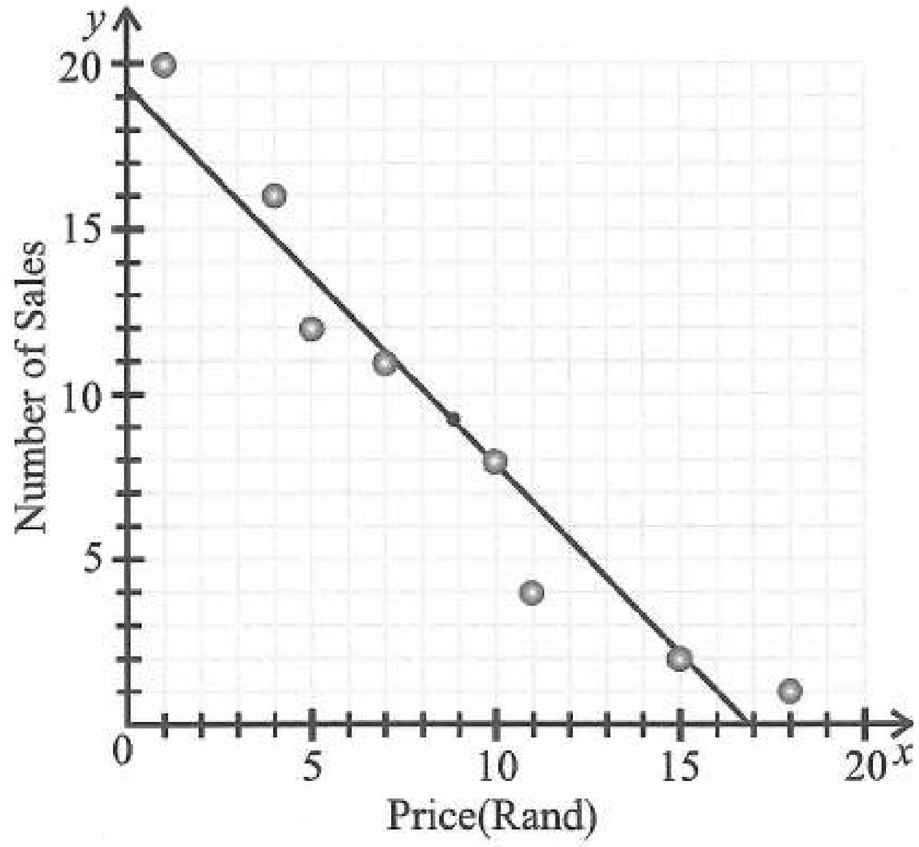
a) $y = -1{,}14x + 19{,}39$

b) $(8{,}875 ; 9{,}25)$

c) 10000

d) R17

e)



3)

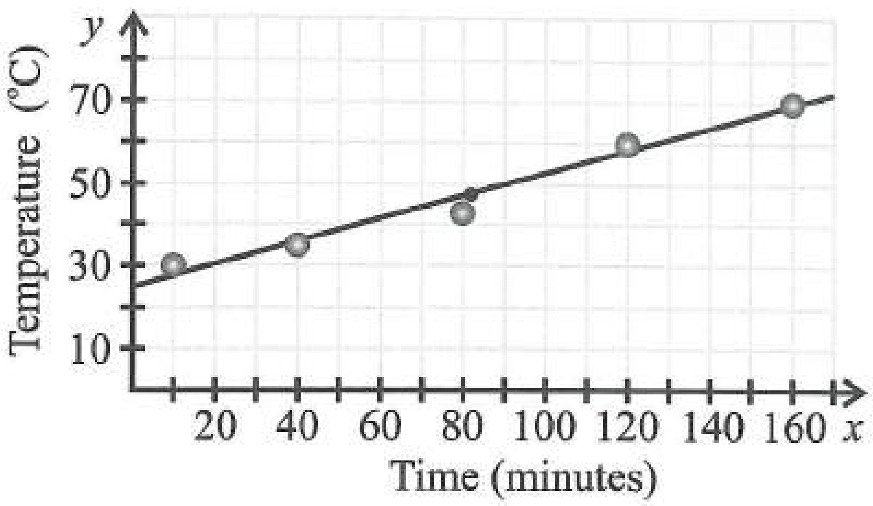
a) $y = 0,28x + 24,81$

b) $(82;47,6)$

c) $24,81°\text{C}$

d) 204,25 minutes

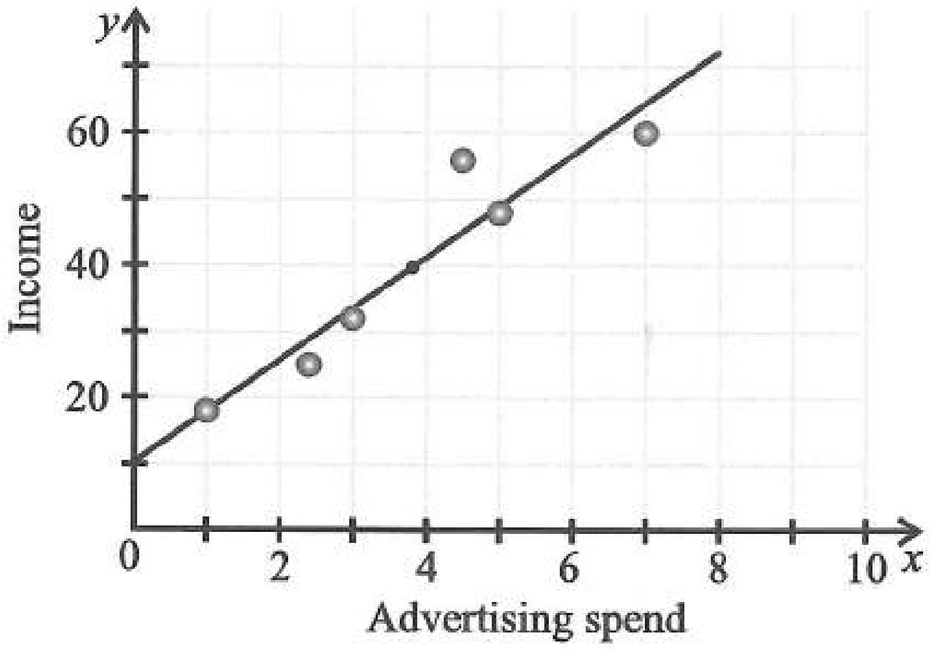
e)



Mixed Exercise:

1)

a)



b) $(3,82;39,83)$

c) $y = 7,69x + 10,49$

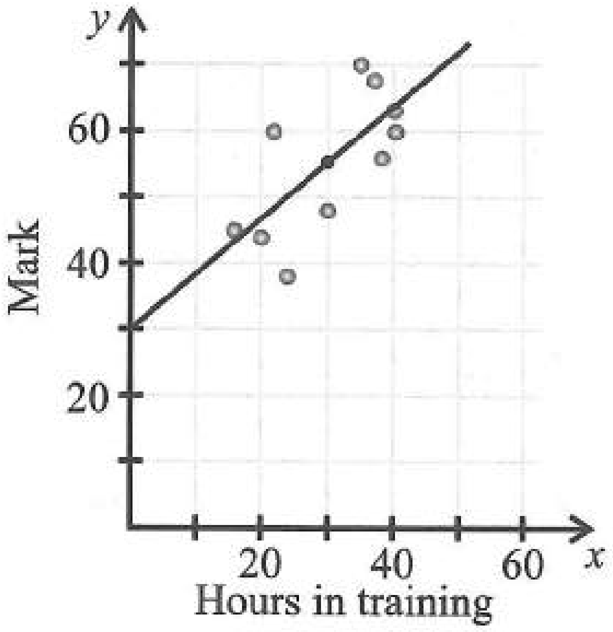
d) $r = 0,95$

e) The correlation coefficient is a positive value very close to 1.

f) Income = R37 405

2)

a)



b) $(30,1;55,3)$

c) $y = 0,84x + 29,88$

d) $r = 0,69$

e) 48,36

f) No, the correlation coefficient is much less than 1. There is a positive correlation. However, it is not strong. The training programme should be revised to meet the required level.

Chapter 14 Probability (Answers)

Grade 11 Revision

Exercise 1:

1)

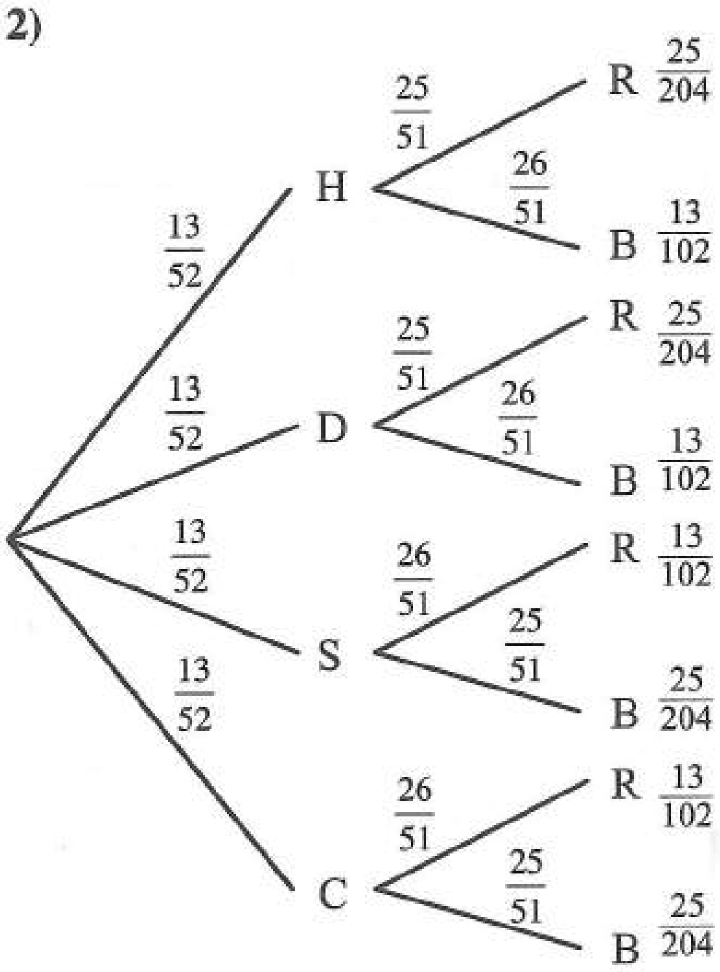
a) Neither

b) Complementary

c) Mutually Exclusive

d) Complementary

2)



$\frac{13}{52}$
H
$\frac{25}{51}$
R $\frac{25}{204}$
$\frac{26}{51}$
B $\frac{13}{102}$
$\frac{13}{52}$
D
$\frac{25}{51}$
R $\frac{25}{204}$
$\frac{26}{51}$
B $\frac{13}{102}$
$\frac{13}{52}$
S
$\frac{26}{51}$
R $\frac{13}{102}$
$\frac{25}{51}$
B $\frac{25}{204}$
$\frac{13}{52}$
C
$\frac{26}{51}$
R $\frac{13}{102}$
$\frac{25}{51}$
B $\frac{25}{204}$

a) $P(2\text{ Black}) = 2\left(\frac{25}{204}\right) = 24{,}51\%$

b) $P(\text{Spade and Red}) = \frac{13}{102} = 12{,}75\%$

c) $P(\text{Spade and Black}) = \frac{25}{204} = 12{,}25\%$

d) $P(S' \text{ or } R') = \frac{13}{102} + \frac{25}{204} = 25\%$

3)

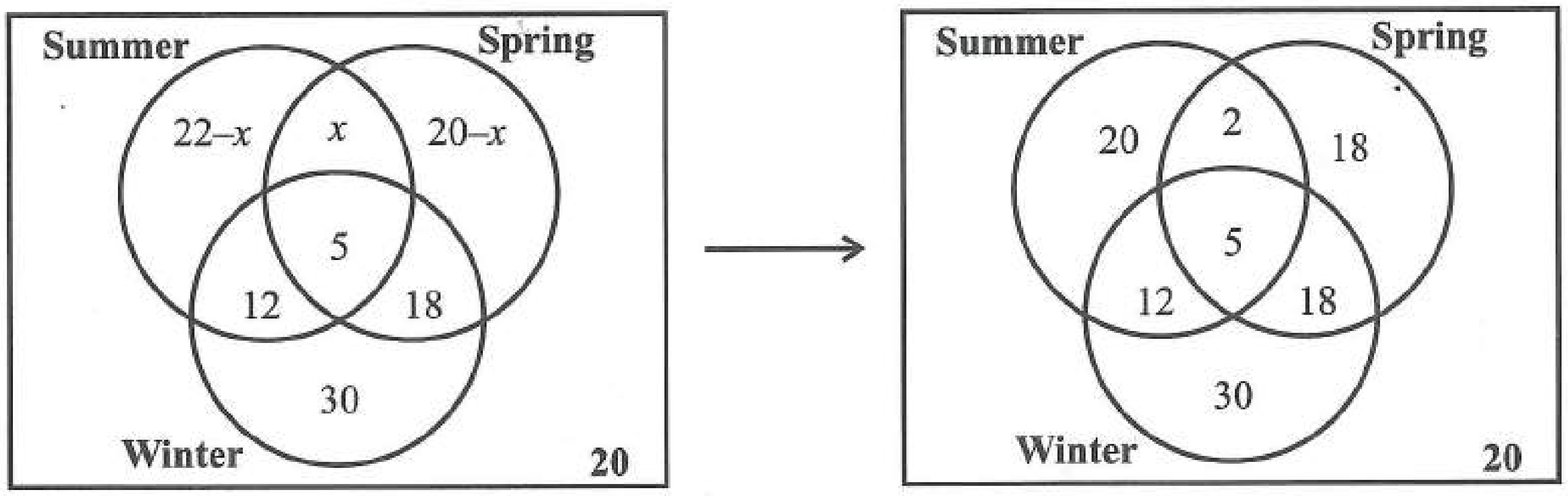
a) Mutually Exclusive

b) Neither

c) Independent

d) Independent

4)



Summer
Spring
Winter
22–x
x
20–x
5
12
18
30
20
Summer
Spring
Winter
20
2
18
5
12
18
30
20

$22 - x + x + 20 - x + 5 + 12 + 18 + 30 + 20 = 125$

$\therefore x = 2$

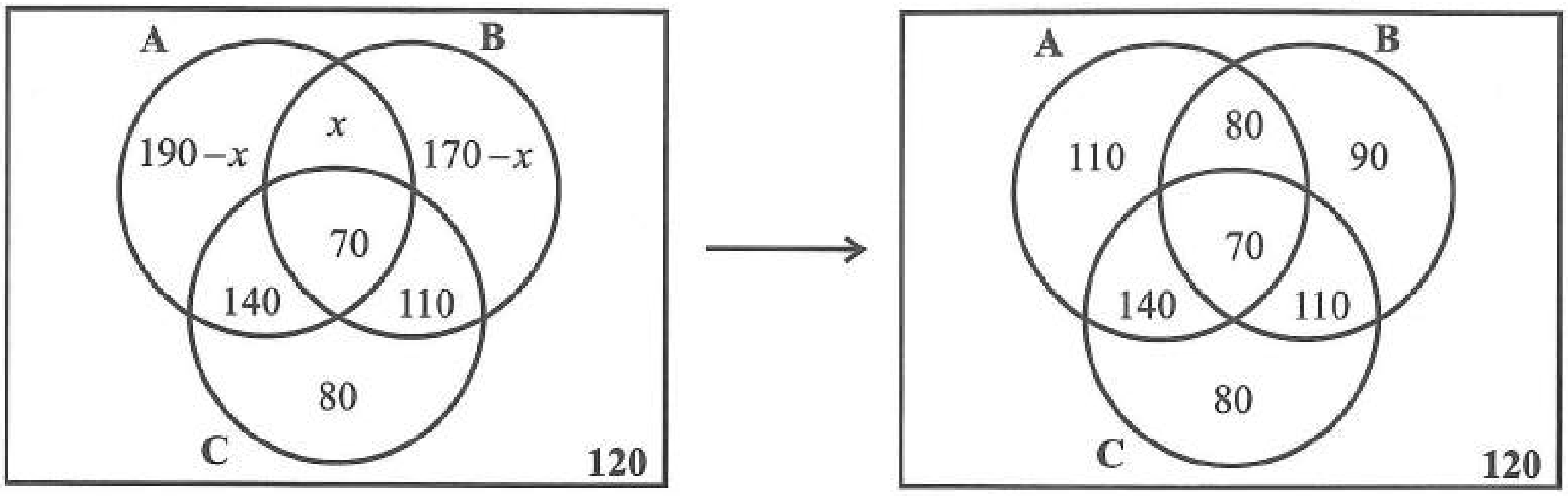
a) 0,84

b) 0,496

c) 0,136

d) 0,144

5)


A
B
190 − x
x
170 − x
70
140
110
80
C
120
A
B
110
80
90
70
140
110
80
C
120

$800 = 190 - x + x + 170 - x + 140 + 70 + 110 + 80 + 120$

$\therefore x = 80$

a) 0,15

b) 0,85

c) 0,5

d) 0,7125

6)

	Male	Female	Total
Had side effects	1500	**1500**	3000
No side effects	**1000**	1000	**2000**
Total	**2500**	**2500**	5000

a) 0,2

b) 0,5

The Fundamental Counting Principle

Exercise 2:

1) 288

2) 32

3)

a) 17 576 000

b) 11 232 000

c) 5 745 600

d) 5 616 000

e) 2 160 000

f) 7 862 400

4)

a) 15

b) 225

c) 120

Factorial Notation

Exercise 3:

1)

a) 4 989 600

b) 453 600

c) 9,09%

2)

a) 20 160

b) 720

c) 25%

3)

a) 479 001 600

b) 103 680

c) 0,02%

4)

a) 39 916 800

b) 172 800

c) 7 257 600

d) 0,43%

Exercise 4:

1) 504
2) 2730
3) 43 680
4) 336

Mixed Exercise:

1) $0{,}7 = 70\%$

2)
- **a)** 0,99
- **b)** No

3)
- **a)** 720
- **b)** 240

4)
- **a)** 39 916 800
- **b)** 241 920
- **c)** 25,45%
- **d)** 28

5)
- **a)** $a = 450$, $b = 319$, $c = 298$ and $d = 748$
- **b)** 19,48%
- **c)** They are independent

6)
- **a)** 1 000
- **b)** 650
- **c)** $\frac{1}{650000}$
- **d)** 96,15%

7)
- **a)** $\frac{1}{8}$
- **b)** $\frac{7}{8}$

Chapter 15 Exemplar Preliminary Exams (Answers)

Paper 1

Question 1:

1.1

1.1.1 $x=-1$ or $x=2$

1.1.2 $x=-1{,}39$ or $x=-2{,}89$

1.1.3 $x\leq -1$ or $x\geq \frac{2}{3}$ this is also written as $x\in(-\infty;-1]$ $x\in\left[\frac{2}{3};\infty\right)$

1.2

1.2.1 $\frac{x}{y}=2$ or $\frac{x}{y}=-4$

1.2.2 $x=4$ and $y=2$ or $x=8$ and $y=-2$

1.3 27

Question 2:

2.1

2.1.1 $x=36$

2.1.2 $x=\pm 22{,}63$

2.2 $S_9=16352$

2.3 Proof.

Question 3:

3.1

3.1.1 $r=\frac{1}{2}$

3.1.2 Converges as $-1<r<1$

3.1.3 $S_\infty=96$

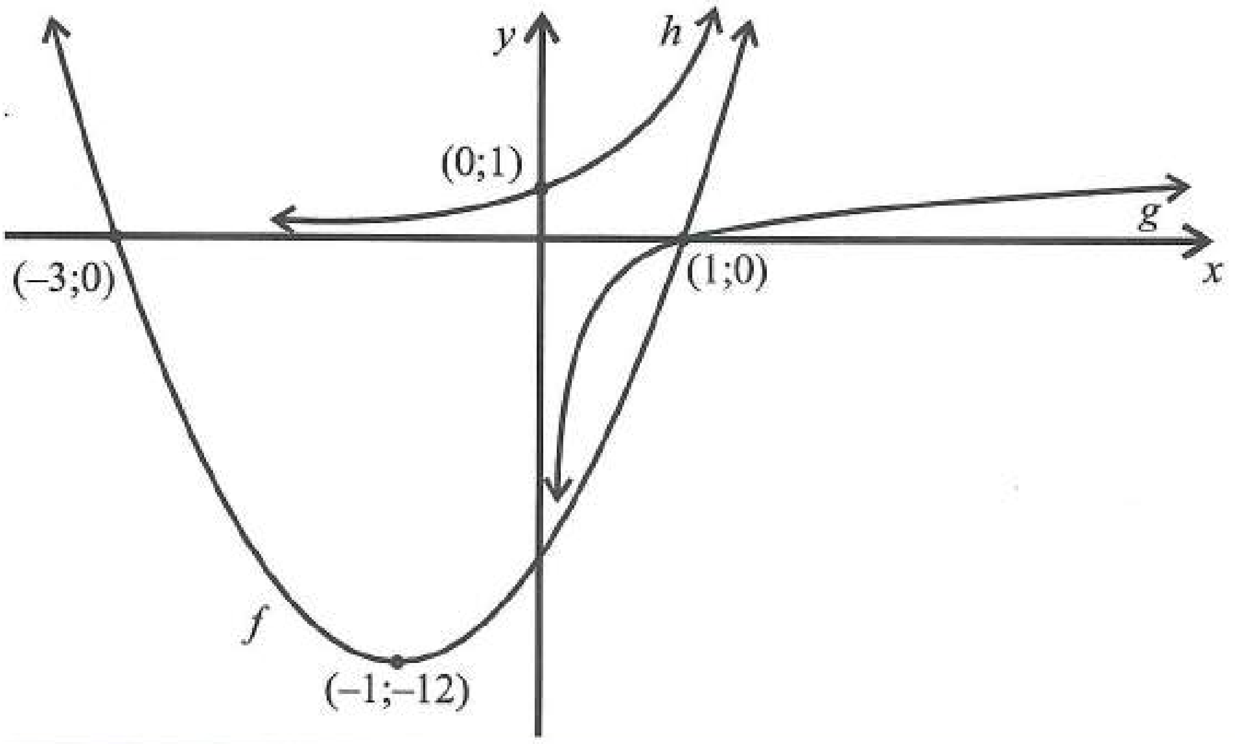
3.2 $x=9$ and $y=3$ or $x=\frac{9}{2}$ and $y=\frac{3}{4}$

Question 4:

4.1 $f(x)=-\frac{1}{10}x^2+6x$ and $h(x)=-3x+180$ and $g(x)=3x$

4.2

4.2.1 and 4.2.2



4.2.3 $p(x)=3(x-1)^2-12$

4.24. Proof

Question 5:

5.1 $A(2;0)$ and $B(0;-3)$

5.2 $y=-4$

5.3 $p(x)=2^{2x}+4$

5.4 $p^{-1}(x)=\frac{1}{2}\log_2(x-4)$

5.5 $h(x)=-\frac{1}{2}\log_2(x-4)$

5.6 -1

Question 6:

6.1

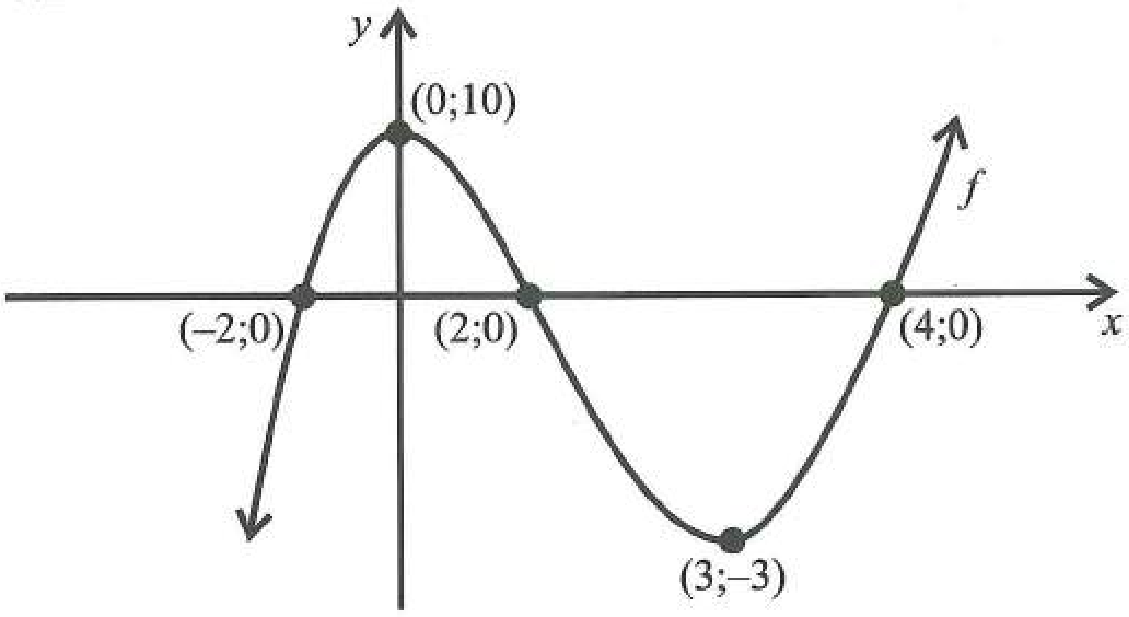
6.1.1 Proof

6.1.2 R3 783,83

6.2 31 years and 10 months

Question 7:

7.1



7.2

7.2.1 $h''(-1)=2$

7.2.2 $h'(x)=-5x^2+8x$

7.2.3 $x=0$ or $x=-\frac{8}{5}$

Question 8:

8.1 $f'(x)=-2x$

8.2

8.2.1 $D_x\left[-5+8x^{\frac{1}{2}}\right]=\frac{4}{x^{\frac{1}{2}}}$

8.2.2 $\frac{dy}{dx}=-\frac{3}{2x^4}-\frac{5}{8}$

Question 9:

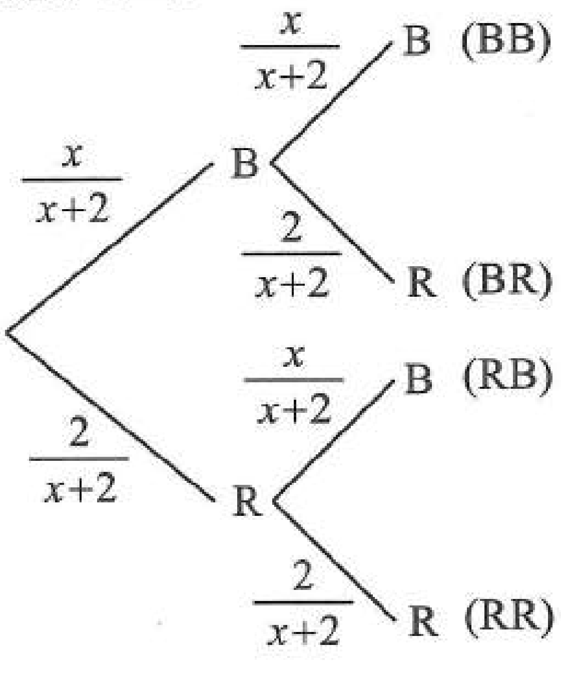
9.1 $\text{BC}=-2k+12$

9.2 $\text{B}(3;6)$

9.3 18 units^2

Question 10:

10.1 $x=6$


$\frac{x}{x+2}$
B (BB)
$\frac{x}{x+2}$
B
$\frac{2}{x+2}$
R (BR)
$\frac{x}{x+2}$
B (RB)
$\frac{2}{x+2}$
R
$\frac{2}{x+2}$
R (RR)

10.2

10.2.1 362 880

10.2.2 2,78%

10.3

10.3.1 60

10.3.2 40%

Paper 2

Question 1:

Proof

Question 2:

2.1

2.1.1 $Q\hat{P}R = 75^\circ$

2.1.2 $\hat{R} = 25^\circ$

2.1.3 $\hat{T}_1 = 25^\circ$

2.2 Proof

2.3 Proof

2.4 Proof

Question 3:

3.1 $\frac{EF}{ED} = \frac{1}{3}$

3.2 $ED = 48\text{cm}$

3.3 $\triangle EFB \,|||\, \triangle EDC$

3.4 $DC = 36\text{cm}$

3.5 Proof

Question 4:

Proofs

Question 5:

5.1 9,58 units^2

5.2 4,23 units

5.3 1,97 units

Question 6:

6.1 $x = 11$ or $x = 3$

6.2 $x = 13$

6.3 $x = 13$

6.4 $x = -\frac{1}{3}$

Question 7:

7.1 $M(3;1)$

7.2 $k = -6$ or $k = -2$

7.3 $y = \frac{7}{9}x - \frac{4}{3}$

7.4 $(12;8)$

7.5 $m = -4$

Question 8:

8.1 $A(-3;3)$

8.2 $r = \sqrt{17}$ units

8.3 $y = \frac{13}{16}x + \frac{43}{4}$

Question 9:

9.1 $y = -6,43x + 434,47$

9.2 $r = -0,77$

9.3 R331,59

9.4 It is reliable

Question 10:

10.1

Distance (d) (km)	Frequency	Midpoint
$0 < d \le 5$	5	2,5
$5 < d \le 10$	41	7,5
$10 < d \le 15$	77	12,5
$15 < d \le 20$	58	17,5
$20 < d \le 25$	39	22,5
$25 < d \le 30$	17	27,5
$30 < d \le 35$	3	32,5

10.2 $\overline{x} = 15,58$

10.3 $\sigma = 6,33$

10.4 $10 < d \le 15$

Question 11:

11.1

11.1.1 Proof

11.1.2 Proof

11.2 Proof

Question 12:

12.1

12.1.1 $\sqrt{1-p^2}$

12.1.2 $\frac{3}{2}p$

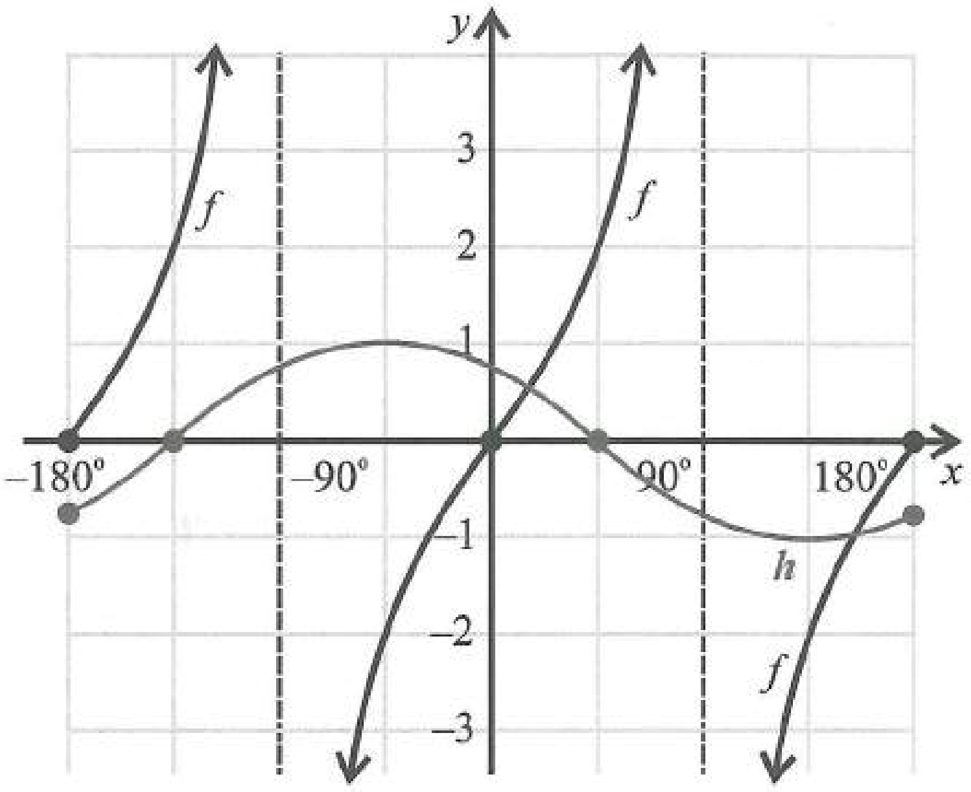
12.2 $x = 14{,}60^\circ + 360^\circ k$ or $x = 165{,}40^\circ + 360^\circ k$; $k \in \mathbb{Z}$

12.3 $x = 26{,}57^\circ$

Question 13:

13.1 $a = 2$

13.2



13.3 $b = 2$ and $c = 1$

Application of the Sine, Cosine and Area Rule (Investigation)

This investigation requires you to make extensive use of the Sine, Cosine and Area Rule. Before attempting the investigation you are encouraged to revise these concepts from your Grade 11 books or on page 119 of this textbook. You may also find it useful to obtain a box of matches in order to construct the figures within the investigation.

Question 1:

1.1 How many different types of triangles can be made with a perimeter of 12 matches? Name the types of triangles and their dimensions. (6)

1.2 Using the matches as units draw all the possible triangles. (3)

1.3 Which of the triangles would have the greatest area? (10)

[19]

Question 2:

2.1 What regular polygons can be made using all 12 matches? (12)

2.2 Draw all of the polygons that you have found using all of the matches. (4)

2.3 Determine the areas of all of the possible polygons with a perimeter of 12 matches. (10)

2.4 What can you conclude from your answers in 2.3 above? (1)

[27]

Question 3:

Summarise your findings from the investigation and make generalisations or conjectures from what you have observed. Remember to motivate all of your answers. (4)

[4]

Total: [50]

The marking rubric below will assist you in understanding what is required from this investigation.

Criteria	0	1	2	3	4	5
	Did not attempt	Attempted but with many errors	Attempted but with 3 errors	Attempted but with 2 errors	Attempted but with 1 error	No error and well presented
Named triangles with measurements						
Drawing of triangles with measurements						
Found the areas of all triangles						
Named polygons with measurements						
Drawing of polygons with measurement						
Found area of dodecagon						
Found area of Hexagon						
Found area of square						
Found area of triangle						
Conclusion of investigation						

Grade 11 Revision of Statistics (Assignment)

Question 1:

1.1 How does the term *quartile* relate to how data values are grouped when using a five-number summary? (1)

1.2 What is the name for the difference between the minimum and maximum values in a five-number summary? (1)

1.3 What is the name for the difference between the third quartile and the first quartile in a five-number summary? (1)

1.4 Using this textbook or any other resource, research outliers of a data set and state how they are related to the whiskers of a box and whisker plot? (1)

[4]

Question 2:

Fifteen Grade 12 learners were asked how many times they went to the movies in one year. The results are tabulated below:

10	15	12	6	3	14	20	8	12	10	5	13	14	9	24

2.1 Give the five number summary for the data. (5)

2.2 Draw a box and whisker diagram for the data. (3)

[8]

Question 3:

For each description, invent a seven-value data set so that all the values are less than 10 and meet the following conditions:

3.1 The box and whisker represents data with a median that is not inside the box. (2)

3.2 The box and whisker represents data with an interquartile range of zero. (2)

3.3 The box and whisker is missing its right whisker. (2)

[6]

Question 4:

The mathematics marks (expressed as a %) of Grade 12 learners at a school in Gauteng are shown below:

23	25	32	31	33	34	41	42	43	44	49	50	48
51	53	54	54	55	58	59	59	60	61	62	62	63
66	66	66	66	66	66	66	70	71	72	73	74	75
75	75	75	76	76	78	80	95	88	86	89	92	82

4.1 Determine the mean of the data. (1)

4.2 Construct a frequency table from the above data. Make provision for class intervals, frequency, midpoint of class interval, frequency x midpoint and cumulative frequency columns. The class intervals should be $20 \le x < 30$, $30 \le x < 40$ etc. (6)

4.3 Construct an ogive from your table. (4)

4.4 Using a calculator and your table, determine the estimate mean. (2)

4.5 How does your answer in 4.4 compare to your answer in 4.1? Which answer is more accurate? (2)

4.6 Determine the standard deviation using a calculator and the values from your table. (2)

4.7 Draw a histogram for the data and discuss the symmetry of the data. (3)

[20]

Question 5:

The table below shows the Mathematics and Physical Science marks of 5 Grade 12 learners. Research scatter plots and the least squares regression line and then answer the questions which follow:

Learner	Mathematics (%) (x)	Physical Science (%) (y)
1	34	46
2	49	52
3	60	65
4	71	75
5	94	90
Sum		
Mean		

5.1 Determine the sum and mean for the Mathematics and Science marks. (4)

5.2 Write a brief description of the least squares regression line. (3)

5.3 Using a calculator determine the equation of the least squares regression line and draw a scatter plot of the above data. On your diagram show the least squares regression line and the mean point. (5)

[12]

Total: [50]

Grade 11 Revision of Functions (Assignment)

Question 1:

The graph of $f(x)=3^{x-p}+q$ cuts the y-axis at $(0;20)$. The line of $y=-7$ is a horizontal asymptote of f.

1.1 Determine the equation of f. (5)

1.2 Determine the value of r if $(r;10)$ is a point that lies on f. (5)

1.3 If h is a reflection of f about the y-axis, determine the equation of h. (2)

1.4 Sketch the graph of f. (3)

[15]

Question 2:

2.1 On the same set of axes, sketch the graphs of $f(x)=\tan x$ and $g(x)=-\cos x$ for $x\in\left[-180^\circ;180^\circ\right]$. (7)

2.2 Write down the period of f and g. (2)

2.3 Write down the domain of f and the range of g. (4)

2.4 Using your graphs, determine the value(s) of x, if $x\in\left[0^\circ;180^\circ\right]$ for which:

2.4.1 $f(x)+g(x)=1$ (2)

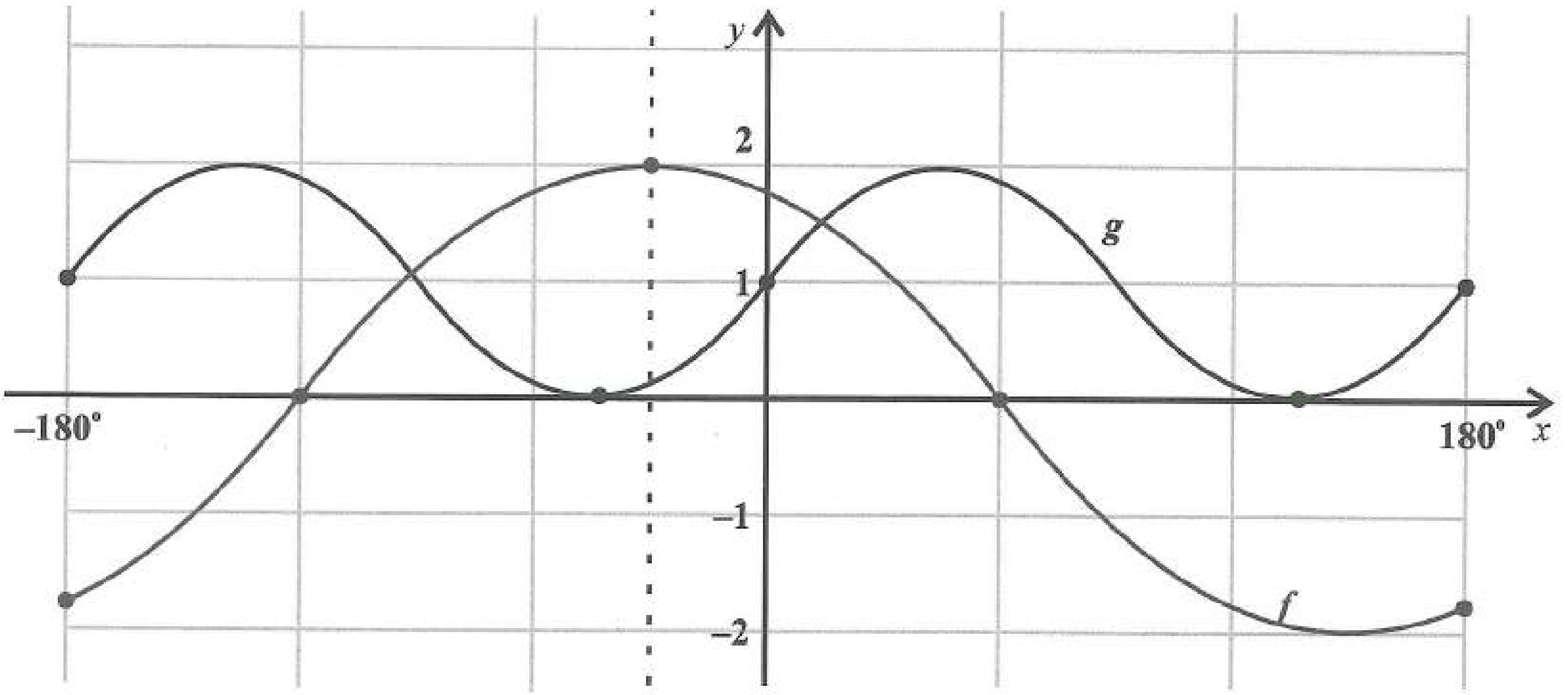
2.4.2 $f(x)>g(x)$ (2)

2.4.3 $\dfrac{f(x)}{g(x)}>0$ (1)

[18]

Question 3:

The sketch below shows the graph of $f(x)=a\cos x(x+p)$ and $g(x)=\sin kx+q$ for $x\in\left[-180^\circ;180^\circ\right]$.



3.1 Determine the value(s) of a, p, k and q, if the dotted line is the line of $x=-30^\circ$. (4)

3.2 Determine the value of $f\left(0^\circ\right)$ without using a calculator. (3)

3.3 If the y-axis is moved to the position of the dotted line, determine the new equation of f. (1)

[8]

Question 4:

The equation of a hyperbola is: $g : x \to \frac{2}{x}; x \neq 0$. Write down the equations of the following functions $f(x)$, which are obtained by translating $g(x)$ according to the following conditions:

4.1 A translation of –5 units vertically. (1)

4.2 A translation of 3 units to the right. (1)

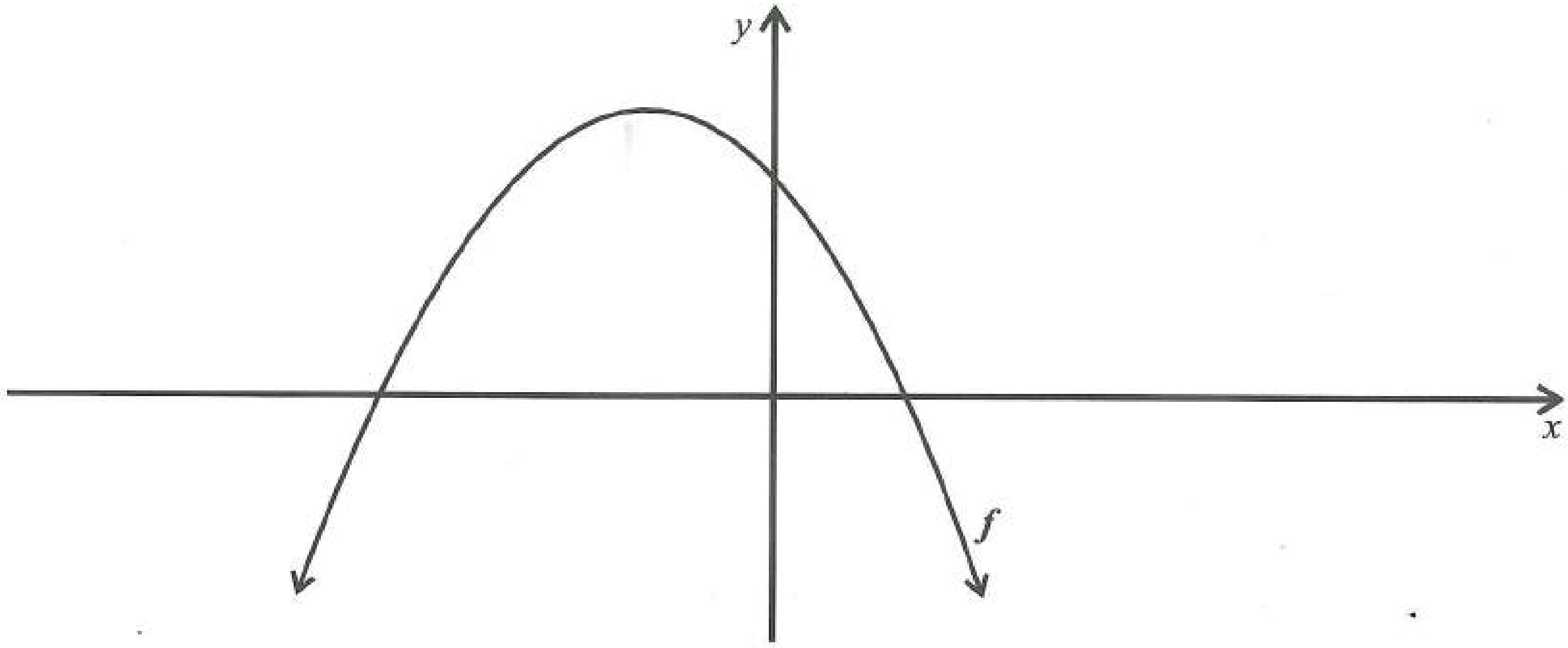
4.3 The graph has a horizontal asymptote of $y = 1$ and a vertical asymptote of $x = -1$. (1)

4.4 The asymptotes intersect at $(2;-3)$. (1)

[4]

Question 5:

$f(x) = -2x^2 - 8x - 1$ is sketched below:



5.1 Determine the turning point of *f*. (3)

5.2 Determine the value(s) of *k* for which $-2x^2 - 8x - 1 + k = 0$ has non-real roots. (2)

[5]

Total: [50]

Euclidean Geometry (Concurrency) Project

This project requires you to do your own research. All references from your research must be included. You will have to make accurate constructions and you will need to ensure that you have the following writing and measuring instruments:

1) Clutch pencil / sharp pencil
2) Eraser
3) Ruler
4) Protractor
5) Compass

Part 1: Terminology

Concurrency forms an integral part of Euclidean Geometry. Through your own research, define the following terms (Remember to include your references):

1.1 Concurrent lines (1)

1.2 Point of concurrency (1)

1.3 Median of a triangle (1)

1.4 Altitude of a triangle (1)

1.5 Perpendicular bisector of a side of a triangle (1)

1.6 Bisector of an angle of a triangle (1)

[6]

Part 2: The medians of the sides of a triangle

2.1 Construct triangle ABC. On the triangle, construct medians AF, CE and BG. What do you notice about the medians of a triangle? (3)

2.2 Clearly show and label point D, the point of concurrency. Research and write down the special name given to this point. (3)

2.3 Measure and record the lengths of the line segments AD; DF; BD; DG; CD and DE. (3)

2.4 By investigating the ratio of the line segments that you measured in 2.3 above, i.e. AD:DF; BD:DG and CD:DE, write down a conjecture for the medians of a triangle. (2)

[11]

Part 3: The bisectors of the angles of a triangle

3.1 Construct triangle ABC. On the triangle, construct AF, BE and CD so that they bisect $\hat{A}$, $\hat{B}$ and $\hat{C}$ respectively. What do you notice about the bisectors of the angles of a triangle? (3)

3.2 Clearly show and label point I, the point of concurrency. Research and write down the special name given to this point. (3)

3.3 Construct a circle centre H so that AB, BC and AC are all tangents to the circle. Write down an observation that you can make about point H and the circle which you have drawn. (3)

3.4 Write down a conjecture for the bisectors of the angles of a triangle. Your conjecture must include reference to a circle and tangents. (2)

[11]

Part 4: The perpendicular bisectors of the sides of a triangle

4.1 Construct triangle PQR. On the triangle construct perpendicular bisectors EC, FC and CG of sides PQ, PR and RQ with midpoints E, F and G respectively.
What do you notice about the perpendicular bisectors of a triangle? (3)

4.2 Clearly show and label point C, the point of concurrency. Research and write down the special name given to this point. (3)

4.3 Construct a circle which passes through the vertices of triangle PQR. How does point C relate to the circle? Research and write down the special name given to this circle. (3)

[9]

Part 5: The altitudes of a triangle

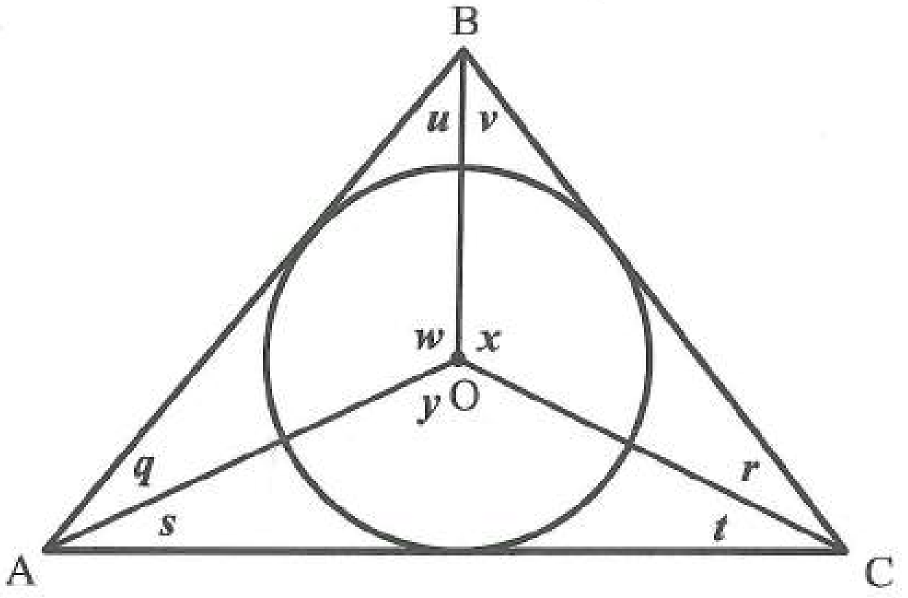
5.1 Construct triangle ABC. On the triangle, construct altitudes AD, BE and CF. What do you notice about the altitudes of a triangle? (3)

5.2 Clearly show and label point O, the point of concurrency. Research and write down the special name given to this point. (3)

[6]

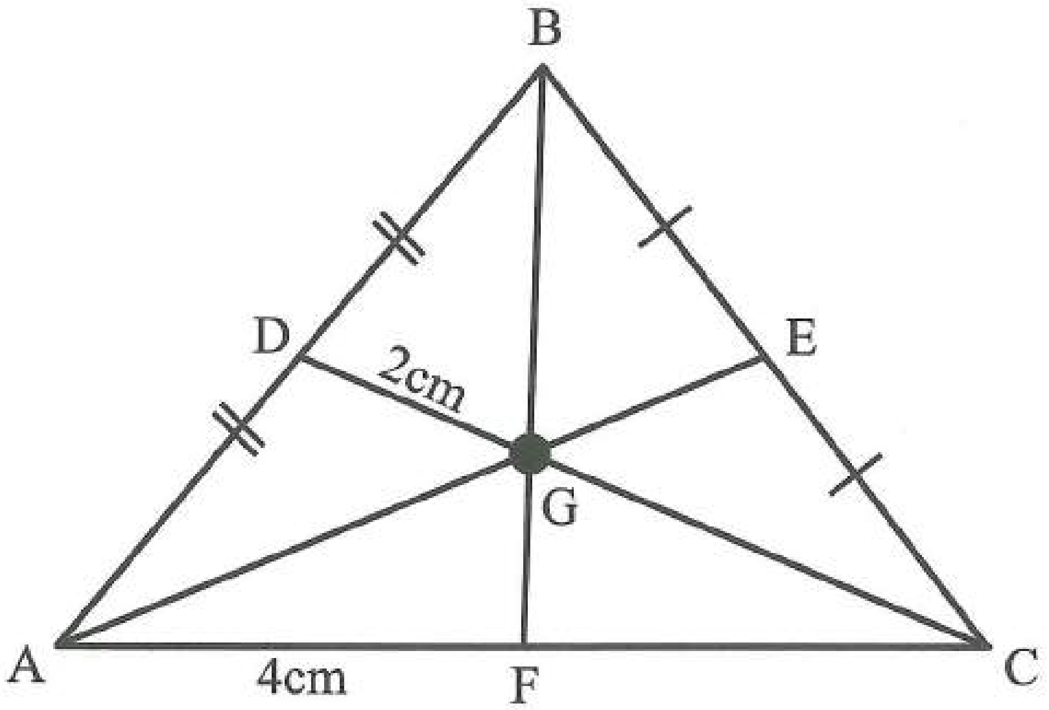
Part 6: Application

6.1 In the diagram below, O is the centre of the inscribed circle of triangle ABC. If $B\hat{A}C = 20^\circ$ and $A\hat{B}C = 80^\circ$ determine the values of q, s, t, u, v, w, x, y and r.



(4)

6.2 In triangle ABC below, AD = BD and BE = CE. If DG = 2 cm and AF = 4 cm, determine the lengths of CD and CF if BGF is a straight line.



(3)

[7]

Total: [50]

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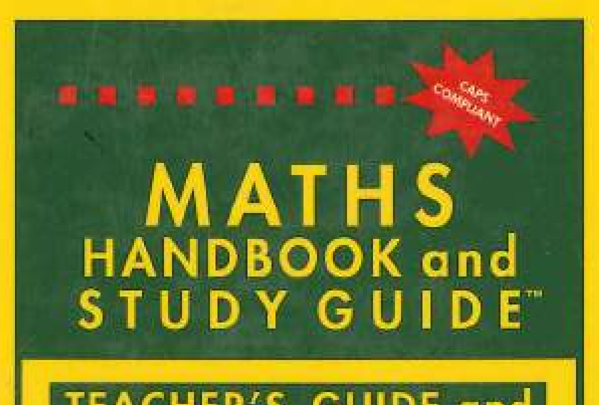

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